

PH2213 : Examples from Chapter 9 : Linear Momentum

Key Concepts

Methods in this chapter give us tools for analyzing:

- collisions (pool balls, car wrecks, football tackle, etc)
- explosions (recoil)
- time-varying forces

Key Equations

Momentum of an object: $\vec{p} = m\vec{v}$

Total momentum of a collection of objects $\vec{P} = \sum \vec{p}_i = \sum m_i \vec{v}_i$

Total momentum is **conserved** in collisions/explosions (consequence of Newton's third law)

In most collisions, some mechanical energy is lost. These are called **inelastic** collisions. The maximum possible loss occurs in **totally inelastic collisions** where the two objects stick and move off together after the collision. In very special cases, both momentum **and** mechanical energy may be conserved, and those are called **elastic** collisions.

Restatement of Newton's laws: $\sum \vec{F} = d\vec{p}/dt$ or integrating both sides: $\vec{p}_2 = \vec{p}_1 + \int_1^2 \vec{F} dt$

Impulse: $\vec{J} = \Delta\vec{p}$ (just a definition; \vec{J} isn't anything new, it's just another name for $\Delta\vec{p}$)

Average net force acting: $\vec{F}_{avg} = \Delta\vec{p}/\Delta t$

Center of mass: $\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \dots}{m_1 + m_2 + \dots}$ or $\vec{r}_{cm} = \frac{\sum m_i \vec{r}_i}{\sum m_i}$

For a collection of interacting objects, all the internal forces cancel out in pairs, leaving $\sum \vec{F}_{ext} = M\vec{a}_{cm}$ (I.e. the external forces like gravity, etc, act on the collection as a whole. The collection's center of mass moves as if all the mass were located at a single point at the center of mass.)

Very specialized 1-D elastic collisions : (warning, this is **not** the norm...)

If one object is at **rest**, and is struck by a second object in a special way such that both momentum **and** energy are conserved (i.e. a 1-D elastic collision where only one of the objects was initially moving), we can derive some special equations for the velocities of the two objects after the collision.

We'll label quantities here as '1' (the object that was initially moving) and '2' (the object that was initially at rest). We'll also use the subscript 'i' to represent the initial condition (i.e. just before the collision) and 'f' the final condition (i.e. just after the collision). Then:

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} \text{ and } v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i}$$

Again, this **only** applies when the object labelled 2 was initially at rest, and when mechanical energy is conserved in the collision. This is **not** the normal situation.

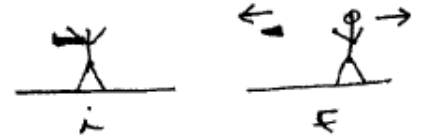
Common Errors

- momentum is conserved as a **vector** which means in 2 or 3-d collision, we may need to break the CoM equations into separate x, y, (and z) components
- mechanical energy is **rarely** conserved during collisions
- if a collision is present, break the problem into steps where we can focus **just** on the collision, where we can **only** count on momentum being conserved. Other methods (work, energy, CoE, etc) are available up to the point of the collision, and are again available AFTER the collision; just NOT DURING the collision itself.
- Average force: $\vec{F}_{avg} = \Delta\vec{p}/\Delta t$ is a vector equation that let's us compute what average force must have been acting **on an object** when **that object** changes **its** momentum suddenly as a result of a collision

1. 1-D Momentum Example : Firing a gun on a frictionless surface

Suppose a hunter is standing on a completely frictionless surface ($\mu = 0$). He aims to the left and fires the gun. What happens to him?

Momentum is always conserved. Initially nothing is moving, but after the bullet is fired, we have a (small) mass travelling to the left with some (high) velocity, so the person must recoil in the opposite direction at some speed.



Suppose the person has a mass of 100 kg , the bullet has a mass of 10 gm (0.01 kg), and flies away at an initial speed of 1000 m/s .

During the **tiny** Δt of the firing, this is 1-D motion, so we'll define a coordinate axis with the +X direction heading off to the right.

Momentum is conserved as a **vector**, but at the instant of the firing, everything is happening in the X direction, so we'll just focus on the X components of everything here.

Initially, the total momentum is zero (nothing is moving).

Immediately after the firing, we have the bullet of mass 0.01 kg moving to the **left** at a speed of 1000 m/s , and the hunter has a mass of 100 kg and will be moving with some velocity V .

The total momentum immediately **after** the firing is: $\sum m_i v_i = (0.01\text{ kg})(-1000\text{ m/s}) + (100.0\text{ kg})(V)$

This must equal the total initial momentum (which was zero), so conservation of momentum requires that $(0.01\text{ kg})(-1000\text{ m/s}) + (100.0\text{ kg})(V) = 0.0$ or $-10 + 100V = 0$ so $V = +0.10\text{ m/s}$.

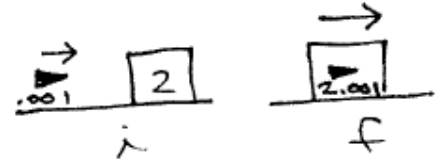
After firing the gun, the hunter must be sliding to the right across this frictionless surface at a speed of 10 cm/s .

Energy : Let's look at the energy before and after the firing. Focusing on the tiny Δt of the firing itself, nothing is changing in the Y direction (yet), so we have no change in gravitational potential energy. Initially, nothing is moving, so our initial kinetic energy is zero. Work-energy says that the total kinetic energy **after** will equal the total kinetic energy **before**, plus any works done: $K_{after} = K_{before} + \sum W$. After the firing, we have two moving objects. The bullet has $K = \frac{1}{2}mv^2 = (0.5)(0.01)(1000)^2 = 5000\text{ J}$, and the shooter has $K = \frac{1}{2}mv^2 = (0.5)(100)(0.1)^2 = 0.5\text{ J}$ so work-energy requires that $(5000\text{ J}) + (0.5\text{ J}) = \sum W$ or the explosion of the powder inside the bullet must have provided 5000.5 J of energy. (Actually more, since some energy would have been converted into friction and heat.)

Summary : As we will see as we go through these examples, momentum is always conserved, but **mechanical** energy is usually **not** conserved. In the case of a collision, some mechanical energy is often 'lost' (converted into heat, for example).

2. 1-D Momentum Example : bullet into block of wood

Suppose we have a 1 *gm* bullet moving at 1000 *m/s* that enters a 2 *kg* block of wood that is sitting on a frictionless surface, initially at rest. The bullet embeds itself in the wood. What happens? How does the block move after this collision? How much mechanical energy was lost in this collision?



As in the previous example, although momentum is conserved as a vector, here everything is happening in the X direction, so we'll just focus on the X components of momentum.

Total momentum **before** the collision: $\sum m_i v_i = (0.001 \text{ kg})(1000 \text{ m/s}) + (2.0 \text{ kg})(0) = 1 \text{ kg m/s}$.

Total momentum **after** the collision: we now have a single object, formed from the block of wood plus the bullet, so it has a mass of 2.001 *kg* now. After the collision, it will be moving with some velocity *V*, so the total momentum immediately after the collision will be $\sum m_i v_i = (2.001 \text{ kg})(V)$.

Momentum is conserved, so the total momentum before the collision must equal the total momentum after the collision: $1 \text{ kg m/s} = (2.001 \text{ kg})(V)$ or $V = (1/2.001) \text{ m/s}$ or $V = 0.49975 \text{ m/s}$.

Quite a lot of energy is often lost in collisions. Let's look at the total mechanical energy before and after this collision.

Initially we have the kinetic energy of the bullet: $K = \frac{1}{2}mv^2 = (0.5)(0.001 \text{ kg})(1000 \text{ m/s})^2 = 500 \text{ J}$. The block is not moving yet, so has no kinetic energy. Our total mechanical energy just before the collision is 500 *J*. (No springs here; nothing is changing elevation, so no U_g either.)

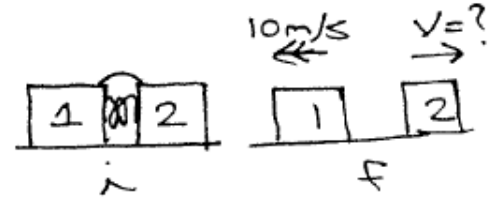
Immediately after the collision, we have the 2.001 *kg* object moving at 0.49975 *m/s* and that represents a kinetic energy of $K = \frac{1}{2}mv^2 = (0.5)(2.001)(0.49975)^2 = 0.25 \text{ J}$.

So we went from 500 *J* to only 0.25 *J* of mechanical energy here: a loss of 499.75 *J* which is a 99.95 percent loss. Where does this energy go? The friction of the bullet with the wood, slowing it to a stop, probably converted most of it to heat.

Summary : As we will see as we go through these examples, momentum is always conserved, but **mechanical energy** is usually **not** conserved. In the case of a collision, some (maybe most) of the mechanical energy is often 'lost' (converted into heat, for example).

3. 1-D Momentum Example : boxes and spring

Suppose we push two boxes together with a spring between them, then let them go. (Frictionless floor, of course.) Let's say that we have a 1 kg box on the left, and a 2 kg box on the right. The spring is storing some amount of energy (we don't know how much yet) but we do observe that when we let the boxes go, the lighter block goes flying off to the left at 10 m/s. What happens to the heavier block? How much energy must have been stored in the spring initially?



Momentum is conserved as a vector. Here, everything is happening in the X direction, so we'll just focus on the X components of momentum.

Momentum before the 'event' (the releasing of the boxes): nothing is moving, so $\sum m_i v_i = 0$ at this point.

Momentum after the event: we have the 1 kg box moving at 10 m/s to the left, and the 2 kg box moving at some velocity V , so the total X momentum at this point is: $\sum m_i v_i = (1 \text{ kg})(-10 \text{ m/s}) + (2 \text{ kg})(V)$.

Momentum is conserved, so the total momentum before the collision must equal the total momentum after the collision: $0 = (1)(-10) + (2)(V)$ or $V = +5 \text{ m/s}$, telling us that the heavier block must now be moving off to the right at 5 m/s.

Energy :

Initially, we have some energy stored in the spring.

After the spring releases this energy and the blocks are flying apart, we have mechanical energy in the form of the two moving boxes: $K_{after} = \sum \frac{1}{2} m_i v_i^2 = \frac{1}{2}(1 \text{ kg})(10 \text{ m/s})^2 + \frac{1}{2}(2 \text{ kg})(5 \text{ m/s})^2 = 50J + 25J = 75 J$. This is the amount of energy that must have been stored in the spring initially.

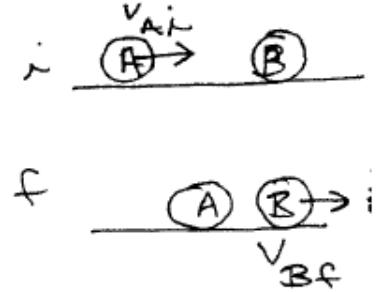
Summary : When we have something like a spring involved in our momentum calculations, momentum is conserved of course but mechanical energy isn't, although it's obvious where the extra energy came from (the spring). In chemical explosions (dynamite, bullet, etc), that energy would be initially stored in the chemicals themselves and is released in some (usually very fast) reaction.

4. 1-D Momentum Example : Pool Ball Collision (Elastic)

Note : these pool-balls are special frictionless versions that are not rotating - they're just idealized point masses that have mass and velocity. We'll get to real rotating objects in the next chapter.

Suppose I shoot the cue ball at 10 m/s directly at a stationary ball and I observe that after the collision, the pool ball has come to a stop and the other ball is now moving off to the right at some velocity V . How fast is it moving? Look at the mechanical energy before and after this collision.

Typical pool balls have masses of 0.16 kg .



Momentum is conserved as a vector. Here, everything is happening in the X direction, so we'll just focus on the X components of momentum.

Total momentum before collision: $\sum m_i v_i = (0.17 \text{ kg})(10 \text{ m/s}) + (0.17 \text{ kg})(0) = 1.7 \text{ kg m/s}$

Total momentum after collision: $\sum m_i v_i = (0.17 \text{ kg})(0 \text{ m/s}) + (0.17 \text{ kg})(V)$.

Momentum is conserved, so $1.7 = 0.17V$ or $V = 10 \text{ m/s}$. Apparently ball B must be moving off with the same speed as the incoming cue ball had.

Let's look at the total mechanical energy before and after this collision:

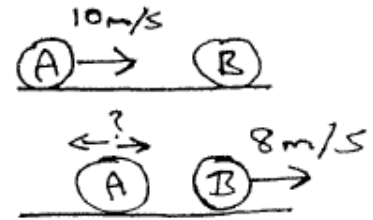
$$\sum K_{before} = \frac{1}{2}(0.17)(10)^2 + 0 = 8.5 \text{ J}$$

$$\sum K_{after} = 0 + \frac{1}{2}(0.17)(10)^2 = 8.5 \text{ J also.}$$

Summary : These types of collisions are referred to as **elastic collisions** : momentum is conserved (like always) **and** energy is conserved. This sometimes happens with collisions between very solid objects that do not deform, or when something slides into a spring. Elastic collisions are **not** the normal situation for most real-world colliding objects. Some energy is almost always lost, so unless specifically noted we can't normally assume that energy will be conserved in collision problems.

5. 1-D Momentum Example : Pool Ball Collision (Inelastic)

Let's alter the previous example. Suppose that after the collision, we observe ball B moving off to the right at 8 m/s . What must have happened to the cue ball? Look at the mechanical energy in this case too.



Momentum is conserved as a vector. Here, everything is happening in the X direction, so we'll just focus on the X components of momentum.

Total momentum before collision: $\sum m_i v_i = (0.17)(10) + (0.17)(0) = 1.7\text{ kg m/s}$

Total momentum after the collision: well, ball B is moving more slowly now, so it only has a momentum of $mv = (0.17)(8) = 1.36\text{ kg m/s}$, which is less than the 1.7 kg m/s of momentum we had initially. This 'leftover' momentum must be being carried away by the cue ball, so it has some velocity V after the collision. The total momentum after the collision then must be $\sum m_i v_i = (0.17\text{ kg})(V) + (0.17\text{ kg})(8\text{ m/s}) = 1.36 + 0.17V$.

Momentum is conserved, so the total momentum after the collision must equal the total momentum before the collision: $1.36 + 0.17V = 1.7$ or $V = +2.0\text{ m/s}$. The cue ball must be travelling to the right (following the other ball) at a speed of 2 m/s .

Let's look at the total mechanical energy before and after this collision.

$$\sum K_{\text{before}} = \frac{1}{2}(0.17)(10)^2 + 0 = 8.5\text{ J}$$

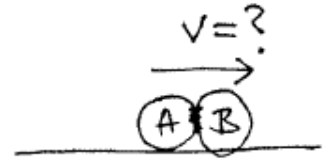
$$\sum K_{\text{after}} = \frac{1}{2}(0.17)(2)^2 + \frac{1}{2}(0.17)(8)^2 = 0.34\text{ J} + 5.44\text{ J} = 5.78\text{ J}$$

In this collision, 32% of the initial mechanical energy was 'lost' (and would have gone into heat probably).

Summary : We apparently lost some energy in this collision. This is the more usual case with collisions, and these are called **inelastic collisions** : momentum is conserved (as always), but some mechanical energy is lost (often being converted into heat).

6. 1-D Momentum Example : Totally Inelastic Collision

Let's modify our previous examples so that the pool balls are covered with Velcro or super-glue causing them to stick together during the collision, and move off as a single object. How fast does the combined object move after the collision? How much energy is lost now?



Total momentum before the collision: $\sum m_i v_i = (0.17)(10) + (0.17)(0) = 1.7 \text{ kg m/s}$

Total momentum after the collision: now we have a single blob of mass $0.17 + 0.17 = 0.34 \text{ kg}$ moving at some velocity V : $\sum m_i v_i = (0.34 \text{ kg})(V)$.

Momentum is conserved, so $1.7 \text{ kg m/s} = (0.34 \text{ kg})(V)$ or $V = 5 \text{ m/s}$.

Let's look at the total mechanical energy before and after this collision.

$$\sum K_{before} = \frac{1}{2}(0.17)(10)^2 + 0 = 8.5 \text{ J}$$

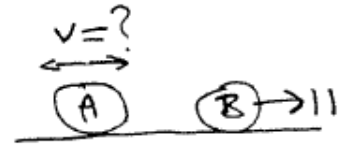
$$\sum K_{after} = \frac{1}{2}(0.34)(5)^2 = 4.25 \text{ J}$$

Compared to the previous examples, in this collision, 50% of the initial mechanical energy was 'lost' (and would have gone into heat probably).

Summary : It turns out this 'sticking together' scenario is the one where the largest possible amount of energy gets lost, and it is referred to as a **totally inelastic collision**.

7. 1-D Momentum Example : Impossible Collision

We have our same pool-ball collision event happening here, but suppose we see ball B moving off at 11 m/s to the right. What must have happened to ball A? Look at energy in this case too.



Total momentum before the collision: $\sum m_i v_i = (0.17)(10) + (0.17)(0) = 1.7 \text{ kg m/s}$

Total momentum after the collision: we have ball B moving to the right at 11 m/s and ball A moving at some velocity V , so: $\sum m_i v_i = (0.17)(V) + (0.17)(11) = 0.17V + 1.87 \text{ kg m/s}$.

Momentum is conserved, so $1.7 \text{ kg m/s} = (0.17 \text{ kg})(V) + (1.87 \text{ kg m/s})$ or $V = -1 \text{ m/s}$.

Allegedly, the cue ball has bounced backwards, and is now moving at 1 m/s to the **left**.

Is this possible though?

Let's look at the total mechanical energy before and after this collision.

$$\sum K_{before} = \frac{1}{2}(0.17)(10)^2 + 0 = 8.5 \text{ J}$$

$$\sum K_{after} = \frac{1}{2}(0.17)(1)^2 + \frac{1}{2}(0.17)(11)^2 = 0.085 \text{ J} + 10.285 \text{ J} = 10.37 \text{ J}$$

That is **more** energy than what we started with though, so this collision is impossible unless we have some source of energy hiding in the problem. That's possible: maybe we have a little blasting cap between them, or we've coated them with a chemical that explodes on contact... Unless we have some source of this energy, though, this scenario is not possible and the original statement of the problem is bogus - there's no way that ball B can move off at 11 m/s unless we have some hidden source of energy.

8. 1-D Momentum Example : Elastic Collision 1

If one object is at rest, and is struck by a second object in a special way such that both momentum **and** energy are conserved (i.e. a 1-D elastic collision where only one of the object was initially moving), we can derive some special equations for the velocities of the two objects after the collision.

We'll label quantities here as '1' (the object that was initially moving) and '2' (the object that was initially at rest). We'll also use the subscript 'i' to represent the initial condition (i.e. just before the collision) and 'f' the final condition (i.e. just after the collision). Then:

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} \text{ and } v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i}$$

Note that no matter what the masses are, the initially stationary object will move off in the same direction that object 1 came in with, but object 1 has more options: it might continue forward, bounce back, or come to a stop, depending on the value of $m_1 - m_2$.

Example: moving pool ball ($m_1 = 0.17 \text{ kg}$, and $v_{1i} = 10 \text{ m/s}$) hits a stationary bowling ball ($m_2 = 6 \text{ kg}$). (Actually the mass of bowling balls is quite variable, but 6 kg appears to be in the middle of the normal range.)

Then **after** the collision, the pool ball will have a velocity of:

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} = \frac{0.17 - 6}{0.17 + 6} (10 \text{ m/s}) = -9.45 \text{ m/s}$$

That's negative, so the pool ball has bounced back with almost the same speed it came in with.

The bowling ball, after the collision, will have a velocity of:

$$v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i} = \frac{2 \times 0.17}{0.17 + 6} (10 \text{ m/s}) = 0.55 \text{ m/s}$$

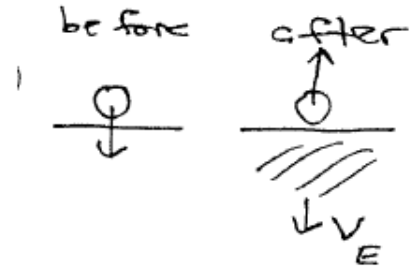
and will thus be moving (slowly) forward (in the same direction that the incoming pool ball original had).

Summary : Be **very careful** using these equations. They require that **both** energy and momentum be conserved in the collision, which is **not** the normal real-world situation.

9. 1-D Momentum Example : Elastic Collision 2

Suppose we drop a metal ball bearing onto a metal floor and that the collision conserves energy as well as momentum (i.e. assume we do have that rare case of an elastic collision occurring here). How fast will the ball bearing be moving after the event? How is momentum being conserved here?

Suppose the ball bearing has a mass of $m = 0.001 \text{ kg}$ and hits the floor with $v = 10 \text{ m/s}$. Basically, the ball bearing is 'colliding' with the Earth and bouncing off it. We'll use a coordinate system with +Y pointing upward. Using the specialized 1-D elastic collision equations from the previous problem, $m_1 = 0.001 \text{ kg}$, $v_{1i} = -10 \text{ m/s}$, and $m_2 = 6 \times 10^{24} \text{ kg}$.



$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} = \frac{0.001 - 6E24}{0.001 + 6E24} (-10 \text{ m/s}) = +9.9999999\dots \text{ m/s}.$$

$$v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i} = \frac{2 \times 0.001}{0.001 + 6E24} (-10 \text{ m/s}) = -3.33 \times 10^{-27} \text{ m/s}.$$

To about 27 significant figures, the ball bearing appears to have bounced back with the same speed it hit the ground with (the factor is very, very slightly less than 1, but is so close to 1 that we'd never be able to measure the difference). And the velocity of the earth after the collision is so tiny that we'd never be able to measure that either.

In order for the Earth to have a significant velocity after the collision, the mass of the incoming object would need to be a significant fraction of the mass of the earth (or be moving incredibly fast: maybe a really big, fast asteroid...).

Summary : when one of the objects involved in a collision is **hugely** more massive than the other, momentum is still conserved, but the calculations may require so many decimal places that a typical calculator may not be accurate enough. And since in most collisions some unknown amount of energy gets lost too, none of the methods we've covered so far is of much use.

10. Average Force : colliding pool balls

Let's go back to our earlier example where the cue ball struck a stationary ball, with the result that the cue ball came to a stop and the other ball moved off.

We generalized Newton's Laws and showed that we can relate a change in momentum to an average force: $\vec{F}_{avg} = \Delta\vec{p}/\Delta t$. The incoming cue ball had an initial X momentum of $mv = (0.17 \text{ kg})(10 \text{ m/s}) = 1.7 \text{ kg m/s}$ and a final X momentum of zero (it came to a stop). Collisions between pool balls happen very rapidly. Using a very high speed camera, it's found that the collision itself takes place in about 0.0006 s (i.e. 0.6 milliseconds). How much force did the cue ball 'feel' during this collision?

This is all 1-D motion in just the X direction, so we'll drop the vectoriness:

$$F_{avg} = \Delta p / \Delta t = \frac{0 - 1.7 \text{ kg m/s}}{0.0006 \text{ s}} = -2833 \text{ N}.$$

Compare this to the weight of the pool ball, or the normal force between the ball and the surface of the table: $mg = (0.17 \text{ kg})(9.81 \text{ m/s}^2) = 1.67 \text{ N}$. The force between the balls during the collision is about 1700 times larger than any of the other forces acting.

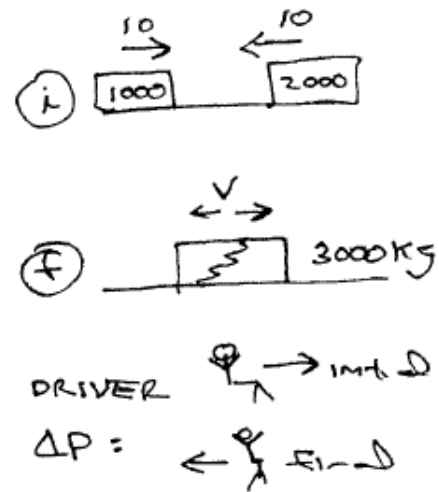
Alternate Solution

We can find the force the old fashioned way too, using Newton's Laws and equations of motion. If we assume a constant acceleration, the cue ball is going from an initial velocity of $v = +10 \text{ m/s}$ to a final velocity of 0 m/s in the given time interval, so we could use $v = v_o + at$ to find the acceleration: $(0) = (10) + (a)(0.0006)$ or $a = -16,667 \text{ m/s}^2$. The force needed to create this acceleration would be $F = ma = (0.17 \text{ kg})(-16,667 \text{ m/s}^2) = -2833 \text{ N}$.

Summary : The contact forces involved in most collisions are much larger than any other forces present, so during a collision we can generally completely **ignore** everything else and just **focus on the collision** : i.e. just focus on the conservation of momentum that is occurring during the collision. They occur so fast, that we can ignore everything else. When we have problems where objects are moving, sliding up or down ramps, for example, and then a collision happens, we can (and should) break the problem into parts and during the instant that the collision occurs, we just worry about conservation of momentum and ignore any other effects. Gravity and friction and other forces are still present, but the forces implied by the collision itself just overwhelm everything else during that brief time interval.

11. Average Force : car and truck colliding

Suppose we have a car with a mass of 1000 kg (including the driver) that is moving to the right at 10 m/s , and a 2000 kg truck (including the driver) moving to the left at 10 m/s . They collide, with the collision taking about 0.1 s . Assume this collision is completely inelastic (i.e. the two vehicles 'merge' and become essentially a single object of mass 3000 kg). What is the speed of the wreck immediately after the collision (before friction or anything else starts to act to slow it down)? If the drivers of each vehicle has masses of 100 kg , how much force will the car driver feel? The truck driver?



This is a 1-D collision, so we'll have a +X axis pointing to the right and just worry about the X components of momentum.

Total momentum before collision: $\sum m_i v_i = (1000\text{ kg})(10\text{ m/s}) + (2000\text{ kg})(-10\text{ m/s}) = -10,000\text{ kg m/s}$.

Total momentum after collision: now we have the single 3000 kg wreck moving off at some velocity V , so: $\sum m_i v_i = (3000\text{ kg})(V)$.

Momentum is conserved, so $-10000\text{ kg m/s} = (3000\text{ kg})(V)$ or $V = -3.33\text{ m/s}$. The wreck is apparently moving off to the left, immediately after the collision occurs.

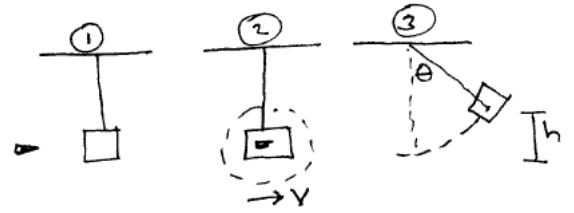
Let's look at the force that **each driver** will feel during the collision.

Car Driver : The driver of the car is initially moving to the right at 10 m/s but immediately after the collision is moving at 3.33 m/s to the left. This represents a change in momentum of $\Delta p = p_{\text{after}} - p_{\text{before}} = (100\text{ kg})(-3.33\text{ m/s}) - (100\text{ kg})(10\text{ m/s}) = -333 - 1000 = -1,333\text{ kg m/s}$. This change occurs in a time interval of 0.1 s , so represents an average force of $F_{\text{avg}} = \Delta p / \Delta t = (-1,333\text{ kg m/s}) / (0.1\text{ s}) = -13,333\text{ N}$. $F = ma$ so we can get an idea of what acceleration the driver is undergoing during the crash: $-13,333\text{ N} = (100\text{ kg})(a)$ or $a = -133\text{ m/s}^2$ or about $13g$'s.

Truck Driver : The driver of the truck is initially moving to the left at 10 m/s and immediately after the collision is moving at 3.33 m/s , still to the left. This represents a change in momentum of: $\Delta p = p_{\text{after}} - p_{\text{before}} = (100\text{ kg})(-3.33\text{ m/s}) - (100\text{ kg})(-10\text{ m/s}) = -333 + 1000 = +667\text{ kg m/s}$. This change occurs in a time interval of 0.1 s , so represents an average force of $F_{\text{avg}} = \Delta p / \Delta t = (+667\text{ kg m/s}) / (0.1\text{ s}) = +6670\text{ N}$. $F = ma$ so we can get an idea of what acceleration the driver is undergoing during the crash: $+6670\text{ N} = (100\text{ kg})(a)$ or $a = +67\text{ m/s}^2$ or about $6.7g$'s : half of what the car driver experienced, in this case.

12. Conservation of Momentum : Ballistic Pendulum

Suppose we hang a 1 kg block of wood from the ceiling on a 1 m long string. A 5 gm (0.005 kg) bullet comes in from the left at 300 m/s and embeds itself into the block. What happens? Conservation of momentum occurs during the collision, so immediately after the collision the combined block and bullet ‘object’ will be moving to the right with some speed. That means it has some kinetic energy at that point, and we earlier looked at the pendulum problem: if it’s moving at some speed at the bottom, how does that relate to the angle it will swing out to?



First, let’s focus on the collision. It occurs in a tiny Δt and we can ignore everything else and just worry about conservation of momentum.

This is basically a 1-D collision in the X direction. The total momentum before the collision is $\sum m_i v_i = (0.005 \text{ kg})(300 \text{ m/s}) + (1.0 \text{ kg})(0) = 1.5 \text{ kg m/s}$.

Immediately after the collision, we have a combined 1.005 kg object moving at some velocity V , so $\sum m_i v_i = (1.005 \text{ kg})(V)$.

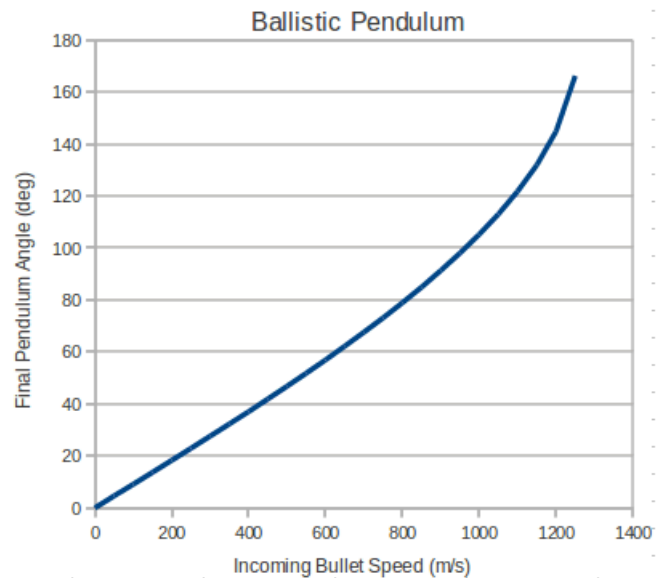
Momentum is conserved, so $1.5 = 1.005V$ or $V = 1.4925 \text{ m/s}$.

Refer to the pendulum example from chapter 6 or 7. We found that the velocity at the bottom was related to the angle θ the pendulum will swing out to via: $v^2 = 2gL(1 - \cos\theta)$. Rearranging this somewhat: $v^2/(2gL) = 1 - \cos\theta$ or $\boxed{\cos\theta = 1 - v^2/(2gL)}$.

Using the numerical values we have here: $\cos\theta = 1 - \frac{1.4925^2}{(2)(9.81)(1)} = 1.0 - 0.1135 = 0.8865$ from which $\theta = 27.6^\circ$.

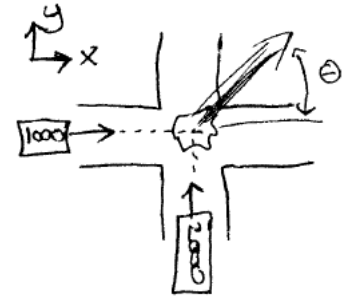
The faster the bullet is going, the larger this angle will be, but since V enters as itself squared and we have to do an inverse cosine to find the angle, the relationship between the incoming speed and the final angle is not linear (although it’s pretty surprisingly close for a while...)

In the figure, we vary the incoming bullet speed, do the computations above, and compute the final angle the pendulum will swing out to. (If the final angle exceeds 90 degrees, the ‘cable’ will need to be able to support a negative value of tension, which means it needs to be a solid rod or stick or something, not a rope or string.)



13. Conservation of Momentum : 2-D car accident

Suppose car A has a mass of 1000 kg and is travelling to the east at 20 m/s , and car B (a truck) has a mass of 2000 kg and is travelling at 15 m/s to the north. They both enter an intersection together and collide, forming a combined wreck that moves off at some velocity. Find this speed and the angle the wreck moves off with.



Momentum is conserved as a vector, and we have things occurring in 2 dimensions now, so we'll need to separately look at the X and Y components.

After the accident, the (combined) wreck is moving off at some velocity V that has components V_x and V_y .

Conservation of momentum in the X direction : $\sum m_i v_{xi}$ is the same before and after the collision. Before the collision, we have the car moving in the X direction, but the truck has **no** X component of velocity (and thus no X component of momentum). Thus before the collision: $\sum p_x = (1000\text{ kg})(+20\text{ m/s}) + 0 = 20,000\text{ kg m/s}$.

After the collision, the X component of momentum will be $\sum p_x = (3000\text{ kg})(V_x)$.

This must equal the X component of momentum before the collision though, so $(3000\text{ kg})(V_x) = (20,000\text{ kg m/s})$ or $V_x = 6.67\text{ m/s}$.

Conservation of momentum in the Y direction : $\sum m_i v_{yi}$ is the same before and after the collision. Before the collision, we have the car moving in the X direction, so it has **no** Y momentum. The truck is moving north, so its momentum is entirely in the Y direction. Thus before the collision: $\sum p_y = (1000\text{ kg})(0\text{ m/s}) + (2000\text{ kg})(15\text{ m/s}) = 30,000\text{ kg m/s}$.

After the collision, the Y component of momentum will be $\sum p_y = (3000\text{ kg})(V_y)$.

This must equal the Y component of momentum before the collision though, so $(3000\text{ kg})(V_y) = (30,000\text{ kg m/s})$ or $V_y = 10.00\text{ m/s}$.

We now have the two components of the velocity the wreck has immediately after the collision. The overall magnitude of this velocity is $V = \sqrt{V_x^2 + V_y^2} = \sqrt{44.44 + 100} = 12\text{ m/s}$ and it will have an angle of $\tan \theta = V_y/V_x = 10/6.67 = 1.5$ from which $\theta = 56.3^\circ$.

Energy Loss : Before the collision, we have two moving objects with a total kinetic energy of $\frac{1}{2}(1000\text{ kg})(20\text{ m/s})^2 + \frac{1}{2}(2000\text{ kg})(15\text{ m/s})^2 = 200,000\text{ J} + 225,000\text{ J} = 425,000\text{ J}$.

After the collision, we have the single combined object moving at the speed we just determined, so the kinetic energy present right after the collision is $\frac{1}{2}(3000\text{ kg})(12\text{ m/s})^2 = 216,000\text{ J}$.

Roughly half the incoming energy was 'lost' in the collision (going into heat, deforming the vehicles, etc).

14. Golf-ball Launch

Let's look at the collision represented by a golf club striking and launching a golf ball. We observe that the ball lands 100 m away from where it was hit, and that the ball was launched into the air at an angle of 45 degrees above the horizontal, and would like to get an idea of **how fast the club head** must have been moving when it struck the ball to achieve this result.

To simplify things, let's assume we have a 1D collision: that is, the club is moving in some direction and right after the collision the club and the ball are still moving along the same direction.

When the club strikes the stationary ball, momentum will be conserved. With real golf balls, mechanical energy won't be conserved (in fact there is a material property called the coefficient of restitution that is related to the fraction of the energy lost that is specified for official balls).

In this version of the problem, let's assume that energy **is** conserved though.

So: for the collision part of the problem, we'll assume that we have a **1-D elastic collision**.

Suppose we observe that the ball is launched at a 45 degree angle and is observed to land 100 m away from its launch point. Then: how fast must the club head have been moving to produce this result?

Some random data we'll use: let the club head have a mass of 0.5 kg and the ball have a mass of 0.05 kg. (These aren't quite right for the real objects, but close enough.) And we also know how far the ball travelled (100 m) and its launch angle relative to the horizontal (45°).

We'll label the club with subscript 'c' and the ball with 'b' and properties just after the collision will have a prime symbol. For example, v_c would be the velocity of the club head before the collision, and v'_c would be its velocity just after the collision.



The collision: we have the club head moving at v_c striking the ball which is at rest. Just after the collision, the club head will be moving at v'_c and the ball at v'_b .

Conservation of momentum in this rotated x direction:

$$\Sigma(mv_x)_{before} = \Sigma(mv_x)_{after} \text{ or: } (0.5)(v_c) + (0.05)(0) = (0.5)(v'_c) + (0.05)(v'_b)$$

Multiplying this equation by two yields: $v_c = v'_c + 0.1v'_b$

Conservation of energy

(Remember, mechanical energy is normally **not** conserved during collision; so this assumption is probably not valid...)

From other examples we've done, the collision occurs so rapidly that the force related to the collision will be **much larger** than any other forces present, so we can ignore any minute

changes in potential energies or any 'other work' during the collision. CoE basically simplifies to $K_{before} = K_{after}$ where K is the total mechanical energy of the objects involved.

Here then: $\frac{1}{2}(0.5 \text{ kg})(v_c)^2 + 0 = \frac{1}{2}(0.5 \text{ kg})(v'_c)^2 + \frac{1}{2}(0.05)(v'_b)^2$

Multiplying this equation by 4 yields:

$$(v_c)^2 = (v'_c)^2 + 0.1(v'_b)^2$$

We now have two equations, but three unknowns but we can use the information we know about the post-collision flight path to eliminate one of them:



We know how far the ball travels so we can go back to one of our projectile motion equations from chapter 3. If an object is launched at some initial speed and angle, it will land this far away: $R = (v_o^2 \sin(2\theta))/g$

Here, we know the range $R = 100 \text{ m}$ and the angle was 45 degrees, so $100 = (v_o^2 \sin 90)/9.8$ from which $v_o = 31.3 \text{ m/s}$.

This is what we've been calling v'_b (the speed of the golf ball right after the collision with the club), so we now know that $v'_b = 31.3$.

Making this substitution in the earlier boxed equations, we have:

$$v_c = v'_c + 3.13, \text{ and}$$

$$(v_c)^2 = (v'_c)^2 + 97.969$$

Replacing v_c in the second equation using the expression in the first equation:

$$(v'_c + 3.13)^2 = (v'_c)^2 + 97.969$$

Expanding out the first term:

$$(v'_c)^2 + 6.26v'_c + 9.7969 = (v'_c)^2 + 97.969$$

Note that we have exactly the same squared term on each side, so we can cancel that out, leaving:

$$6.26v'_c + 9.7969 = 97.969 \text{ from which } v'_c = 14.085 \text{ m/s}$$

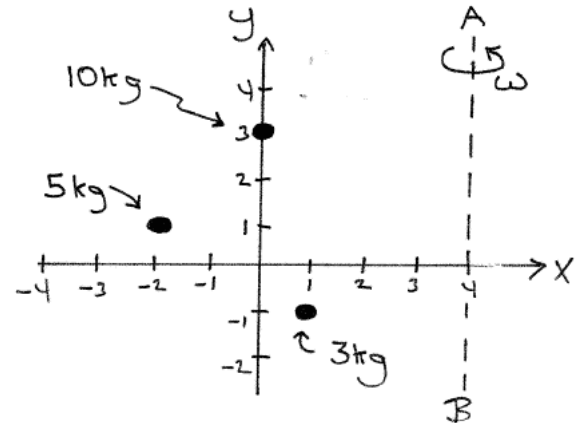
That's the velocity of the club just after the collision but CoM connected that velocity to the velocity of the club just before the collision: $v_c = v'_c + 3.13 = 14.085 + 3.13 = 17.2 \text{ m/s}$.

The club head must have been moving at 17.2 m/s (about 38 miles/hr) when it struck the ball.

That's not an unreasonable result, despite all our assumptions. From what I've found, a professional golfer might hit the ball with a club-head speed up to 100 miles/hr (resulting in an even longer range for the ball).

15. **Center of Mass**

The object in the figure is constructed from three tiny dense objects, connected by strong by massless rods (which means that all the mass is concentrated at the three points shown). Find the center of mass of the object.



(Ignore that dotted line AB for now - we'll come back to this when we talk about rotation and moments of inertia.)

Center of mass (and later moment of inertia) calculations involve summing products, and it is convenient to organize these into a table:

i	m_i	x_i	y_i	$m_i x_i$	$m_i y_i$
1	3	1	-1	3	-3
2	5	-2	1	-10	5
3	10	0	3	0	30
	$\sum m_i = 18$			$\sum m_i x_i = -7$	$\sum m_i y_i = 32$

Adding the appropriate columns:

$$M = \sum m_i = 3 + 5 + 10 = 18 \text{ kg}$$

$$\sum m_i x_i = 3 - 10 + 0 = -7 \text{ kg m}$$

$$\sum m_i y_i = -3 + 5 + 30 = 32 \text{ kg m}$$

$$X_{cm} = \frac{1}{M} \sum m_i x_i = \frac{1}{18}(-7) = -0.389 \text{ m}$$

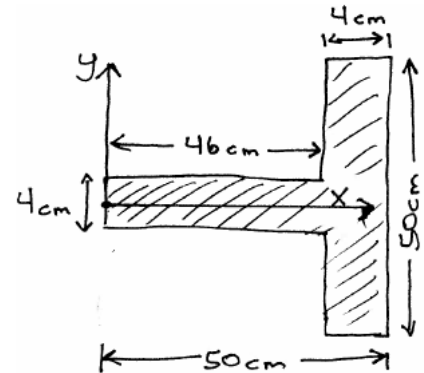
$$Y_{cm} = \frac{1}{M} \sum m_i y_i = \frac{1}{18}(32) = 1.78 \text{ m}$$

If we toss this object in the air, it may tumble and spin around, but the center of mass point will move through the air in a parabola exactly according to our 2D equations of motion.

16. Center of Mass for a Composite Object (1)

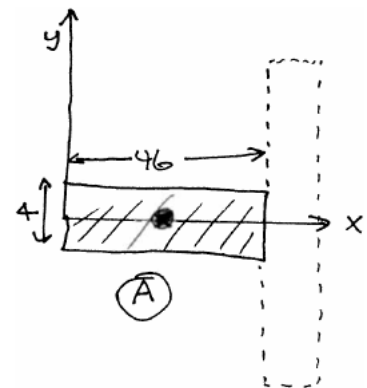
Due to the way the center of mass is defined, we can break the sum into partial sums that represent the different 'parts' of an object and combine those parts as if we replaced each part with a point mass located at that part's center of mass.

Suppose we have the T-square shown in the figure. Where is its center of mass? The object is made of thin flat metal that is 1 mm (0.001 m) thick with a density of $\rho = 10,000\text{ kg/m}^3$ (which is a typical density for metals).

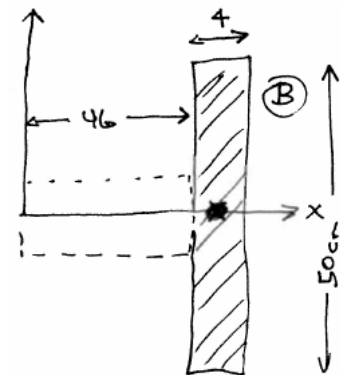


For a rectangle of uniform density, the center of mass is right at its geometric center, so we can think of this object as two rectangles, A and B. We can find the mass and center of mass of each rectangle, and then use the Center of Mass formula to find the CM of the composite object.

Object A : this rectangle is 46 cm long and 4 cm wide so its CM will be at $x = 23\text{ cm}$, $y = 0\text{ cm}$. What is its mass? Its volume will be $V = (0.46\text{ m})(0.04\text{ m})(0.001\text{ m}) = 1.84 \times 10^{-5}\text{ m}^3$ so its mass will be $m = \rho V = (10,000\text{ kg/m}^3)(1.84 \times 10^{-5}\text{ m}^3) = 0.184\text{ kg}$.



Object B : this rectangle is 50 cm tall and 4 cm wide. Its CM will be at its geometric center which means to get to the X coordinate of the CM we need to shift 46 cm to the right and then another 2 cm to the right so $x = 48\text{ cm}$ and $y = 0\text{ cm}$. What is the mass of this part? Its volume will be $V = (0.50\text{ m})(0.04\text{ m})(0.001\text{ m}) = 2.00 \times 10^{-5}\text{ m}^3$ so its mass will be $m = \rho V = (10,000\text{ kg/m}^3)(2.00 \times 10^{-5}\text{ m}^3) = 0.200\text{ kg}$.



(continued)

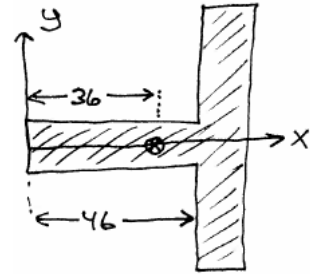
Center of Mass : $X_{cm} = \frac{1}{M} \sum m_i x_i$ where this sum now just has two terms since we've replaced each of our two rectangles with point masses located at their corresponding centers of mass. We have a 0.184 kg point-mass located at $x = 0.23 \text{ m}$, $y = 0 \text{ m}$ and a 0.200 kg point-mass located at $x = 0.48 \text{ m}$, $y = 0 \text{ m}$.

M in the CM equation is the total mass of the object so $M = 0.184 + 0.200 = 0.384 \text{ kg}$.

$$X_{cm} = \frac{1}{0.384 \text{ kg}} ((0.184 \text{ kg})(0.23 \text{ m}) + (0.200 \text{ kg})(0.48 \text{ m})) = 0.3602 \text{ m} \text{ or } X_{cm} = 36.02 \text{ cm}.$$

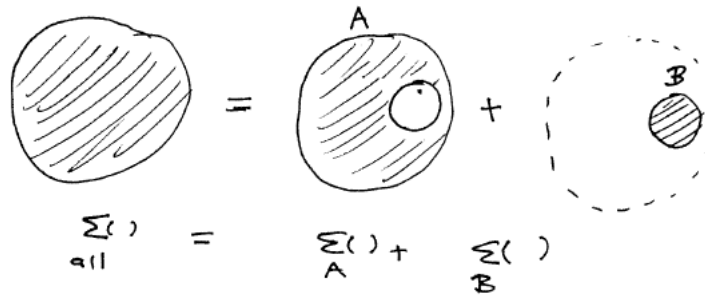
Each of the two parts had their centers of mass at $y = 0$ so the corresponding Y_{cm} calculation will just end up with $Y_{cm} = 0$.

The center of mass of this object then will be about where the mark is shown on the figure.



17. **Center of Mass for a Composite Object (2)** Due to the way the center of mass is defined, we can break the sum into partial sums that represent the different 'parts' of an object and combine those parts as if we replaced each part with a point mass located at that part's center of mass. We can use this same process to compute the CM of objects with holes cut in them.

Suppose we want to determine the center of mass of a flat disk with a circular hole cut out. The CM of a uniform disk will be right at its geometric center. We can think of a solid disk as being constructed from two parts: the disk with a hole in it (which is what we're interested in) plus a small disk that would just fill in that hole. We can break the sum (or integral) over the whole object into a sum over just the part we're interested in, plus the part that fills the hole.



Suppose the outer disk has a radius of 1 meter and the hole is a disk with a radius of 20 cm whose center is located 50 cm out from the center of the big disk. Assume the disk is made of thin metal with a density of $\sigma = 10 \text{ kg/m}^2$ (that's mass per area, not mass per volume; the symbol σ is often used to represent this type of 'density').

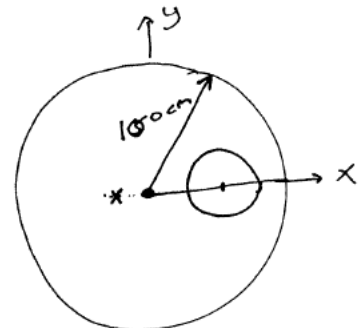
The mass of the complete disk (without a hole) will be its area times the mass/area so $M = (\pi r^2)\sigma = (3.14159..)(1.00 \text{ m})^2 \times (10 \text{ kg/m}^2) = 31.415 \text{ kg}$.

The mass of part B (the plug that will fit in the hole) will be $m_b = \sigma(\pi r^2) = (10 \text{ kg/m}^2) \times \pi(0.20 \text{ m})^2 = 1.257 \text{ kg}$.

That means the part we are interested in (the disk with the hole cut in it) will have a mass of $m_a = 31.415 - 1.257 = 30.158 \text{ kg}$.

The center of mass is defined as $x_{cm} = \frac{1}{M} \sum m_i x_i$. Like we did in the previous problem, for a composite object, we can break it up into two (or more) parts where each part has been replaced by a point mass of the same mass, located at the center of mass of that part.

So the center of mass of the overall (filled) object can be written as $X_{cm} = \frac{1}{M}(m_a x_a + m_b x_b)$ where M is the total mass (31.415 kg), m_a is the mass of part A (30.158 kg) located at the center of mass of part A x_a (which we do not know, so it remains a variable), m_b is the mass of part B (1.257 kg) located at its center of mass ($x = 0.50 \text{ m}$). BUT the center of mass of the (filled) object is at the origin, so $0 = \frac{1}{31.415 \text{ kg}} \times (30.158 x_a + 1.257 * 0.50)$ which becomes just algebra: $0 = 31.158 x_a + 0.6285$ or $x_a = -0.0202 \text{ m}$. Apparently the center of mass of the thing with the hole in it (object A) is about 2 cm to the left of the center of the outer disk (marked with a little X in the figure).



18. Bat-turn Revisited

In chapter 5, we examined a stunt from a Batman movie that the Mythbusters show tried to perform. The car is travelling down a street and desires to make a high-speed, right-angle turn. The car basically needs to travel along a tight circular arc taking it from one street into another, perpendicular street as shown in the top figure. Circular motion implies a radial acceleration of $a_c = v^2/r$ which from Newton's laws would require a radial force of $F = ma = mv^2/r$.

If $m = 1000 \text{ kg}$, $v = 30 \text{ m/s}$ and $r = 10 \text{ m}$, this implies a force of $F = (1000)(30)^2/10 = 90,000 \text{ N}$. Normally, cars make turns like this (at lower speeds) via the static friction between their tires and the road, but the maximum amount of static friction available would be $f_{s,max} = \mu_s F_N$. If the road is flat and horizontal (i.e. not banked), $F_N = mg = 9800 \text{ N}$ here so if we set the force we need equal to the force we have, we can determine what μ_s we'll need: $F_{need} = 90000 = F_{have} = 9800\mu_s$ which implies we'd need a μ_s of 9.2 or higher to be able to make this high speed turn. Typically, car tires have a μ_s of about 1 so it's unlikely that even the 'bat-tires' would be able to make a turn like this.

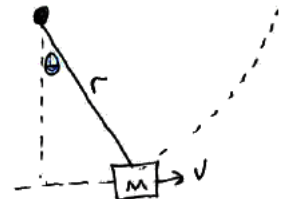
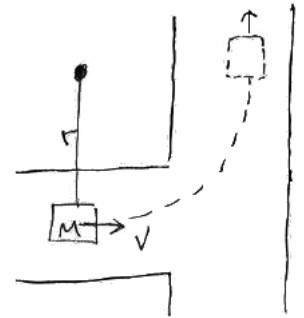
The 'solution' (in the movie) was for the car to fire a grappling hook at a fire hydrant or lamp post on the corner, and then the tension in the cable would provide the force needed to make the turn. For the Mythbusters episode, they purchased steel cable sufficient to provide more than this tension without breaking but were unsuccessful: the cable continued to break.

What went wrong here?

Realistically, the cable isn't actually taught (isn't providing any tension) when it's first fired. By the time it does go taught, the car has continued past the desired point. Then suddenly, the car's momentum switches from it's original direction to a direction implied by the circle of radius equal to the cable length. This implies that we have a sudden change in momentum that occurs very rapidly, and we should compute the force involved a different way: $\vec{F}_{avg} = \Delta\vec{p}/\Delta t$.

Once the car's direction has changed and is now moving along the circular path, the 90,000 N of tension in the cable will provide the force needed, but during the tiny Δt of the transition (the 'collision'), it may need to provide dramatically more force. Let's estimate the force needed.

Suppose the car travels 1 meter further before the cable goes taught. This implies $\sin \theta = (1 \text{ m})/(10 \text{ m})$ or $\theta = 5.74^\circ$.



The momentum vector turns by the same angle. Looking at the bottom figure, that implies that $\sin(\theta/2) = (\Delta p/2)/p$ or $\Delta p = 2p \sin(\theta/2) = (2)(1000)(30) \sin(2.2.87^\circ) = 3000 \text{ kg m/s}$.

The magnitude of the force during this sudden change in the car's direction will be $F = \Delta p/\Delta t$ so what would Δt be? How long does it take for the cable to go from 10 m long but slack, to 10 m long under tension? One way of estimating this is to use the speed of sound in the material, which is about 9000 m/s in steel. This gives us an approximate time for the 'information' about the tension to get from one end of the cable to the other, which for a 10 m long cable would be about 1 millisecond.

For that brief time interval, the cable has to provide a force of about $F = (3000 \text{ kg m/s})/(0.001 \text{ s}) = 3,000,000 \text{ N}$. That's about **thirty times** more force that the cable was designed to provide. We could achieve that using 30 cables all connecting the car to the grappling hook but the bundle of cables would be too heavy to launch.

Ultimately, this attempt was doomed from the start at the time the episode was done: there simply weren't any lightweight materials that could provide that much force without breaking. Today, carbon fibers can be more than 10 times stronger than steel and also 5 times lighter. Manufacturing such a cable might be challenging today, but the idea of doing such a turn via cable tension is certainly more plausible today. (That 3 million newton's of force on the 1000 kg vehicle represents an acceleration of 3000 m/s^2 or about $300 g/s$, so the car might not survive intact, but at least the cable would...)

