Problem 5.4: The coefficient of static friction between hard rubber and normal street pavement is about 0.90. On how steep a hill (maximum angle) can you leave a car parked?

The figure shows the situation described in this problem. We have an object (the car) of (unknown) mass $M$ sitting on an inclined surface and want to determine how big the angle $\theta$ can be before static friction can’t hold the car in place.

There are three forces acting on the car: $F_g$ straight down, $F_n$ (labelled as just $n$ in the figure) oriented perpendicular to the street, and some static friction $f_s$, oriented to oppose the motion (which, if there were no friction, would be for the car to slide down the slope).

We have friction here, so will ultimately need the normal force, so let’s look in the rotated Y direction first: $\Sigma F_y = 0$ so $F_n - mg \cos \theta = 0$ or $F_n = mg \cos \theta$. The maximum amount of static friction that can be present will be $f_{s,max} = \mu_s F_n$, so $f_{s,max} = \mu_s mg \cos \theta$.

Take note of how that term depends on the angle of the ramp (street). As the angle of the ramp increases, the cosine term gets smaller, so the amount of static friction present decreases as the street gets steeper and steeper.

In the rotated X direction, we want the car to be stationary, so $\Sigma F_x = ma_x = 0$. Here, the component of the force of gravity downslope will be $mg \sin \theta$, and note how that term depends on the angle of the street: as the angle increases, this term gets larger.

So: as the angle increases, the force of gravity down along the incline increases and the amount of static friction present decreases. Eventually there won’t be enough friction to hold the car in place anymore. That will occur when the downslope force of gravity becomes equal to the upslope force of (static) friction: $mg \sin \theta = \mu_s mg \cos \theta$.

Note here that at this point both $m$ and $g$ cancel out, leaving us with $\sin \theta = \mu_s \cos \theta$ or rearranging a bit: $\tan \theta = \mu_s$.

For this problem then, if $\mu_s = 0.9$ then $\tan \theta = 0.9$ from which $\theta = 41.987...^o$ or about $\theta = 42^o$.

This is actually a method for determining the coefficients of friction. If we put one object on another, then gradually increase the angle, eventually we’ll get to a point where the object on top starts sliding. That angle is the point where the downslope force of gravity has just exceeded the upslope force of static friction, giving us a simple way of finding $\mu_s$ between two materials.

(Note: we can determine the coefficient of kinetic friction in a similar way but it requires adjusting the angle until the object on the ramp slides down at a constant speed. At that special angle, the object is not accelerating, so $a = 0$ again but now the frictional force involved is $f_k$ instead of $f_s$, leading to $\tan \theta = \mu_k$.)
Problem 5.10: A wet bar of soap slides freely down a ramp 9.0 m long inclined at 8°. How long does it take to reach the bottom. Assume $\mu_k = 0.060$. (The hint on the course website said to assume the initial speed at the top of the ramp was zero.)

The figure shows the situation described in this problem. We have an object (the bar of soap) of (unknown) mass $M$ sitting on an inclined surface and want to determine how long it takes to slide down the ramp. We can apply Newton’s Laws here to determine the downslope acceleration of the object, then use our 1-D equations of motion to relate that acceleration to the time it takes to reach the bottom of the ramp.

In the rotated $X$ direction, we have a component of $F_g$ downslope and the force of kinetic friction upslope. The $X$ component of $F_g$ will be $mg \sin \theta$ (see the course website section on resolving vectors into components). $\Sigma F_x = ma_x$ in this direction then becomes: $-f_k + mg \sin \theta = ma_x$.

The force of friction depends on the normal force $f_k = \mu_k F_n$ so we’ll need to determine $F_n$. Looking in the (rotated) $Y$ direction, $\Sigma F_y = ma_y = 0$ so $F_n - mg \cos \theta = 0$ or $F_n = mg \cos \theta$.

That makes $f_k = \mu_k F_n = \mu_k mg \cos \theta$.

Substituting that into the boxed equation above, we have: $-\mu_k mg \cos \theta + mg \sin \theta = ma_x$.

I left everything symbolic up to this point because here we see that $m$ appears in every term in this equation, so we can divide the entire equation by $m$ and eliminate it:

$-\mu_k g \cos \theta + g \sin \theta = a_x$.

Rearranging terms a bit, we end up with: $a_x = g (\sin \theta - \mu_k \cos \theta)$.

We did this problem entirely generically up to this point, so that’s a generic equation for the downslope acceleration of an object down an incline in the presence of kinetic friction.

For our specific problem, we have $\theta = 8^\circ$ and $\mu_k = 0.06$ so $a_x = (9.8 \text{ m/s}^2)(\sin (8^\circ) - 0.06 \cos (8^\circ)) = 0.7816\ldots \text{ m/s}^2$.

Now that we have the acceleration, there are a couple ways to determine how long it takes to reach the bottom. The most direct would probably be to use the soap’s equation of motion in the $X$ direction: $x = x_o + v_o x t + \frac{1}{2} a_x t^2$. Setting our origin at the top of the ramp, and with the soap released from rest, we have: $x = 0 + 0 + \frac{1}{2} a_x t^2$ so when it reaches the bottom ($x = 9 \text{ m}$) we have $9 = 0.39081\ldots t^2$ or $t = 4.799 \text{ s}$ (or just $t = 4.8 \text{ s}$ rounding to two significant figures).
**Problem 5.14**: Police investigators examining the scene of an accident involving two cars measure 72-m-long skid marks of one of the cars, which nearly came to a stop before colliding. The coefficient of kinetic friction between rubber and the pavement is about 0.80. Estimate the initial speed of that car, assuming a level road.

We have 1-D motion here, with a car starting at some initial speed $v_o$ and coming to a final speed of zero. We’re on flat, horizontal ground so let’s define our axes so that $+X$ is in the direction of motion of the car, and $+Y$ is vertically upward.

**X direction**: $\Sigma F_x = ma_x$ so $-f_k = ma_x$

**Y direction**: $\Sigma F_y = 0$ so $F_n - mg = 0$ or $F_n = mg$.

We have kinetic friction here, so $f_k = \mu_k F_n$ but we just determined that $n = mg$ so $f_k = \mu_k mg$ here.

Substituting this into the X equation: $-\mu_k mg = ma_x$ and note that the mass $m$ appears in every term in the equation. We can divide the entire equation by $m$ and eliminate it (fortunate since they didn’t give us the mass of the car): $a_x = -\mu_k g = -(0.80)(9.8 \text{ m/s}^2) = -7.84 \text{ m/s}^2$.

We have the final velocity, the acceleration, and the displacement over which the event occurred, so we can use $v^2 = v_o^2 + 2a_x \Delta x$ to determine the initial velocity: $0 = (v_o)^2 + (2)(-7.84)(72)$ or $0 = v^2 - 1128.96$ from which $v = \sqrt{1128.96} = 33.6 \text{ m/s}$ (about 75 miles/hr).
Problem 5.20:
Two blocks made of different materials connected together by a thin cord, slide down a plane ramp inclined at an angle \( \theta \) to the horizontal as shown in the figure. The masses of the blocks are \( m_a = m_b = 5.0 \text{ kg} \) and the coefficients of kinetic friction are \( \mu_a = 0.20 \) and \( \mu_b = 0.30 \). Determine (a) the acceleration of the blocks and (b) the tension in the cord, when the angle is \( \theta = 23^\circ \).

We’ve done these connected-object type problems before. In general, Newton’s Laws state that the forces acting on an object produce the acceleration of that object. That’s happening separately for each of these two objects, so we’ll end up with an equation for each one of them, but there are quantities that they share: namely they both have the same acceleration down the ramp, and there is a single tension in the string connecting them.

The blocks are moving down the slope, so we’ll make that our +\( X \) direction.

**Applying Newton’s Laws to the upper block (B)**

We have the tension in the string pulling this block down-slope (our +\( X \) direction); the force of friction acting to oppose the motion; force of gravity straight down, and normal force acting perpendicular to the ramp.

Resolving \( F_g \) into components in the rotated coordinate system, we have: \( F_{gx} = mg \sin \theta = (5 \text{ kg})(9.8 \text{ m/s}^2) \sin (23^\circ) = 19.146 \text{ N} \) and \( F_{gy} = mg \cos \theta = (5 \text{ kg})(9.8 \text{ m/s}^2) \cos (23^\circ) = 45.105 \text{ N} \).

Applying Newton’s Laws:

\[ \Sigma F_y = ma_y = 0 \text{ so } F_n - F_{gy} = 0 \text{ or } F_n = F_{gy} = 45.105 \text{ N}. \]

\[ \Sigma F_x = ma_x \text{ so } F_T + F_{gx} - f_k = ma. \] But \( f_k = \mu_k F_n \) so for this block, \( f_k = (0.3)(45.105 \text{ N}) = 13.53 \text{ N} \). This leads to \( F_T + 19.146 - 13.53 = 5a \) or just \( F_T + 5.616 = 5a \).

**Applying Newton’s Laws to the lower block (A)**

We have the tension in the string pulling this block up-slope (our −\( X \) direction); the force of friction acting to oppose the motion; force of gravity straight down, and normal force acting perpendicular to the ramp.

Resolving \( F_g \) into components in the rotated coordinate system, we have: \( F_{gx} = mg \sin \theta = (5 \text{ kg})(9.8 \text{ m/s}^2) \sin (23^\circ) = 19.146 \text{ N} \) and \( F_{gy} = mg \cos \theta = (5 \text{ kg})(9.8 \text{ m/s}^2) \cos (23^\circ) = 45.105 \text{ N} \).

Applying Newton’s Laws:

\[ \Sigma F_y = ma_y = 0 \text{ so } F_n - F_{gy} = 0 \text{ or } F_n = F_{gy} = 45.105 \text{ N}. \]

\[ \Sigma F_x = ma_x \text{ so } -F_T + F_{gx} - f_k = ma. \] But \( f_k = \mu_k F_n \) so for this block, \( f_k = (0.2)(45.105 \text{ N}) = 9.021 \text{ N} \). This leads to \( -F_T + 19.146 - 9.021 = 5a \) or just \( -F_T + 10.125 = 5a \).
Now we have our two equations with two unknowns (the two boxed equations). We can just add the two equations together to cancel out the $F_T$ terms, leaving us with $15.741 = 10a$ or $a = 1.574 \text{ m/s}^2$ (which is positive, so apparently the boxes are accelerating as they slide down the ramp).

Now that we know $a$, we can find the tension from either of the boxed equations. Using $F_T + 5.616 = 5a$ we have $F_T = 5a - 5.616 = (5)(1.574) - 5.616 = 2.25 \text{ N}$. 
Problem 5.26: A 75 kg snow-boarder has an initial velocity of 5.0 m/s at the top of a 28° incline as shown in the figure. After sliding down the 110 m long incline (on which the coefficient of kinetic friction is $\mu_k = 0.18$) the snow-boarder then slides along a flat surface (on which the coefficient of kinetic friction is $\mu_k = 0.15$) and comes to a rest after a distance $x$. Use Newton’s Laws to find the snow-boarder’s acceleration while on the incline and while on the flat surface. Use these to determine $x$. (Note: the hints on the course website gave the coordinate directions to use to get the signs right.)

We have to break this problem into two problems: one part where the person is sliding down the ramp, then a second part where they’re sliding across the flat ground. The accelerations are constant in each of these parts but are not equal to each other.

The first part of this problem is essentially identical to the earlier problem with the bar of soap sliding down an inclined surface. I’ll just refer you there. We left everything symbolic in that problem and ended up finding that the acceleration downslope is $a_x = g(\sin \theta - \mu_k \cos \theta)$.

On the incline, we have $\mu_k = 0.18$ and the angle is $\theta = 28^\circ$ so substituting in those values we find that $a_x = 3.0433 \text{ m/s}^2$.

When they reach the bottom of the slope, how fast will they be going? $v^2 = v_o^2 + 2a_x \Delta x$ so $v^2 = (5)^2 + (2)(3.0433)(110)$ from which $v = 26.35 \text{ m/s}$.

Now they move onto the flat ground with this same speed and we want to determine how far they’ll travel before coming to a stop.

(Continued on next page)
Let’s apply Newton’s Laws to this part of the problem now. We have the force of gravity straight down, normal force perpendicular to the ground (i.e. pointing straight up), and the force of kinetic friction horizontally to the left, opposing the person’s motion. In this new coordinate system, \( \Sigma F_x = ma_x \) becomes just \(-f_k = ma_x\).

But \( f_k = \mu_k F_n \) and we can find \( F_n \) trivially here:

\[
\Sigma F_y = ma_y = 0 \text{ so } F_n - mg = 0 \text{ or } F_n = mg.
\]

That makes \( f_k = \mu_k F_n = \mu_k mg \).

Substituting this into the boxed equation just above, we have \(-\mu_k mg = ma_x\) or \(a_x = -\mu_k g\). While on this flat ground part of the trip, the (magnitude of the) person’s acceleration is just \( g \) times the coefficient of kinetic friction.

With the particular numbers we have here: \( a_x = -(0.15)(9.8 \text{ m/s}^2) = -1.47 \text{ m/s}^2 \).

How far will the person travel? We know their initial speed \( v_0 = 26.35 \text{ m/s} \) and their acceleration, and the final speed (when they stop) will be \( v = 0 \): \( v^2 = v_0^2 + 2a_x \Delta x \) so \((0)^2 = (26.35)^2 + (2)(-1.47)(\Delta x) \) from which \( \Delta x = 236 \text{ m} \).
Problem 5.34: What is the maximum speed with which a 1200 kg car can round a turn of radius 80 m on a flat road if the coefficient of static friction between the tires and road is 0.65? Is this result independent of the mass of the car?

Here we have an object moving in a circular path, which means there is a centripetal acceleration. If there is an acceleration, there needs to be a force to support it, and this force is the static friction between the tires and the road (since the tires are rolling along the road, not slipping).

In the vertical direction, \( \Sigma F_y = 0 \) leads to \( F_n - mg = 0 \) or \( F_n = mg \). The amount of static friction present will be something between 0 and the maximum amount of force that static friction can provide which is \( f_{s,max} = \mu_s F_n = \mu_s mg \) here.

In the radial direction, \( \Sigma F_r = ma_r \) becomes: \( f_s = mv^2/r \). The faster the car goes, the higher the centripetal force needed to keep the car in that circle. Eventually we reach a speed where we’ve max’ed out the amount of force available from friction. At that point: \( f_{s,max} = mv^2/r \) but we found above that \( f_{s,max} = \mu_s mg \) so making that substitution, this limiting scenario occurs when: \( \mu_s mg = mv^2/r \). Note that the mass cancels at this point, so our result actually won’t depend on the mass of the vehicle.

\( \mu_s g = v^2/r \) so \( v^2 = \mu_s rg \) (a result we derived in class).

For the current situation, \( \mu_s = 0.65 \) (not very good) and \( r = 80 \) m so \( v^2 = (0.65)(9.8)(80) = 509.6 \) from which \( |v| = 22.57 \) m/s (about 50 miles/hr).
Problem 5.38: How fast (in rpm) must a centrifuge rotate if a particle 8 cm from the axis of rotation is to experience an acceleration of 125,000 g’s?

An object moving in a circle of some radius $r$ at a speed $v$ will undergo a radial acceleration of $a_r = v^2/r$. We’re looking for the frequency (how many revolutions the object makes in one minute) but we can relate that to the speed and radius also: $T = (2\pi r)/v$ and $f = 1/T$ so $f = \frac{v}{2\pi r}$.

**Determining the speed**

$a_r = v^2/r$ so $v = \sqrt{ra_r}$. Here $r = 8\text{ cm} = 0.08\text{ m}$. The desired acceleration is 125,000 g’s so $a_r = (125000)(9.8) = 1.225 \times 10^6 \text{ m/s}^2$.

$v = \sqrt{ra_r} = \sqrt{(0.08 \text{ m})(1.225 \times 10^6 \text{ m/s}^2)} = 313.05 \text{ m/s}$. That means that the object is spinning around at a speed that’s just short of the speed of sound in air (340 m/s).

**Determining frequency of rotation**

Now that we have the speed of the object, we can relate that to the frequency of rotation:

$f = \frac{v}{2\pi r} = \frac{313.05 \text{ m/s}}{2\pi(0.08 \text{ m})} = 622.8 \text{ s}^{-1}$.

This means that the particle is making 622.8 complete revolutions in each second.

The problem asked for the frequency in revolutions per minute though, so:

$f = 622.8 \text{ rev/sec} \times \frac{60 \text{ sec}}{1 \text{ min}} = 37,370 \text{ rev/min}$ or 37,370 rpm.
Problem 5.44: A bucket of mass 2.0 kg is whirled in a vertical circle of radius 1.1 m. At the lowest point of its motion, the tension in the rope supporting the bucket is 25.0 N. (a) Find the speed of the bucket. (b) How fast must the bucket move at the top of the circle so that the rope does not go slack?

(a) As the bucket moves through the point at the bottom, \( \Sigma F = ma \) and at this point we have the tension in the rope up, the force of gravity down, and the acceleration will be \( a_c = \frac{v^2}{r} \) upward (towards the center of the circular path) so (using a coordinate system with positive upwards): \( T - mg = ma = \frac{mv^2}{r} \). Substituting in what we know at this point: \( 25 - (2)(9.8) = (2)(\frac{v^2}{1.1}) \)
from which \( v^2 = 2.97 \) or \(|v| = 1.72 \text{ m/s} \).

(b) At the top of the arc, Newton’s laws still apply of course but here we have the tension in the rope downward, gravity downward, and these forces combine to produce a radial acceleration of \( a_c = \frac{v^2}{r} \), also downward at this point. Let’s define the positive direction to be downward right at this instant. \( \Sigma F = ma \) then becomes: \( T + mg = ma = \frac{mv^2}{r} \) or \( T + mg = \frac{mv^2}{r} \). Look what happens if \( v = 0 \) though. Suppose we try to whirl the bucket around so that it momentarily comes to a stop at the top of the arc. If \( v = 0 \) that means that \( T + mg = 0 \) or \( T = -mg \) but that means we need a negative tension in the rope. Loose things like string and rope can’t support a negative tension (a solid rod can) so any attempt to do this with just rope will fail. The limiting case here will be when the object passes by the top of the arc in such a way that \( T \) drops to zero, in which case \( T + mg = \frac{mv^2}{r} \) becomes \( mg = \frac{mv^2}{r} \) or (cancelling the common \( m \) on both sides): \( g = \frac{v^2}{r} \) or \( v = \sqrt{rg} \) which for this problem is \( v = \sqrt{(1.1 \text{ m})(9.8 \text{ m/s}^2)} = 3.28 \text{ m/s} \).

If we manage to swing the bucket in an arc so that it passes by the point on the top at this speed, no tension is needed in the rope. Anything slower and the rope will go slack before the bucket can reach that point.