

PH2213 : Examples from Chapter 8 : PE/Conservation of Energy

Key Concepts

In the previous chapter, the concepts of **energy** and **work** were introduced and we saw how this gave us another way to solve many motion problems more easily than using the full Newton's Laws plus equations of motion. In this chapter, we further simplify the solution of these problems through the concepts of potential energy and conservation of energy.

Key Equations

Gravitational potential energy: $U_g = mgh$ (where h means vertically upward relative to some reference level). (NOTE: this is only an approximation for U_g near the surface of a planet or moon.

General gravitational potential energy: $U_g = -GMm/r$ where M and m are the masses of the two objects and r is the distance between their centers.

Elastic potential energy (springs, for example): $U_s = \frac{1}{2}kd^2$ where d represents the amount a spring has been extended or compressed from its rest length

Conservation of energy: $(K + U_g + U_s)_2 = (K + U_g + U_s)_1 + \sum W_{other}$. The three terms in the parentheses are collectively called the mechanical energy. The mechanical energy at position 1, plus the work done by any 'other' forces (other than gravity and springs) as the object moves 1 to 2 is equal to the mechanical energy at position 2.

Power is the **rate** at which work is being done: $P = \Delta W/\Delta t$ or $P = dW/dt$. If we have a constant force, it does work at a rate of $P = \vec{F} \cdot \vec{v}$

Warnings and Common Errors

- Since $K = \frac{1}{2}mv^2$, it can **never** be negative. (Do not ignore.)
- Since $U_s = \frac{1}{2}kd^2$, it can **never** be negative. (Do not ignore.)
- Conservation of energy can be treated like a budget: we have some amount of energy, and it can move between the various cells/buckets as an object moves, but using it to solve problems involves keeping track **very carefully** of the two positions you are considering, and what belongs on what side of the equation.
- A common error using Conservation of Energy is to double-count gravity. In this chapter, we have **removed** springs and gravity from the $\sum W$ part of the equation and replaced them with U_g and U_s
- Warning: don't confuse the CoE equation (which has **moved** the work done by gravity and springs into U terms) with the work-kinetic energy approach from the previous chapter (which **does** still include the work done by gravity and springs).
- force, work, power are all connected but separate (not even the same units)
- $U_g = mgh$ only depends on the height above some reference level
- $U_s = \frac{1}{2}kd^2$ only depends on how much the spring is stretched or compressed **relative to its** original, relaxed length just sitting in space with nothing attached

1. Example: Gravitational Potential Energy

Suppose we launch a 10 kg object vertically upward from the ground ($y = 0$). When the object is at a height of $y_1 = 5.0$ m, we measure its speed to be $v_1 = 10$ m/s upward.

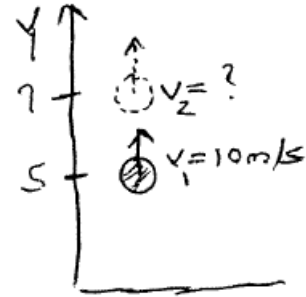
- How fast will the object be moving when it reaches a height of $y_2 = 7.0$ m above the ground?
- What is the maximum height the object will reach before stopping and returning to the ground?
- How fast will the object be moving when it hits the ground?

We can do all these using conservation of energy.

(a) How fast is it moving at $y = 7$ m?

We have no 'other' forces doing work here: just gravity, and we've already replaced the work done by gravity by the concept of gravitational potential energy. Also, we have no springs present here, so $U_s = 0$ at both locations. Our conservation of energy equation: $(K + U_g + U_s)_2 = (K + U_g + U_s)_1 + \sum W_{other}$. becomes: $(\frac{1}{2}mv_2^2 + mgy_2 + 0) = (\frac{1}{2}mv_1^2 + mgy_1 + 0) + 0$ or just:

$$\frac{1}{2}mv_2^2 + mgy_2 = \frac{1}{2}mv_1^2 + mgy_1$$



Every term in this equation includes the factor m , so we can divide the entire equation by m and simplify further, but let's solve it numerically at this point first. $m = 10$ kg, $v_1 = 10$ m/s, $y_1 = 5$ m and $y_2 = 7$ m so the equation becomes:

$$\frac{1}{2}(10)v_2^2 + (10)(9.81)(7) = \frac{1}{2}(10)(10)^2 + (10)(9.81)(5) \text{ or } (5 \text{ kg})v_2^2 + 686.7J = 500.0J + 490.5J.$$

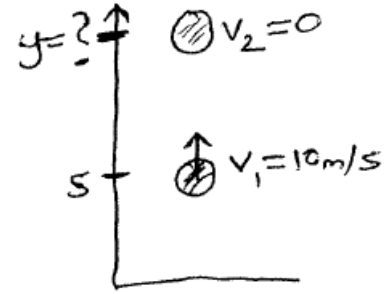
On the left side, we have the kinetic and potential energies at the higher position; on the right side we have the kinetic and potential energies at the lower position. We can look at conservation of energy as a **budget**: we have some number of joules of energy in various forms (kinetic, gravitational potential, elastic potential) and that energy can change form, switching from one type to the other, but overall we always have the same total amount of energy. Solving that last equation leads to (a) $v = 7.79$ m/s

If we back up to the boxed equation and leave it symbolic - this time dividing out the common factor of m and multiplying the equation by 2, we have: $v_2^2 + 2gy_2 = v_1^2 + 2gy_1$ or: $v_2^2 = v_1^2 + 2g(y_1 - y_2)$ but $\Delta y = y_2 - y_1$ so we could write this as $v_2^2 = v_1^2 - 2g\Delta y$. This is right from our old 1-D equations of motion: $v^2 = v_o^2 + 2a\Delta y$ where here $a = -g$ (an object in flight under the influence of gravity will have an acceleration of magnitude g and a direction downward). $\Delta y = 2$ so this becomes: $v_2^2 = (10 \text{ m/s})^2 + (2)(-9.81 \text{ m/s}^2)(2.0 \text{ m})$ or $v_2^2 = 100 - 39.24$ and finally $v_2 = 7.79$ m/s (same result).

(b) Find the maximum height

The object will continue to fly upward until it eventually stops and starts falling back towards the ground. At that maximum height then, the velocity is zero which means the kinetic energy is zero.

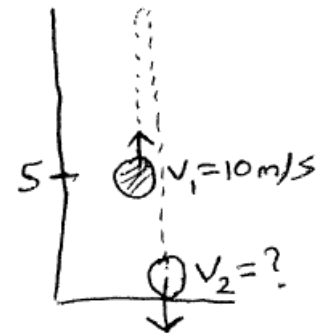
In this problem, then, we're looking for the value of y_2 for which $K_2 = 0$.



Starting from our generic conservation of energy equation: $(K + U_g + U_s)_2 = (K + U_g + U_s)_1 + \sum W_{\text{other}}$. again we have no springs here, so the U_s terms are both zero. We also have no 'other' work going on, so we have just: $(K + U_g)_2 = (K + U_g)_1$. In part (a), we found that the right hand side of this equation is 990.5 J and at the highest point we have $v_2 = 0$ so $K_2 = 0$ so in effect we're looking for the height where all 990.5 J has been converted into gravitational potential energy: $(0 + U_g)_2 = 990.5 \text{ J}$. $U_g = mgy$ though, so this becomes: $mgy_2 = 990.5 \text{ J}$ or $(10 \text{ kg})(9.81 \text{ m/s}^2)(y_2) = 990.5 \text{ J}$ which is $y_2 = 10.097 \text{ m}$.

(c) How fast will the object be going the instant just before it hits the ground?

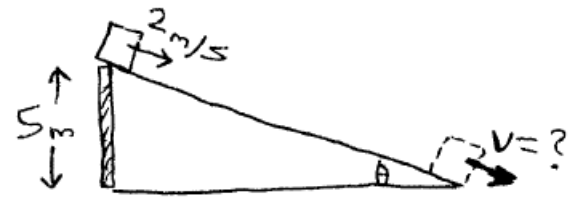
The object has a total mechanical energy of 990.5 J , as we found in part (a). There are no other works here, so it will always have this amount of mechanical energy. In particular, at the point where it is just about to hit the ground, $y = 0$ so it has no more gravitational potential energy. All of the energy is now in the form of the kinetic energy of the object, so $K_2 = 990.5 \text{ J}$ from which $\frac{1}{2}(10 \text{ kg})(v_2)^2 = 990.5 \text{ J}$ or $v_2 = 14.07 \text{ m/s}$. (Since the equation involved the square of the velocity, we really don't know anything about the direction from this analysis, but our real-world knowledge tells us that the object is falling downward just before it hits the ground, so technically the **velocity** at this point would be $v_2 = -14.07 \text{ m/s}$.



Summary : Conservation of energy can be used to find the speed of an object, but technically it's just the speed: there's no information about the direction. It could be moving up, down, left, right, or at any angle really. Newton's Laws and the equations of motion gave us exact, signed values including directions. So although using conservation of energy generally makes solving these problems easier, we have lost information that we may need to infer from other knowledge.

2. Example: Box Sliding Down Ramp

Let's say we have a long frictionless ramp. We don't know how long it is, but we put one end on the floor, and elevate the other end exactly 5 m above the floor. We place an object at the top of the ramp and give it an initial push, giving it an initial speed down the ramp of $v_1 = 2 \text{ m/s}$. How fast is it going when it reaches the end of the ramp? (We don't know the mass of the object.)



In previous chapters, we solved problems like this with Newton's Laws and equations of motion, but in those problems we knew the angle of the ramp so that we could resolve the forces into components as needed. Here we don't know how long the ramp is, so we can't determine that angle. We could just leave everything symbolic and hope for the best, and in fact if you do that you do end up with the correct answer: everything we don't know ends up cancelling out in various ways.

It's quite a bit cleaner to look at this problem from the conservation of energy approach though. Let position 1 be the initial position at the top of the ramp, and position 2 be the point we're interested in at the bottom of the ramp. Then:

$$(K + U_g + U_s)_2 = (K + U_g + U_s)_1 + \sum W_{other}.$$

We have no springs, so $U_s = 0$ on both sides. We also don't have any other work here. The normal force is perpendicular to the ramp (i.e. perpendicular to the displacement vector of the object) so does no work, and we have no friction here.

If we let $y = 0$ be the level of the floor, then U_g at position 2 will be zero, and U_g at the initial position will be $U_g = mgy = (M)(9.81 \text{ m/s}^2)(5 \text{ m})$.

Filling in what we know now, the equation becomes:

$$\left(\frac{1}{2}Mv_2^2 + 0 + 0\right) = \left(\frac{1}{2}M(2 \text{ m/s})^2 + (M)(9.81 \text{ m/s}^2)(5.0 \text{ m}) + 0\right) + 0 \text{ or: } \frac{1}{2}Mv_2^2 = \frac{1}{2}(M)(4) + 49.05M$$

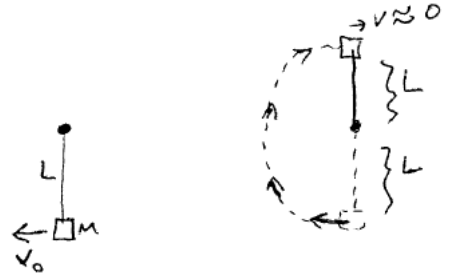
Dividing out the common factor of M and multiplying both sides by 2: $v_2^2 = 4 + 98.1 = 102.1$ and finally $v_2 = 10.1 \text{ m/s}$. (As usual with using energy to solve problems, we only know the magnitude of the velocity, not its direction or sign, but from reality we know the object is sliding down along the ramp.)

So here we find that when we have no friction, the length of the ramp doesn't matter, only how far above the floor the object is when it starts. Whether the ramp is 10 meters long or 10,000 meters long, it will get to the end of the ramp moving the same speed.

Summary : The same result we got above applies to curved ramps, such as roller coasters. This is one of the advantages of using conservation of energy and allows us to solve problems that would be difficult or even impossible using the machinery we've developed so far.

3. Example : Mythbuster's Swing (again)

In an old Mythbuster's episode, they tried to have someone swing on a swing-set fast enough that they made a complete 360° loop around the top. How fast does a person need to be moving at the bottom of the arc in order to do this? What will the tension be in the chain at the bottom and top?



We basically have a pendulum here, and the goal is for the mass at the end of the rope/string/chain to make it all the way to the top of the circle.

In order to make a full circle, the mass has to still have some v when it reaches the top. As long as it has some infinitesimal v there, it will swing over, so let's look at that threshold value and assume $v = 0+$ at that point. What does v need to be at the bottom?

Let's do this symbolically, using a rope of length L and assume we have our usual point mass m at the end of the rope. We'll use conservation of energy to solve this:

$$(K + U_g + U_s)_2 = (K + U_g + U_s)_1 + \sum W_{other}.$$

Do we have any 'other' work here? We do have the tension in the rope, but each millimeter along the path this tension is perpendicular to the displacement, so overall the tension does **no work** and we can ignore it.

It's common practice to place the $h = 0$ level at the lowest point in the motion, so that $U_g = 0$ there. At the top of the circle, the mass is at $y = 2L$, so at that location $U_g = mgh = 2mgL$. The kinetic energy at the bottom will be $\frac{1}{2}mv^2$ (this is the v we are trying to find), and at the top, the kinetic energy will be essentially zero (since the mass just barely makes it to the top - the speed at that point is just a hair higher than zero).

Conservation of energy then becomes:

$$\boxed{(0 + 2mgL + 0) = (\frac{1}{2}mv^2 + 0 + 0) + 0} \text{ or just } 2mgL = \frac{1}{2}mv^2. \text{ Dividing the entire equation by } m \text{ and multiply by 2: } 4gL = v^2 \text{ or } v = 2\sqrt{gL}.$$

A typical swing-set might have $L = 2 \text{ m}$ which would give us a speed at the bottom of $v = (2)\sqrt{(9.81)(2.0)} = 8.86 \text{ m/s}$ (or about 20 miles/hr).

Tension at Bottom : What will be the tension in the rope at the bottom of the arc? At that point, we have the mass m moving in a circle of radius L at a speed v , so there will be a radial acceleration (directed towards the center of the circle) of $a_r = v^2/r$. We just found that $v^2 = 4gL$ and the radius of the circle is L , so substituting in those values: $a_r = (4gL)/L = 4g$. $\sum F = ma$ and what are the forces acting on the mass at that point? We have the tension in the rope (acting **upward**) and the weight of the mass mg acting downward. So $\sum F = T - mg$ but $\sum F = ma = m(4g)$ so we must have $T - mg = 4mg$ and finally $T = 5mg$. That means that the tension in the rope will be five times what it would be if the person were just sitting there on the swing, not moving. (In the TV episode, when they attempted this experiment, the rope just broke...)

Tension at Top : What will be the tension in the rope at the top of the arc? Same arguments as before, but now the velocity is essentially zero, so $a_r = v^2/r$ is also zero. The two forces acting on the mass will be tension (acting **downward** now - tension in a rope or wire always acts to pull things inward, as if trying to shorten the rope). The weight of the mass will also be acting downward. So $\sum F = -T - mg$. $\sum F = ma$ but here $a = 0$ so that means that

$-T - mg = 0$ or $T = -mg$. Well that's not possible, since it means that we must have a **negative** value for tension. That **is** possible if we replace the rope/chain/wire with something solid like a rod, in which case this 'outward' version of tension is referred to as compression.

Starting speed needed to allow swing to actually make complete circle

Suppose we have a rope or chain here which doesn't allow negative tensions. At the top of the circle, the mass is moving with some speed v which means we have some radial acceleration v^2/r (directed towards the center of the circle, i.e. downward). The two forces creating this acceleration are tension (downward) and gravity (downward). Letting positive point downward: $\sum F_r = ma_r$ implies that $T + mg = mv^2/r$. The tension can't be negative: the smallest it can be is zero, so if $T = 0$ this implies the $0 + mg = mv^2/r$ or $v = \sqrt{gr}$ is the minimum speed the mass must be moving at the top in order to stay on this circular path.

If that's so, how fast is the mass moving at the **bottom** of the circle?

$$(K + U_g + U_s)_{bot} = (K + U_g + U_s)_{top} + \sum W_{other}$$

and using the same coordinate system as the first part, that means that:

$$\left(\frac{1}{2}mv_{bot}^2 + 0 + 0\right) = \left(\frac{1}{2}mv_{top}^2 + mg(2L) + 0\right) + 0$$

Cancelling a common factor of m in every term and multiplying the whole equation by 2:

$v_{bot}^2 = v_{top}^2 + 2g(2L)$ but we just found that the speed at the top must be $v_{top}^2 = gL$ so making that substitution, we have

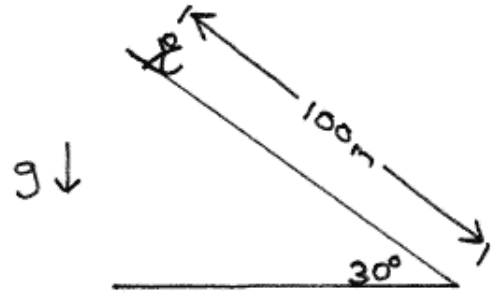
$$v_{bot}^2 = gL + 4gL = 5gL \text{ so } v_{bot} = \sqrt{5gL} \text{ which is a bit faster than we found previously.}$$

What's the tension in the rope/chain at the bottom of the arc now?

$\sum F_r = ma_r$ so $T - mg = mv^2/r$ so $T = mg + mv^2/r$ but we just found that $v^2 = 5gL$ (and $r = L$) so finally $T = 6mg$. At the bottom of the arc, the rope or chain must be able to support **six** times the weight of the person in order for them to actually perform this stunt.

4. Example: ramp with friction

An 80 kg skier at the top of a 100 meter long, 30° slope, is initially moving down the slope at a speed of 3.0 m/s. How fast is the skier moving when they reach the bottom of this slope? The coefficient of kinetic friction is $\mu_k = 0.06$. $v = \underline{29.8 \text{ m/s}}$

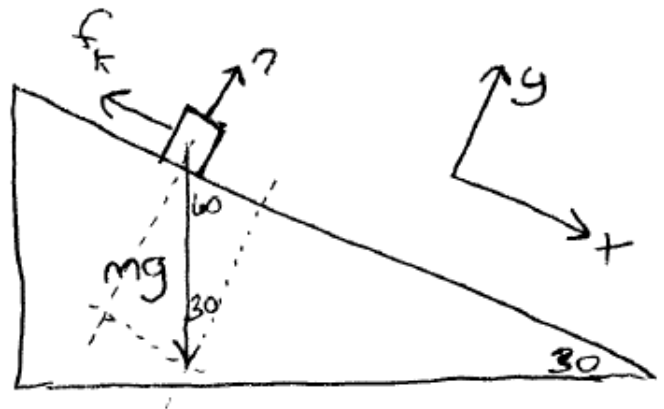


This is a problem we've solved before using Newton's Laws. Here we'll use conservation of energy. $E_2 = E_1 + \sum_{\text{other}} W$ where $E = K + U_g + U_s$. We don't have any springs, so the U_s terms will go away. We can use U_g to handle the change in the skier's elevation from the top to the bottom. We **do** have some 'other' force doing work here though: friction.

We will use $g = 9.8 \text{ m/s}^2$ to be consistent with the sample problem.

Since we need to determine the force of friction, we'll do a free-body diagram to determine the normal force so we can find f_k :

The figure on the right shows our rotated coordinate system and the resolving of the vectors as needed. **Please review** the section on the course website on **Resolving Vectors**.



Looking in our rotated Y direction:

We have the normal force 'up' and a component of the weight 'down'. Because of the way we defined our coordinate system, the skier is **only** moving in the X direction. There is no velocity or acceleration in the Y direction, so $\sum F_y = 0$, giving us: $F_n - mg \cos 30 = 0$ or $F_n = mg \cos 30 = (80 \text{ kg})(9.8 \text{ m/s}^2) \cos 30 = 678.96 \text{ N}$. Thus friction will have a magnitude of $f_k = \mu_k F_n$ or $f_k = \underline{40.74 \text{ N}}$.

We can directly go to our conservation of energy equation now, and go back to non-rotated coordinates, where +Y is pointing directly upward.

$$(K + U_g + U_s)_2 = (K + U_g + U_s)_1 + \sum_{\text{other}} W.$$

Define $y = 0$ to be the final level of the skier (i.e. at bottom of ramp).

U_g : $U_2 = 0$, and $U_1 = mgh = (80)(9.80)(50 \text{ m}) = 39200 \text{ J}$. (Can you see from the figure why $h = 50$ at the starting position?)

U_s : both zero since we don't have any springs anywhere here

$$K_1 = \frac{1}{2}mv^2 = (0.5)(80)(3)^2 = 360 \text{ J}$$

Work done by friction: $w_f = \vec{f}_k \cdot \vec{d} = (40.74)(100)(-1) = -4074 \text{ J}$.

So inserting what we know:

$$\frac{1}{2}(80)v^2 + 0 + 0 = (360 \text{ J}) + (39200 \text{ J}) + (-4074 \text{ J}) = 35486 \text{ J} \text{ which becomes: } 40v^2 = 35486$$

and $v^2 = 887.15$ or finally $v = 29.785 \text{ m/s}$

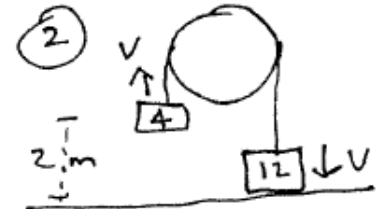
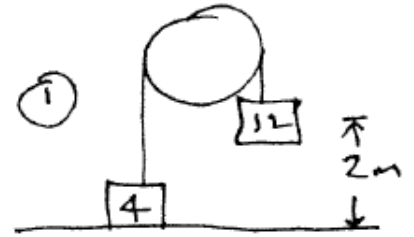
Note : we technically can do this with the skier's mass being unknown. If m were different, what would change? **Every term** in the conservation of energy equation here involved m directly: this is obvious in the case of K and U_g but the work done by friction is the force of friction times the distance but f_k is proportional to F_n , which in turn is directly proportional to m , so yes - the mass doesn't actually matter here. (All the numbers would be individually different, but the v at the bottom would be unchanged.)

5. Example : Atwood Machine

Here, we re-do the Atwood machine example from previous examples. In this scenario, we have a 4 kg block sitting on the floor and a 12 kg block suspended on the other end of a string and (temporarily) located 2 m above the floor. When we release the heavy block, it will fall downward pulling the lighter block upward. How fast will the blocks be moving the instant just before the heavier block hits the floor?

Conservation of energy: $(K + U_g)_2 = (K + U_g)_1 + W_{\text{other}}$ (we've dropped the spring term since we have no springs here)

Let's choose $y = 0$ to be the level of the floor.



In the lecture examples from the previous chapter, we looked at work-energy for each mass separately and then added them together at the end (which resulted in the work done by the tension cancelling out). Let's go ahead start this solution looking at the entire system: both blocks.

Initial Position : Initially, nothing is moving, so we have no kinetic energy in the system: $K_1 = 0$. We do have some gravitational potential energy though: that 12 kg block is located 2 m above the floor, so at the start we have $U_g = (12)(9.81)(2) = 235.44\text{ J}$.

Final Position : At the final position, we have the light block moving upward at some speed v and the heavy block moving downward at the exact same speed v (they have to be the same since the string connecting them doesn't stretch). So we have a total kinetic energy of $\sum \frac{1}{2}mv^2 = \frac{1}{2}(4)v^2 + \frac{1}{2}(12)v^2 = 2v^2 + 6v^2 = 8v^2$.

At this time, the 4 kg block has moved vertically upward by 2 m so there is gravitational potential energy of $U_g = mgh = (4\text{ kg})(9.81\text{ m/s}^2)(2\text{ m}) = 78.48\text{ J}$.

What might **other work** be here? We **do** have the tension in the string, but look at the work that it is doing on each block. On the 4 kg block, \vec{T} is up and \vec{d} is up. On the 12 kg block, \vec{T} is up but \vec{d} is down, so when we do the dot product of the force and displacement to find the work done, we have terms of: $+Td - Td$ or overall the tension does zero work. (**NOTE:** if you run into a connected-object problem on a test, be sure to include these works: they are there, they just happen to cancel each other out in this particular scenario. In more complicated situations with multiple pulleys, they may not.)

Finally then, conservation of energy $(K + U_g)_2 = (K + U_g)_1 + W_{\text{other}}$ becomes:

$$(8v^2 + 78.48) = (0 + 235.44) + 0 \text{ or } 8v^2 = 156.96 \text{ and } v = 4.43\text{ m/s.}$$

As usual, this really just tells us the speed of the blocks - nothing about their direction - and we have to infer from what we know about how these blocks will move that the light block is moving at this speed upward, while the heavier block is moving at this speed downward.

6. Example: box sliding into spring (A)

Suppose we have a 10 kg block moving at 4 m/s along frictionless floor towards a spring with $k = 1000\text{ N/m}$. How far in will the spring compress before the block comes to a stop?

Conservation of energy : $(K + U_g + U_s)_2 = (K + U_g + U_s)_1 + \sum_{\text{other}} W_i$

Nothing is moving up or down here - all the motion is horizontal - so let's define $y = 0$ to be this level, which makes $U_g = 0$ on both sides. Also, we have no 'other' work happening here (no friction), so the $\sum W$ term is also zero and our equation simplifies to just: $(K + U_s)_2 = (K + U_s)_1$

Initially, the spring is not (yet) compressed, so U_s starts off at zero. We do have the moving block at this point, so we do have some initial kinetic energy.

As the block pushes in the spring, the spring pushes back, slowing down the block until eventually it comes to a stop when the spring has been compressed by a length d from its original rest length.

Conservation of Energy:

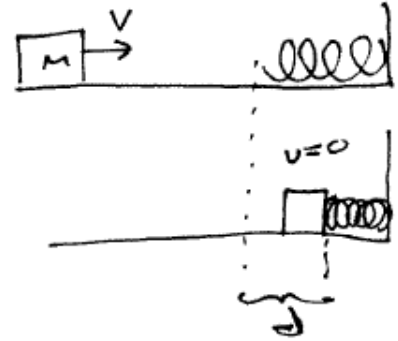
$$(0 + 0 + \frac{1}{2}kd^2) = (\frac{1}{2}(10)(4)^2 + 0 + 0) + 0$$

$$\frac{1}{2}(1000)d^2 = 80 \text{ or } 500d^2 = 80 \text{ or } d^2 = 0.16 \text{ and } d = 0.4\text{ m}$$

Apparently this spring had to become 40 cm shorter before bringing the block to a stop (so the spring had better be pretty long).

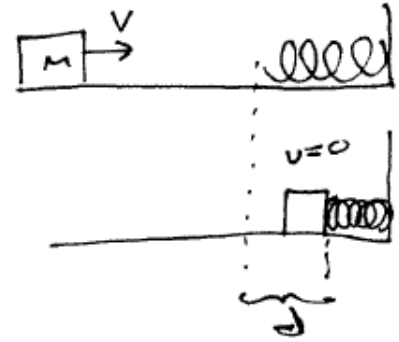
Note : what happens now? We have the spring compressed by 40 cm , which means it is exerting a force of $F = -kx = (-1000)(0.4) = -400\text{ N}$ (400 N to the left), $F = ma$ so $a = F/m = -400/10 = -40\text{ m/s}^2$ or about $4g$'s to the left.

Note : As the spring expands back out, $F = -kx$ so the force keeps changing, so this acceleration is **not constant** , and we can't use our usual equations of motion from chapter 2 to analyze this. Using conservation of energy allowed us to solve for the motion of the block though.



7. Example: box sliding into spring (B)

Let's change up the previous example and add **friction** on the floor under the spring. In this case, our 10 kg block is moving across the floor and is moving at 4 m/s when it first touches the spring (which has $k = 1000 \text{ N/m}$ as in the previous example). How far in will the spring compress now before the block comes to a stop? Suppose that $\mu_k = 0.4$ and $\mu_s = 0.8$ between the box and the floor in this version of the problem.



Conservation of energy : $(K + U_g + U_s)_2 = (K + U_g + U_s)_1 + \sum_{\text{other}} W_i$

Nothing is moving up or down here - all the motion is horizontal - so let's define $h = 0$ to be this level, which makes $U_g = 0$ on both sides.

In this example, we **do** have some other work going on though: friction is removing energy from the box as it moves across the rough floor.

The box hits the spring with some initial energy, and as it continues to move to the right friction is removing energy and the spring is become more and more compressed, meaning that more and more energy is being shifted into the spring. Eventually, all the initial energy will be gone from the box and it will come to a stop after moving d to the right. (So our +X axis is horizontal and pointing to the right here.)

First, how much work did friction do? $W_{\text{friction}} = \vec{f}_k \cdot \vec{d} = |f_k||d| \cos(180^\circ) = -\mu_k F_N d$. What is the normal force here? If we look in the vertical direction, we have F_N up and $F_g = mg$ down and those are the only force components in that direction so apparently here we have $F_N = F_g = mg$. Finally then, $W_{\text{friction}} = -\mu_k mgd$ in this scenario. We have most of those values, so $W_{\text{friction}} = -(0.4)(10)(9.8)(d) = -39.2d$

At the initial position (1):

$$K = \frac{1}{2}mv^2 = \frac{1}{2}(10)(4)^2 = 80 \text{ J}$$

$$U_g = mgh = 0 \text{ (we'll measure all heights relative to the floor)}$$

$$U_s = \frac{1}{2}kd^2 = 0 \text{ (the spring is not yet compressed)}$$

At the final position (2):

$$K = \frac{1}{2}mv^2 = 0 \text{ (box has momentarily come to a stop)}$$

$$U_g = mgh = 0 \text{ (we'll measure all heights relative to the floor)}$$

$$U_s = \frac{1}{2}kd^2 = \frac{1}{2}(1000)d^2 \text{ (the spring is now compressed by } d\text{)}$$

Replacing the corresponding terms then, conservation of energy becomes:

$$(K + U_g + U_s)_2 = (K + U_g + U_s)_1 + \sum_{\text{other}} W_i$$

$$(0 + 0 + 500d^2) = (80 + 0 + 0) - 39.2d \text{ or } 500d^2 + 39.2d - 80 = 0.$$

This has two solutions: $d = +0.3627 \text{ m}$ or $d = -0.4411 \text{ m}$.

The box was sliding to the right and will continue to do so until coming to a stop, so obviously the positive solution is the one we want here.

NOTE: The negative solution is doubly bogus: friction always does negative work and we found that $W_{\text{friction}} = -39.2d$ here, so if we used a negative value for d we'd end up with friction doing **positive** work (i.e. adding energy to the box). That isn't possible. If the box really did displace to the left, we'd have to write the work done by friction as $+39.2d$ so that the negative d value still resulted in friction doing negative work...

What happens now?

At this point the box is sitting up against a compressed spring. The spring will be exerting a force of $F_s = -kx = -(1000 \text{ N/m})(0.3627 \text{ m}) = -362.7 \text{ N}$. The spring is pushing the box to the left this strongly.

If that were the only horizontal force component, the box would immediately start accelerating to the left. Static friction is present though attempting to prevent that. How much f_s can there be though? It's maximum value will be $f_{s,max} = \mu_s F_n = \mu_s mg$ here so $f_{s,max} = (0.8)(10 \text{ kg})(9.8 \text{ m/s}^2) = 78.4 \text{ N}$.

There is not **nearly** enough static friction to hold the block in place, so it will immediately start accelerating to the left.

How fast will the box be moving when it has moved far enough to the left to detach from the spring? Remember, when it was at this position before, it was moving at 4 m/s to the right.

While sliding to the right 0.3627 m , friction did negative work of $W_{friction} = -39.2d = -(39.2)(0.3627) = -14.22 \text{ J}$.

On the way back to it's initial position, friction will do the exact same amount of (negative) work: the force of friction is the same (it's just reversed direction), the displacement is also the same (except it's also reversed direction), so the work done by friction is the same (negative) 14.22 J in either direction.

Thinking of this entirely from an energy standpoint, the box started off with 80 J of kinetic energy when it first touched the spring. It continued to slide to the right. Along the way, friction removed 14.22 J of energy, leaving us with $80 - 14.22 = 65.78 \text{ J}$ to account for, but at this point the spring has been compressed by 0.3627 m meaning that the spring was now storing $U_s = \frac{1}{2}kd^2 = (0.5)(1000)(0.3627)^2 = 65.78 \text{ J}$ of energy.

As the box now slides back out to the left, the force of friction is the same and the displacement is the same so friction removes another 14.22 J of energy along the way, leaving us with $65.78 - 14.22 = 51.56 \text{ J}$ of energy left when the box gets back to its original location. The spring is no longer compressed, so all this energy is in the K of the box as it leaves the spring. $K = 51.56 = \frac{1}{2}mv^2 = (0.5)(10)(v)^2$ so $|v| = 3.21 \text{ m/s}$.

8. Example : spring and incline

Let's mix gravity and springs together. Suppose we start things off with a box pressed up against a spring (so it is storing some energy). The box is released, and the spring will push it outward, eventually releasing all its energy into the box. The box will carry this energy (in the form of kinetic energy) across the floor. When it reaches the ramp and starts to rise up, this energy gets converted into gravitational potential energy, until there's nothing left and the block comes to a stop. How high above the ground will it stop?

The box has a mass of 10 kg , the spring has a spring constant of $k = 1000\text{ N/m}$ and it is initially compressed by $d = 10\text{ cm}$.

Assume we have **no** friction here.

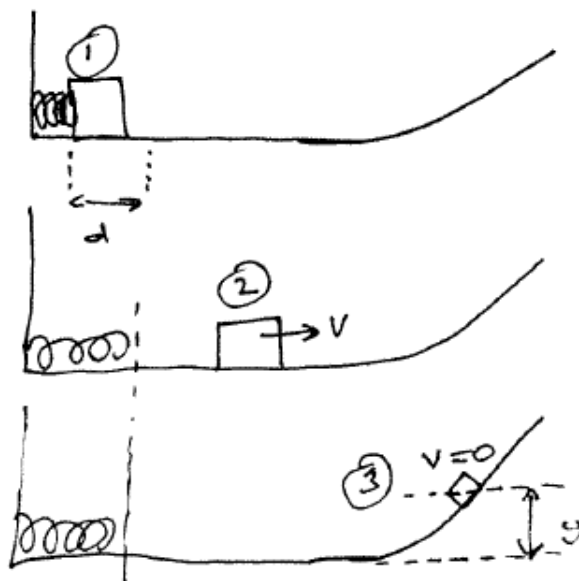
Let's define the flat part of the floor to be our $y = 0$ reference level.

Position 1 : How much mechanical energy do we have? $E_1 = (K + U_g + U_s)_1 = 0 + 0 + \frac{1}{2}(1000)(0.1)^2 = 5\text{ J}$

Position 2 : How fast does the box slide across floor? $E_2 = E_1 + W_{\text{other}} = E_1 + 0 = 5\text{ J}$, so $\frac{1}{2}(10)v^2 + 0 + 0 = 5$ and $v = 1\text{ m/s}$.

Position 3 : How high up the ramp does it go?

$E_3 = E_2 + W_{\text{other}} = E_2 = 5\text{ J}$ so $0 + mgh + 0 = 5$ whence: $(10)(9.81)(h) = 5$ or $h = 0.05\text{ m}$ or just 5 cm above the floor. (We don't know the angle of the ramp, so can't say how far up the ramp it slid before coming to a stop; all we can say is it will stop at whatever point it takes to reach a height of 5 cm above the floor...)

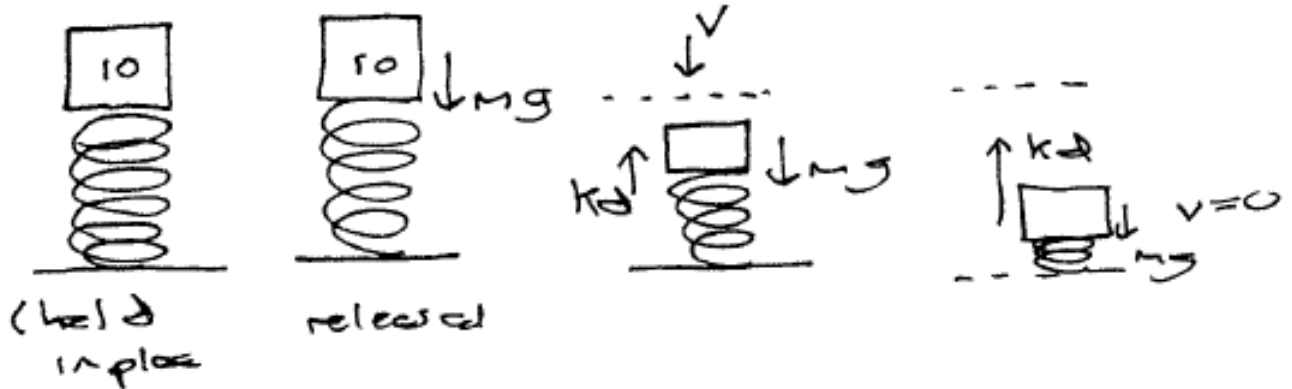


9. Example: spring sitting vertically on the floor

Here, we mix together U_g and U_s in a more complicated way where both are present at once.

Suppose we have a spring with $k = 1000 \text{ N/m}$ sitting vertically on the floor. We lower a 10 kg block until it is just touching the spring, but has **not yet** compressed it. We now let the block go and it falls down, compressing the spring.

(a) How far will the spring compress before bringing the block to a stop?



We have the block changing height, so we'll definitely have some U_g terms to worry about, but that means we have to pick some reference level, but where?

One option is to look ahead: we know the block eventually comes to a stop after compressing the spring by a distance d , so suppose we make that our $y = 0$ reference level. Then in that coordinate system, when the box has been lowered so it just barely touched the spring, it is located at $y = d$ at that point.

Conservation of energy: $(K + U_g + U_s)_2 = (K + U_g + U_s)_1 + \sum_{\text{other}} W$. We have no 'other' works here, so can ignore the $\sum W$ term.

Initial Position : the box is just touching the spring, but is not yet moving. That means $K_1 = 0$ and $U_g = mgy = mgd$. The spring is **not yet** compressed at all, so $U_s = 0$.

Final Position : the box has compressed in the spring and has come to a stop, which means it also has no kinetic energy at that point: $K_2 = 0$. It's now located at $y = 0$ because that's how we defined our Y axis, so $U_g = 0$. We have a compressed spring, so $U_s = \frac{1}{2}kd^2$.

Putting these together:

$(0 + 0 + \frac{1}{2}kd^2) = (0 + mgd + 0) + 0$ or $\frac{1}{2}kd^2 = mgd$ and dividing out a common factor of d : $\frac{1}{2}kd = mg$ and finally $d = 2mg/k$.

For our particular spring, $d = (2)(10 \text{ kg})(9.81 \text{ m/s}^2)/(1000 \text{ N/m}) = 0.1962 \text{ m}$ or 19.62 cm . The block will be compressing the spring this amount when it comes to a stop.

What is the force that the spring is exerting on the box at this point? $F = -kx$ where $x = 0$ denotes the location of the end of the spring when it is **not yet** compressed. Let's let $+X$ point vertically upward here. Then at this point, $x = -d = -0.1962 \text{ m}$ so $F = -kx = (-1000 \text{ N/m})(-0.1962 \text{ m}) = +196.2 \text{ N}$. The force of gravity will be of magnitude $|F| = mg = (10 \text{ kg})(9.81 \text{ m/s}^2) = 98.1 \text{ N}$, and will be directed downward. So at this point, the spring is exerting twice as much force upward as gravity is exerting downward: the net result is that the box will not just sit here, it will accelerate upward. $\sum F = ma$ so $+192.2 - 98.1 = 10a$ or $a = +9.8 \text{ m/s}^2$ (upward).

Intermediate Position : When the box has just initially touched the spring, the spring isn't exerting any force on the box yet. When the box reaches its lowest point, the spring is exerting twice as much force upward as gravity is exerting downward. Let's look for the point where those are equal: how far in has the spring compressed at the point where $|F_s| = |F_g|$. At this point, the spring has compressed in some amount d' which means that $x = -d'$. $F_s = -kx$ and $F_g = mg$ so their magnitudes will be equal when $kd' = mg$ or $d' = mg/k$. Note that this is exactly halfway down to the lowest point, which we found was $d = 2mg/k$.

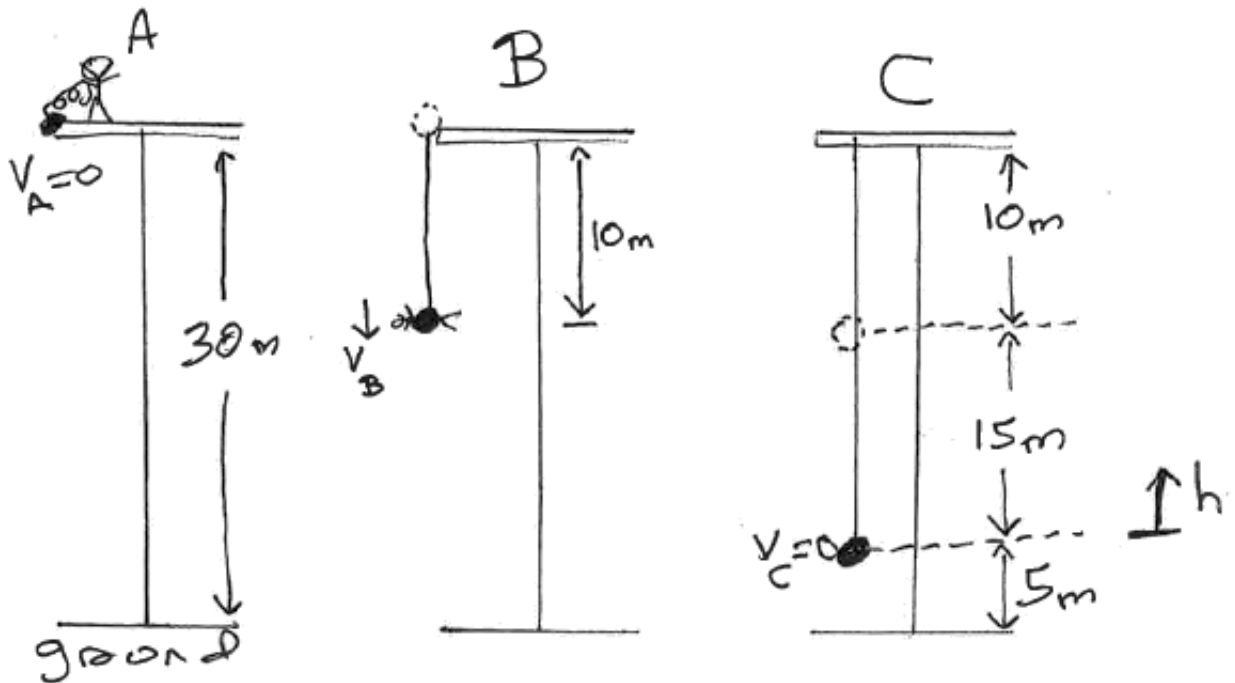
10. Example: Bungee-cord jump

We are tasked with designing a bungee jump and want to determine how strong the cord needs to be. The cord is an elastic material (basically a strong rubber band) and we'll assume (although this should be tested) that it behaves according to Hooke's law, just like rubber bands and springs do.

Elastic materials are not quite springs though: if you try and stretch the material past its rest length, it definitely pulls back ($F_s = -kx$) but unlike a spring, it's not symmetric. If you push a rubber band the other way, it doesn't push back - the material just flops out of the way.

Suppose we have an 80 kg person that is going to step off a bridge that is 30 m above the ground. We have a 10 m long cord and we want the person to stop 5 m before they hit the ground.

We want to determine two parameters here: what spring constant k the material should have, and what maximum tension it will need to undergo without breaking.



At position (A), the person has just stepped off the bridge and is now falling downward due to gravity. The cord won't start providing any force until it's reached and gone beyond it's 'rest length' of 10 m.

At position (B), the cord has reached it's 10 m length and from here down will start to stretch and will be exerting a force upward.

At position (C), which is 5 m above the ground, we want the person to (momentarily) come to a stop. Looking at the other distances involved here, the cord at this point has been stretched by 15 m beyond it's rest length.

Let's apply conservation of energy between points A and C directly:

$$(K + U_g + U_s)_C = (K + U_g + U_s)_A + \sum_{\text{other}} W.$$

There are no 'other' forces here (just gravity and the 'spring' which we can account for using U_g and U_s), so there is no 'other' work that we need to worry about.

We can pick whatever reference level we want for our U_g calculations, and I usually use whatever the lowest point the object reaches, so let's measure all our height values relative to point C.

At the initial position (A):

$$K = \frac{1}{2}mv^2 = 0 \text{ (not moving yet)}$$

$$U_g = mgh = (80 \text{ kg})(9.8 \text{ m/s}^2)(25 \text{ m}) = 19600 \text{ J}$$

$$U_s = \frac{1}{2}kd^2 = 0 \text{ (the cord is not yet stretched)}$$

(Note that for U_s above, we had to use the observation that things like rubber bands and elastic cords aren't entirely like springs: they only store energy when they are **stretched** beyond their rest length.)

At the final position (2):

$$K = \frac{1}{2}mv^2 = 0 \text{ (person has momentarily come to a stop)}$$

$$U_g = mgh = 0 \text{ (this is our reference level)}$$

$$U_s = \frac{1}{2}kd^2 = \frac{1}{2}(k)(15)^2 \text{ (cord is stretched 15 meters now)}$$

Replacing the corresponding terms then, conservation of energy becomes:

$$(K + U_g + U_s)_C = (K + U_g + U_s)_A + \sum_{\text{other}} W_i$$

$$(0 + 0 + \frac{1}{2}(k)(15)^2) = (0 + 17600 + 0) + 0 \text{ or } 112.5k = 17600 \text{ or finally } \boxed{k = 156.44 \text{ N/m}}.$$

How much tension will the cord be under at this point?

Our 'spring' (cord) has a spring constant of $k = 156.44 \text{ N/m}$ and is now stretched by 15 m beyond its rest length, so it's exerting a force (upward here) with a magnitude of: $|F_s| = kd = (156.44 \text{ N/m})(15 \text{ m}) = 2347 \text{ N}$ (about 528 pounds). (And all the other key points in the mechanism like how the cord is connected to the vest the person will be wearing, the vest material itself, and so on, will also need to hold together under that much force - and probably quite a bit more to build in a safety factor...)

How many g's of acceleration will the person feel?

During the initial 10 meters, we just have gravity down and nothing up so the person is just in free-fall. Past this point, the cord is starting to stretch and is exerting more and more force upward. When we reach point C, the cord is exerting a force of 2347 N upward and gravity is exerting a force of $F_g = mg = (80)(9.8) = 784 \text{ N}$ downward, so the net force upward is $\Sigma F_y = +2347 - 784 = 1563 \text{ N}$.

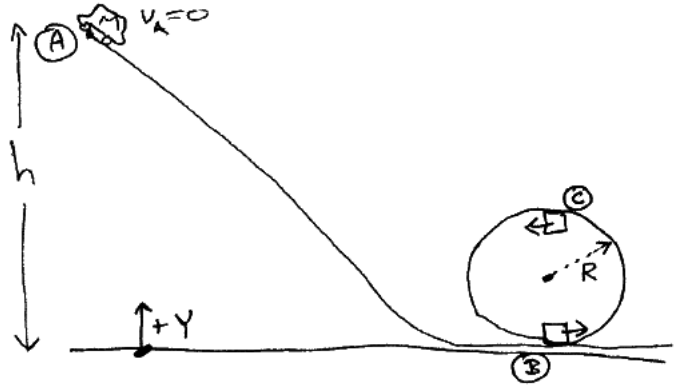
At this instant, $F = ma$ so $1563 = 80a$ or $a = 19.5 \text{ m/s}^2$ upward (about 2 g's).

Considerations

Going back to the beginning, note that the spring constant k depended on the mass of the person, so if a heavier or lighter person wanted to take the same 'trip', they'd need a different k (a different cord) in order to stop at the same point. Alternately, if they used the same cord they'd travel a different distance before coming to a stop.

11. Hot Wheels track loop

The figure at the right shows a Hot Wheels track layout where the toy car starts at some height h above the floor, then goes into a part of the track that's been formed into a circle of radius R .



Version 1 : Suppose the radius of the circle is $R = 10 \text{ cm}$ and we start the car at rest at an initial height of $h = 50 \text{ cm}$ above the floor. How fast will the car be moving when it passes through the point at the top of the circular part of the track (the point labelled **C** in the figure)? The mass of the little cars vary, but assume this one has a mass of $m = 35 \text{ gram}$.

We'll use conservation of energy here, going from point A where the car starts to point C at the top of the circular part of the track.

$$(K + U_g + U_s)_C = (K + U_g + U_s)_A + \sum W_{\text{other}}$$

We have no springs here, so we can drop the U_s terms. We do have two other forces present: the normal force between the track and the car, and the force of kinetic friction. The normal force is always perpendicular to the track and our displacement, millimeter by millimeter, is along the track, so this force is always **perpendicular** to the little displacement steps, meaning F_N does no work. Kinetic friction unfortunately **will** do some work (which is always negative with friction) but let's assume we have no friction here. Then our CoE equation becomes just:

$$(K + U_g)_C = (K + U_g)_A$$

Since the elevation is changing, we need to define a reference level for our U_g calculations. I'll use the floor.

At point A, the car isn't moving so $K_A = 0$ but it does have some gravitational potential energy since it's h above the floor.

At point B, the car is moving at some speed we want to determine, and it also has some gravitational potential energy since it's located one diameter of the circle above the floor. Our CoE equation then becomes:

$$\frac{1}{2}mv_C^2 + mg(2R) = mgh$$

Note here that m appear in every term in this equation so we can divide the entire equation by m . Rearranging a bit: $v_C^2 = 2gh - 4gR$.

With the particular numbers we have here: $g = 9.8 \text{ m/s}^2$, $h = 0.50 \text{ m}$ and $R = 0.10 \text{ m}$ we end up with $v_C = 2.425 \text{ m/s}$.

This type of motion looks very similar to the Mythbuster's swing scenario so let's double check that the car actually **can** make it through the circular part of the track. The smallest normal force will occur at the top of the track so let's compute it and make sure it's 'legal.'

At the top of the track, we've moving with a speed of v_c in a circle of radius R which implies a radial acceleration of $a_r = v^2/r = v_c^2/R$ here. This acceleration is being produced by two

radial forces: F_g pointing down (i.e. towards the center of the circle) and F_N , also pointing towards the center of the circle at this point. If we let our positive direction be towards the center of the circle, then $\sum F_r = ma_r$ becomes $mg + F_N = mv_c^2/R$ or $F_N = mv_c^2/R - mg$

We already found that $v_c^2 = 2gh - 4gR$ though, so substituting in that expression and collecting terms, we find: $F_N = mg(2\frac{h}{R} - 5)$.

For the numbers we've been using, $F_N = mg(2\frac{50}{10} - 5) = 5mg$ which is certainly greater than zero, so we're fine. The car will definitely stay on the track as it goes around the circular loop.

Impact on the track itself

With the particular numbers we've been using, we found that as the car is passing through the top of the loop, the track must be exerting a force of $F_N = 5mg$ on the car, helping to keep it moving in a circle. That means the **car** is also exerting a force on the **track** of that same magnitude. The car is basically pushing the track **upward** with that amount of force. If the car is heavy enough and the track is light enough, that might actually **lift** the track off the ground...

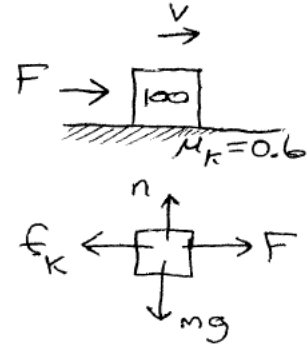
Minimum height h to allow the car to do the loop

That last boxed equation above tells us something else: the normal force can't be negative - the smallest it can be is zero and that will occur when $2\frac{h}{R} - 5 = 0$ or $h = 2.5R$. That's apparently the minimum height we can release the car (at rest anyway) and guarantee that it will stay on the track as it passes around the loop.

The $2R$ part of the height just gets us to the top of the circle so apparently we need to start off half a radius above the top of the circle. (Well, at least that should be enough height as long as we don't have any friction present...)

12. Power : pushing crate

Suppose I am pushing a 100 kg crate across a flat, horizontal floor with friction of $\mu_k = 0.6$, and the crate is moving at a constant speed. If I am pushing the crate horizontally, how much force do I need to apply? How much power am I expending to do this if the crate is moving at 2 m/s? 20 m/s?



The crate is moving at a constant speed, so $\sum \vec{F} = 0$ which means that $\sum F_x = 0$ and $\sum F_y = 0$ separately.

Using the expected coordinate system of +X pointing to the right, and +Y pointing upward in the figure, $\sum F_y = 0$ implies $n - mg = 0$ or $n = mg = (100 \text{ kg})(9.81 \text{ m/s}^2) = 981 \text{ N}$.

In the X direction, we have the person's force to the right, and friction to the left, so $\sum F_x = 0$ implies $F - f_k = 0$ or $F = f_k$. Thus I have to be pushing the crate with exactly the same force that friction is providing. $f_k = \mu_k n = (0.6)(981 \text{ N}) = 588.6 \text{ N}$, so I must be pushing the crate with a force of $F = 588.6 \text{ N}$.

How much power am I exerting? Well one of the expressions we have for power in the case where the force is constant is $P = \vec{F} \cdot \vec{v}$. The power I am exerting to push the crate then depends on how fast the crate is actually moving.

At 2 m/s : $P = \vec{F} \cdot \vec{v}$. Here the force \vec{F} and the velocity \vec{v} are **both** pointing in the +X direction, so the angle between them is zero. $P = \vec{F} \cdot \vec{v} = Fv \cos \phi = Fv \cos 0 = Fv$ so $P = (588.6 \text{ N})(2 \text{ m/s}) = 1177.2 \text{ W}$ (where that W represents the units of power which are **watts** in the metric system). 746 W in the metric system is the same as 1 HP in the English system, so this effort requires a power of 1.58 HP.

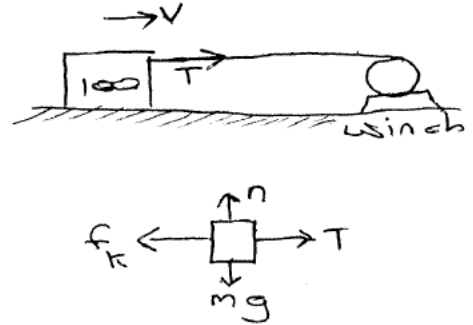
At 20 m/s : same as above but now $v = 20 \text{ m/s}$ so $P = (588.6 \text{ N})(20 \text{ m/s}) = 11772 \text{ W}$, which in English units would be $(11772 \text{ W}) \times \frac{1 \text{ HP}}{746 \text{ W}} = 15.8 \text{ HP}$. (Too much power for a person to put out...)

Power is the rate at which work is being done. In the case of the 2 m/s speed, the person is expending power at 1177.2 W or 1177.2 J/s. The person is basically **adding** 1177.2 joules of energy to the crate every second. Friction is **removing** energy at the same rate though: the power that friction is expending is $P = \vec{f}_k \cdot \vec{v}$ and since the friction vector is in exactly the opposite direction as the velocity vector, the angle between them is exactly 180° so $P = \vec{f}_k \cdot \vec{v} = f_k v \cos 180 = -f_k v = -(588.6 \text{ N})(2 \text{ m/s}) = -1177.2 \text{ W}$.

(This power being put out by friction is basically becoming heat, so the block and/or the floor must be heating up quite a bit here.)

13. Power : motor pulling crate

As a variation of the previous problem, suppose I hook up the crate to a 2 HP motor and use it to pull the crate across the floor. How fast can this motor pull the crate? We'll use the same amount of friction between the crate and the floor, with $\mu_k = 0.6$.



From the previous problem, the power needed goes up as the speed goes up. Our motor is limited here to putting out 2 HP or $(2 \text{ HP}) \times \frac{746 \text{ W}}{1 \text{ HP}} = 1492 \text{ W}$, so that will result in some constant speed at which the crate can move. The motor will accelerate the crate from rest up to this speed, but from there on the crate won't be able to go any faster.

From the figure, the force actually acting on the block pulling it to the right is the tension in the cable, with friction between the crate and the floor acting against this tension.

The crate is moving at some constant speed, so $\sum F_x = 0$ and $\sum F_y = 0$.

Looking at the forces in the Y direction, we have $\sum F_y = n - mg = 0$ so $n = mg = (100 \text{ kg})(9.81 \text{ m/s}^2) = 981 \text{ N}$.

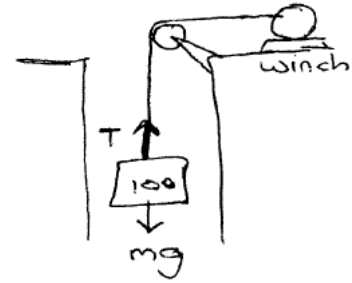
In the X direction, we have the tension T to the right, and friction f_k to the left. $f_k = \mu_k n = (0.6)(981 \text{ N}) = 588.6 \text{ N}$. $\sum F_x = 0$ since the crate isn't accelerating so $T - f_k = 0$ or $T = f_k = 588.6 \text{ N}$.

The power being expended through the cable pulling the crate to the right then is $P = \vec{F} \cdot \vec{v}$ and both the tension and the speed are to the right, so the angle between them is zero leading to $P = Fv$.

Here, we know how much power we have available: 1492 W and we know what the tension in the cable is: $T = 588.6 \text{ N}$ so we can find the speed the crate is moving: $P = Fv$ or $(1492 \text{ W}) = (588.6 \text{ N})(v)$ and $v = 2.53 \text{ m/s}$.

14. **Power : motor raising crate**

As a variation of the previous problem, suppose I use the same 2 *HP* motor to raise a 100 *kg* crate vertically upward (lifting something up from the bottom of a well, perhaps). How fast can this motor raise the crate?



From the previous problems, the power needed goes up as the speed goes up. Our motor is limited here to putting out 2 *HP* or $(2 \text{ HP}) \times \frac{746 \text{ W}}{1 \text{ HP}} = 1492 \text{ W}$, so that will result in some constant speed at which the crate can move. The motor will accelerate the crate from rest up to this speed, and then it will continue to rise at that speed.

In the vertical direction, we have $\sum F_y = 0$ so $T - mg = 0$ or $T = mg = (100 \text{ kg})(9.81 \text{ m/s}^2) = 981 \text{ N}$.

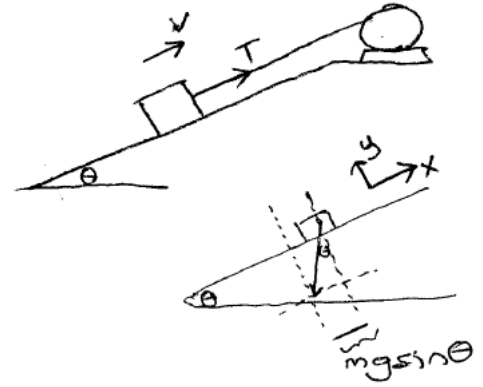
Here then we have a force of 981 *N* pulling the crate at a speed v . This requires a power of $P = Fv = (981)(v)$ watts. We have 1492 *W* available, so $1492 = 981v$ and $v = 1.52 \text{ m/s}$.

The motor lifting the object is working against the full weight of the object, so can't lift it as fast as when it was pulling it across the floor (where it only had to work against friction, and since μ_k is generally less than 1, friction will be somewhat less than the full weight of the crate).

15. Power : tow truck winch

Suppose we have one of those tow trucks where the car is being pulled up a tilted ramp. If the mass of the car is 1000 kg and we're still using our little 2 HP motor, how fast will the car move up the ramp?

In the figure, the block represents the car. Suppose the angle of the ramp is 30° , and that we have **no friction** present in this problem. (The 'rolling friction' of a car is generally pretty low.)



The power is being delivered to the car through the tension in the cable. That tension T is in the same direction the car is moving up the ramp, so the power involved will be $P = \vec{F} \cdot \vec{v} = (T)(v) \cos 0 = Tv$. We know how much power the motor can put out, so we'll need to find the tension in the cable to be able to determine how fast the car will be moving.

As with all these sample problems, the motor initially accelerates the car but as the speed increases, more and more power is required and eventually we reach the capacity of the motor and the car just continues at some constant speed.

If we set up a rotated coordinate system as shown in the figure, $\sum F_x = 0$ (we're at the point where we're moving up the ramp at some constant speed). Resolving the weight of the car into its X and Y components, we see the X component will have a magnitude of $mg \sin \theta$ and will be directed down-slope, while the tension is directed up-slope. $\sum F_x = T - mg \sin \theta = 0$ so $T = mg \sin \theta$.

Putting this into the equation we found for the power: $P = Tv = mgv \sin \theta$.

Let's look at that equation for a second: we see that P is proportional to m so a heavier car will require more power; P is proportional to v , so it would take more power to pull the car at a faster speed. And it's proportional to $\sin \theta$ so as the angle increases more power is needed too.

In our particular case, we have a 1000 kg car being pulled up a 30° ramp using a 1492 W motor, so: $P = mgv \sin \theta$ becomes: $1492 = (1000)(9.81)(v)(0.5)$ or $v = 0.304\text{ m/s}$ (or about 1 foot/sec).

Addendum : what happens when the car reaches the top of the ramp and is now on the flat part of the truck's bed? The motor is creating some tension in the cable still, but we have no friction here, and no component of gravity working against that force, so the car will accelerate forward, potentially crashing into the cab of the truck. The tow truck operator has to carefully decrease the power the motor is putting out as the car reaches the flat part of the bed.

16. Example: Generalized Gravitational Potential Energy

How fast does an object near the surface of the Earth need to be launched vertically upward in order to reach a height of 500 *miles* above the surface of the Earth?

Let's do this two ways: first, we'll assume $U_g = mgh$, which is a good approximation near the surface of the Earth. Later, we'll use the actual form for U_g based on Newton's universal gravitational force.

Assuming $U_g = mgh$: let's define our Y axis to point vertically upward with $y = 0$ at the surface. Then conservation of energy would require that: $(K + U_g + U_s)_2 = (K + U_g + U_s)_1 + W_{other}$.

Position 1 is at the surface and position 2 will be when the object has reached 500 *miles* or 804.5 *km* above that point. We have no springs, and no other work. At the starting position, the object is moving upward with some speed v and is located at $y = 0$ (making $U_g = 0$ at that point). At the 'final' position, the object has come to a stop. Applying that information: $(0 + mgh + 0) = (\frac{1}{2}mv^2 + 0 + 0) + 0$ or $mgh = \frac{1}{2}mv^2$ from which $v = \sqrt{2gh}$.

Here, the height the object reaches is $h = 804,500$ *m* which yields $v = 3971$ *m/s* (about 8800 *miles/hr*).

Using actual form for U_g : Since the force of gravity depends on the location of the object, the actual work done by gravity as the object rises to the given level is slightly different (and slightly less actually). The more general form for potential energy involves doing an integral: $U(r) = -\int F(r)dr$ and applying this to the general force of gravity between two objects ($F = GMm/r^2$, attractive) yields: $U_g = -GMm/r$.

Energy is still conserved: $(K + U_g + U_s)_2 = (K + U_g + U_s)_1 + W_{other}$ but now we use the more correct version for U_g . At the surface of the earth, $U_g = -GMm/R_e$ where R_e is the radius of the Earth. At the height of 500 miles, we have $U_g = -GMm/(R_e + h)$ (where here $h = 804.5$ *km*).

CoE becomes: $(0 - G\frac{Mm}{R_e+h} + 0) = (\frac{1}{2}mv^2 - G\frac{Mm}{R_e} + 0) + 0$

Rearranging terms: $\frac{1}{2}mv^2 = GMm(\frac{1}{R_e} - \frac{1}{R_e+h})$.

The mass of the object itself cancels out on both sides, so apparently doesn't matter, leaving us with:

$$v^2 = 2GM(\frac{1}{R_e} - \frac{1}{R_e+h})$$

One more bit of rearranging: we can also write this as:

$$v^2 = 2\frac{GM}{R_e}(1 - \frac{R_e}{R_e+h}).$$

Using G , the mass of the Earth, and the radius of the Earth from the tables in the front of the book, and using $h = 804,500$ *m* as the final height above the surface, we end up with: $v = 3742$ *m/s* or about 8370 *miles/hr* which is a bit lower than what we got before when we just used $U_g = mgh$.

17. Example: Escape Velocity

In the previous example, we ended up with an equation relating launch speed and maximum height: $v^2 = 2GM\left(\frac{1}{R_e} - \frac{1}{R_e+h}\right)$

The larger h is (the farther away from the planet we want the object to reach), the smaller the second term becomes, leading to a higher launch speed v . What launch speed would be needed for the object to reach ‘infinity’? I.e. completely escape from the planet?

Setting h to infinity, the second term goes to zero, leaving us with just $v^2 = 2GM/R_e$. This is called the escape velocity for a planet or moon. Using the Earth’s mass the radius, the escape speed for an object starting at the surface of the Earth is $v = 11,182 \text{ m/s}$ or about $25,000 \text{ miles/hr}$. For an object trying to take off from the Moon, the escape speed would be $v = 2374 \text{ m/s}$ or about 5300 miles/hr .

Rules of thumb : beyond what height, or what velocity, is the approximation $U_g = mgh$ bad to use?

We have to decide what we mean by ‘bad’ first... Let’s say we’re okay with a 1 percent error in the answer.

Maximum height errors of 1 percent occur when our vertical velocity reaches about 1100 m/s . The errors aren’t linear though. By the time we reach 3600 m/s , the error in our height calculation has risen to 10 percent. As long as our vertical speeds are below roughly 1000 m/s though, we can get away with just using the $U_g = mgh$ approximation.

If we’re looking for the launch speed needed to achieve a given actual height, when h reaches around 120 km (that’s kilometers) our initial launch speed calculations are about 1 percent in error.