

## PH2213 : Examples from Chapter 12 : Static Equilibrium

Equilibrium represents the condition where an object is not accelerating (either linearly or angularly). It can still be moving (and rotating), it just can't be accelerating in any way. That means that Newton's laws reduce to:  $\sum \vec{F} = 0$  and  $\sum \vec{\tau} = 0$ , that is: at every point in the figure, the vector forces acting at that point must add to zero, and the vector torques must also add to zero. This generates a series of equations (usually more than we need) linking together the various forces, tensions, etc present.

(Note: For most of the problems we'll do in this chapter, nothing is moving at all, but technically equilibrium only tells us that all the accelerations are zero. We can still have motion (linear and angular) - it just has to be constant.)

The concepts outlined in this section are the basis for an entire class (**Statics**) so obviously we're just scratching the surface here, but the principles are the same and can be applied to complicated 3-dimensional structures (like a bridge or building) where we basically have a bunch of objects connected at various points. The rules are simple:

**For each object (or point) in the structure**, the sum of the forces (as vectors) acting on that object needs to be zero.

**For each point in the structure**, the sum of the torques (as vectors) acting about that point needs to be zero.

We have a lot of unknowns (the tensions or compressions in each beam, and the torques at each connection) but we have more than enough equations to solve for all of them. You end up with a potentially very large set of simultaneous linear equations, but by then you will have covered the mathematical methods for solving them, and some high-end calculators have that function built-in.

### Torque (general comments)

In full vector notation,  $\vec{\tau} = \vec{r} \times \vec{F}$  where  $\vec{F}$  is the force vector we're analyzing, and  $\vec{r}$  is a vector that points from the rotation point to the point on the object where the force is being applied. The magnitude of the cross product will be  $|\tau| = |\vec{r}| |\vec{F}| \sin \phi$  or simply  $|\tau| = rF \sin \phi$  where  $\phi$  is the angle between  $\vec{r}$  and  $\vec{F}$  when we draw them 'tail-to-tail.'

The **direction** can be found using the right hand rule.

Generally to work 'torque' problems, we need to:

- decide what point we're doing our calculations about. generally, this would be the point the object would rotate about, if it could, under the influence of the applied force(s)
- figure out where the force is being applied. the vector  $\vec{r}$  will point from the location determined in part (a) to the location where the force is being applied
- use one of the methods to compute the magnitude of the torque, given the magnitudes of the force and position vector
- using the right-hand rule, is this torque positive (would induce a counter-clockwise rotation) or negative (would induce a clockwise rotation)

## Computing Torque

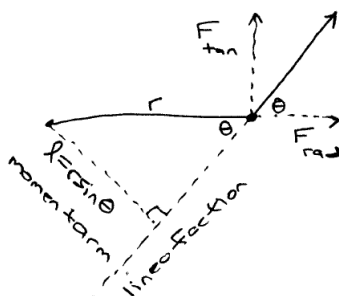
For rotation, we have a few key locations: the axis about which the rotation occurs, the point where the force is being applied, and the direction of that force.

The torque a force produces on an object will be equal to the product of:

- the distance between the axis and the point where the force is being applied
- the component of the force that is tangent to that ‘radius’

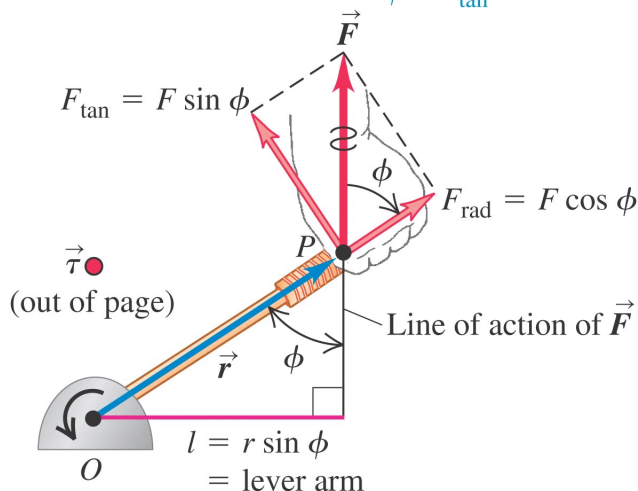
Or:  $|\tau| = |F_{tan}| |r|$

Torque is technically a vector, but if everything is confined to a plane, then the vector-ness just becomes a direction: clockwise or counter-clockwise rotation about the axis. Following the usual polar convention, counter-clockwise is considered positive and clockwise is considered negative.



Three ways to calculate torque:

$$\tau = Fl = rF \sin \phi = F_{tan}r$$



Technically, in full vector form, the torque is the vector product (i.e. the cross product) between the position vector and the force vector:  $\vec{\tau} = \vec{r} \times \vec{F}$ . Recall the magnitude of this product will be:  $|\tau| = rF \sin \phi$  where  $\phi$  is the (inner) angle between the directions of the two vectors.

If we have  $\vec{r}$  and  $\vec{F}$  as vectors, we can just apply that cross product directly to find the torque.

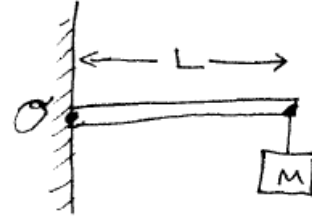
Usually, it's easier to just compute the distance from the axis to the point where the force is being applied and then multiply that by the magnitude of the force, times the sine of the angle between the  $\vec{r}$  and  $\vec{F}$  directions.

Depending on how the force is being applied, sometimes it's convenient to compute the torque differently by rearranging the terms. We can write the torque magnitude as:  $\tau = (r \sin \phi)(F)$ . Looking at the figure below, the length  $r \sin \phi$  is called the **lever arm** (or sometimes the moment arm). and

For some situations, it may be easier to pick out this ‘lever arm’ than to find  $F_{tan}$ , but we just saw they're really the same, so whatever works...

## 1. Equilibrium : Hanging Sign (Outline)

Suppose a massless rod of length  $L$  is attached to the wall at point  $O$  marked in the figure. The rod is perfectly horizontal, and a weight of mass  $M$  is hung from the end. What is the torque about the point  $O$  being created by this mass on the rod? What force and torque must be present where the rod connects to the wall?



**Forces** : The rod is just sitting there, so the sum of all the forces acting on it must add to zero. Here we pretended that the rod itself was massless, but we do have that weight hanging on the end, so we have a force of  $Mg$  downward acting on the rod.

The rod isn't moving, so where is the other force coming from that's keeping it in place? There must be some bolt or weld or duct-tape or something where the rod is connected to the wall. That connection must be providing a counter-balancing force of (in this case)  $Mg$  acting **upward** on the rod. Whatever that connection is, it has to be able to handle that much force without breaking. Taking this a step further, if that bolt or weld is exerting a force of  $Mg$  on the rod, the rod is exerting an equal and opposite force of  $Mg$  on the bolt. For example, if we hang a  $100\text{ kg}$  box on the rightmost end of this rod, there must be a  $981\text{ N}$  force at the connection that is attempting to shear off the bolt.

**Torques** : The rod is not only not moving, it's also not rotating. That means that the sum of all the **torques** on the rod have to be adding up to zero. In chapter 10, we calculated that the hanging mass is exerting a torque on the rod (about the point where the rod is attached to the wall) of  $-LMg$ . The rod isn't (angularly) accelerating, so whatever that connection is, it must be exerting a torque of  $+LMg$  on the rod. That's two constraints on our connection now: it has to be able to withstand a certain amount of shearing force, **and** it has to be able to withstand a certain amount of torque. Both of these go into the selection of what type of connection has to be made at the wall.

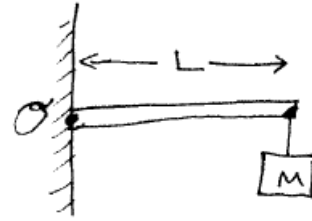
### Digression

If the rod can swing freely around the point  $O$ , then that connection can't provide any torque, so if we assume equilibrium and compute that a non-zero torque is needed at that point, then the initial assumption must be wrong and equilibrium won't be possible.

If it's solidly connected through a bolt or weld, then that type of connection can provide a torque to counter the torque being applied to it. Imagine using a wrench to tighten a bolt: you're providing some torque (a force out at a distance from the bolt, via the wrench). Once the bolt is solidly in place and can't turn anymore, you can still apply the force via the wrench, so you are still providing a torque, but the material properties of the metal in the bolt are providing the needed opposing torque. This is the **rotational analogy** to the **normal force** we encounter when a weight is sitting on a table. Gravity is still applying a force on the mass trying to accelerate is downward, but the molecules in the mass can't pass through the molecules in the table, resulting in what we call the normal force. In the case of rotation, we're continuing to apply a torque via the wrench, but the effect is that this torque is trying to tear apart the material making up the bolt. Eventually if you apply enough torque, you will break the bolt in half, but up to that point the material properties of the crystals making up the metal prevent it from breaking apart.

## 2. Computing Torque (A)

Suppose a massless rod of length  $L$  is attached to the wall at point  $O$  marked in the figure. The rod is perfectly horizontal, and a weight of mass  $M$  is hung from the end. What is the torque about the point  $O$  being created by this mass on the rod?



In this case, we have a force of  $F = Mg$  acting straight down at the end of the rod, which is located  $L$  from the axis of rotation  $O$ .  $F$  is straight down, and  $L$  is straight out to the right, so the angle between those two vectors is  $90^\circ$ . The magnitude of the torque then will be  $|\tau| = rF \sin \phi = (L)(Mg) \sin 90 = LMg$ .

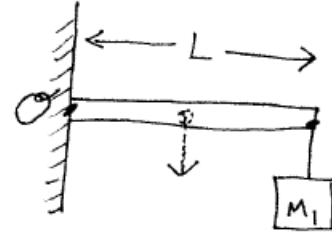
How about the direction? We have  $\vec{r}$  pointing to the right, and  $\vec{F}$  pointing down, so from the right-hand rule, the cross product  $\vec{r} \times \vec{F}$  will be pointing into the page. This would induce a clockwise rotation, so is considered to be a **negative** torque. (Remember for rotational motion, the positive direction is counter-clockwise.)

Finally then, we'd write this torque as  $\tau = -Lmg$ . If we have a  $10 \text{ kg}$  mass at the end of a  $2 \text{ m}$  long rod, the torque would be  $\tau = -(2 \text{ m})(10 \text{ kg})(9.81 \text{ m/s}^2) = -196.2 \text{ N m}$ .

Note the units of torque: it's the units of force times the units of distance. In English units, this would be 'pound-foot' (or 'foot-pound', depending on which engineering book you use).  $1 \text{ N m}$  is the same as  $0.7376 \text{ ft lb}$  so we could also say that this mass is producing a torque of  $-144.7 \text{ ft lb}$ . (Technically, force times distance is also the units for energy (joules), but don't confuse the two. Torque is not energy, it's basically a 'rotational force'.)

### 3. Computing Torque (B)

Let's modify our previous example and give the rod some mass. Suppose a rod of length  $L$  and mass  $M_2$  is attached to the wall at point  $O$  marked in the figure. The rod is perfectly horizontal, and a weight of mass  $M_1$  is hung from the end. What is the torque about the point  $O$  being created by this mass on the rod?



Make sure you've looked at the previous problem first. What is different this time? We still have the mass hanging off the end of the rod, so it's still there creating a torque of  $\tau = -LM_1g$ .

The rod itself has mass now though, and how do we handle that? You can break up the rod into tiny mass elements  $dm$  and compute the torque for each element but in the end the result is that in effect the mass acts as if it were all located at the center of mass of the rod. So in effect, we have a mass of  $M_2$  located at the midpoint of the rod. The weight of the rod is producing a force of  $M_2g$  acting at that point, which is located  $L/2$  from the point of rotation  $O$ . This produces a torque of  $|\tau| = rF \sin \phi = (L/2)(M_2g) \sin 90 = \frac{1}{2}LM_2g$ . Using the same right-hand rule arguments as before, this torque is attempting to cause the rod to rotate clockwise about the point  $O$  so this is considered to have a negative sign, so  $\tau = -\frac{1}{2}LM_2g$ .

From the previous problem, if the rod is  $2\text{ m}$  long and the hanging weight has a mass of  $10\text{ kg}$ , we found it produced a torque of  $-196.2\text{ N m}$ .

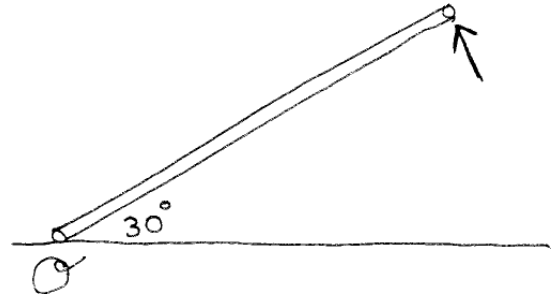
If the rod itself has a mass of  $10\text{ kg}$  also, in effect we have a mass of  $10\text{ kg}$  acting at the midpoint of the rod, so  $r = L/2 = (2\text{ m})/2 = 1.0\text{ m}$  and this creates a torque of:  $|\tau| = rF \sin \phi = (1.0\text{ m})(10\text{ kg})(9.81\text{ m/s}^2) \sin 90 = 98.1\text{ N m}$ . This torque is trying to make the rod rotate clockwise about point  $O$  so again this is a negative torque, so finally  $\tau = -98.1\text{ N m}$ .

So now we have **two** torques acting to try and rotate the rod about point  $O$ , and the total torque present about that point is  $\sum \tau = (-196.2\text{ N m}) + (-98.1\text{ N m}) = -294.3\text{ N m}$ .

(If the rod can pivot about the point  $O$ , these torques are what will produce the angular acceleration of the rod as it starts rotating clockwise about that point.)

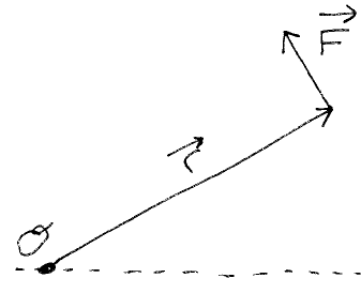
#### 4. Computing Torque (C)

We are lifting a  $50\text{ kg}$  pole that is  $4\text{ m}$  long by pushing up on one end, while the other end remains in place on the ground at the point marked  $O$ . At the point shown in the figure, the pole is making a  $30^\circ$  angle with the ground, and we're pushing with a  $250\text{ N}$  force being applied to the end of the pole, and exactly perpendicular to it. What is the net torque acting on the pole about the end still on the ground? Is this enough torque to raise the pole or will it be heading back towards the ground?

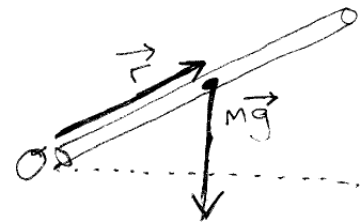


Recalling the previous problem, the pole itself is exerting a torque about  $O$  since we can think of it as a  $50\text{ kg}$  point mass with a weight of  $mg = 490.5\text{ N}$  acting straight downward at the center of gravity of the pole (i.e. the exact center). The person lifting the pole also represents a torque, so we have two torques to calculate here. If the net torque is positive, the pole will have a positive angular acceleration and will be lifting away from the ground, but if the net torque is negative, the angular acceleration will be negative (clockwise) and the pole will fall towards the ground.

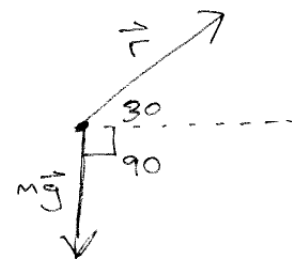
**Torque due to person :** this one is the easiest since the force is being applied perpendicular to the pole. It's a tangential force, then, which makes the angle between  $\vec{r}$  and  $\vec{F}$  exactly  $90^\circ$  so  $|\tau| = rF \sin \phi = (4\text{ m})(250\text{ N}) \sin 90 = 1000\text{ N m}$ . Looking at the right-hand rule between the directions of  $\vec{r}$  and  $\vec{F}$ , the torque will be coming up out of the plane of the page, which makes it a positive or counter-clockwise torque, so  $\tau = +1000\text{ N m}$ .



**Torque due to the weight of the pole :** here we have the weight of the pole  $F = mg = (50\text{ kg})(9.81\text{ m/s}^2) = 490.5\text{ N}$  acting downward at a point midway between the two ends of the pole, so  $|\tau| = rF \sin \phi = (2.0\text{ m})(490.5\text{ N}) \sin \phi$



What is the angle between the vector force  $\vec{F}$  and the vector  $\vec{r}$  which runs from the axis of rotation at  $O$  out to the point where the force is being applied: the midpoint of the pole? In this figure we put the two vectors tail-to-tail and see that if the pole is at an angle  $\theta$  up from the vertical that the angle  $\phi$  between the two vectors (the thing we need to compute the cross product) is  $\phi = 90 + \theta$  or here  $\phi = 90^\circ + 30^\circ = 120^\circ$ .



Thus  $|\tau| = (2.0\text{ m})(490.5\text{ N}) \sin 120^\circ = 849.6\text{ N m}$ . Looking at that lower figure and using the right-hand rule, starting in the direction of  $\vec{r}$  and curling towards the direction of  $\vec{F}$  we see that the direction of this cross product is **into** the page, which makes this a **negative** torque: i.e. a torque in the clockwise direction. You can also see this by just thinking of how the pole would move if it was just sitting there at a  $30^\circ$  angle with no other forces acting on it: it's own weight would cause it to fall, which means motion in the clockwise direction (which is the negative

direction for angular motion). So this torque technically is  $\tau = -849.6 \text{ N m}$ .

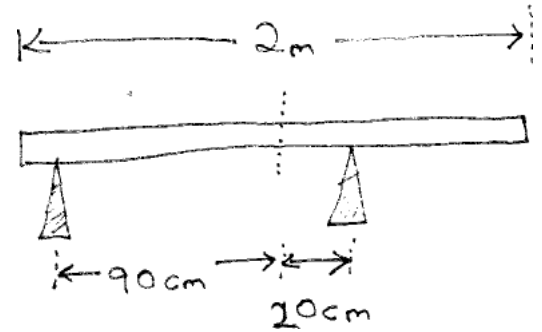
The total torque being applied about the point  $O$  then will be  $\sum \tau = (+1000 \text{ N m}) + (-849.6 \text{ N m}) = +150.4 \text{ N}$ . This is positive, which means the pole will accelerate counterclockwise and will be lifting upward. We are apparently providing more than enough force to raise the pole.

Note: to actually lift the pole vertically off the ground would require a force of at least  $W = mg = (50 \text{ kg})(9.81 \text{ m/s}^2) = 490.5 \text{ N}$  but here we're able to raise it with one end still on the ground using much less effort.

## 5. Equilibrium : Distribution of Forces

Suppose we have a  $2\text{ m}$  long board with a mass of  $10\text{ kg}$  that we are supporting on two jack-stands as shown. The left support is  $10\text{ cm}$  in from the left end of the board (i.e.  $90\text{ cm}$  to the left of the center of mass of the board); and the right support is  $20\text{ cm}$  to the right of the center of mass of the board.

How is the weight of the board being distributed between the two supports?



**Forces** : what are all the forces acting on the board? We have its weight downward and each jack-stand is exerting some amount of force upward on the board. Let  $F_1$  be the force the left support is exerting upward, and  $F_2$  be the force the right support is exerting upward on the board.

All these forces are in the Y (vertical) direction, and the board is not accelerating, so  $\sum F_y = 0$  implies that  $-Mg + F_1 + F_2 = 0$  or  $F_1 + F_2 = Mg = (10\text{ kg})(9.81\text{ m/s}^2) = 98.1\text{ N}$ .

We still have two unknowns here though: Newton's laws have told us what **total** force these two supports must be providing together to balance out the weight of the board downward, but doesn't tell us anything about how the weight is being distributed between the two supports.

**Torques** : Torque is always defined to be about some point. We have to pick a point in the figure, and then compute the torques about that point. The board is not rotating about **any** point, so the torques must add to zero, no matter what point we choose.

In practice, there are two **common choices**:

- pick some point where the object is being supported (like the point where the left jack-stand is touching the board)
- use the center of mass of the object

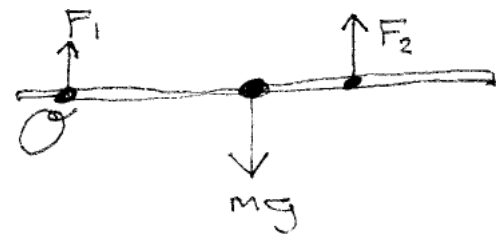
Any point will do, you just have to use that point in all your calculations for the torque on the board. Choosing a point right where a force is being applied means the torque caused by that specific force is zero (since  $r$  will be zero for that force).

**Torques (a)** : Here we pick an axis that is located where the left-side jack-stand is touching the board. The force due to the weight of the board will be acting at the center of mass, which is  $90\text{ cm}$  from the pivot. The force due to  $F_2$  is acting at a point that is  $90 + 20 = 110\text{ cm}$  from the pivot point.

We have two torques to compute then:  $|\tau| = rF \sin \phi$

(a) The torque due to the weight of the board: here we have a force of  $Mg = 98.1\text{ N}$  downward, at a point that is  $90\text{ cm}$  from  $O$ . The angle between  $\vec{r}$  and  $\vec{F}$  is  $90^\circ$ . If this were the only torque present, the board would rotate clockwise, so this is a negative torque. Finally:  $\tau = -(0.9\text{ m})(98.1\text{ N}) = -88.29\text{ N m}$

(b) The torque due to the force  $F_2$ : this force has a magnitude of  $F_2$  (an unknown) and is being applied at a location that is  $90 + 20 = 110\text{ cm}$  away from the pivot at  $O$ , so  $r = 1.1\text{ m}$ . That torque alone would cause the board to rotate counterclockwise, which makes it positive. Finally  $\tau = +(1.1\text{ m})(F_2)$



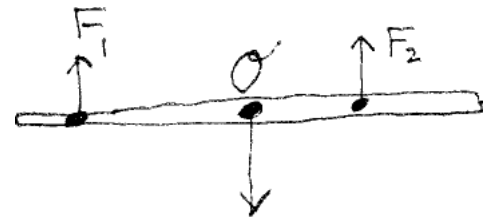
The sum of these torques must be zero, so:  $-88.29 + 1.1F_2 = 0$  or  $F_2 = 80.26 \text{ N}$ .

From earlier,  $F_1 + F_2 = 98.1 \text{ N}$  so  $F_1 = 98.1 - 80.26 = 17.84 \text{ N}$

**Torques (b)** : Let's redo the problem, but choose an axis that is located at the center of mass of the board.

The weight of the board acts as if it were located at the center of mass, which is right where we've picked our point  $O$ , so the weight is creating no torque now.

The force  $F_1$  is located  $90 \text{ cm}$  from point  $O$ , and the force  $F_2$  is located  $20 \text{ cm}$  from point  $O$ .



We have two torques to compute then:  $|\tau| = rF \sin \phi$

(a) The torque due to  $F_1$  : the magnitude of the force is  $F_1$ , the position is  $r = 0.9 \text{ m}$ , and the angle between  $\vec{r}$  and  $\vec{F}$  is  $90^\circ$ . This torque alone would cause the board to rotate counterclockwise, so it is positive. Finally:  $\tau = +(0.9 \text{ m})F_1$ .

(b) The torque due to the force  $F_2$ : this force has a magnitude of  $F_2$  and is being applied at a location that is  $20 \text{ cm}$  away from the pivot at  $O$ , so  $r = 0.2 \text{ m}$ . That torque alone would cause the board to rotate clockwise, which makes it negative. Finally  $\tau = -(0.2 \text{ m})(F_2)$

The sum of these torques must be zero, so:  $0.9F_1 - 0.2F_2 = 0$

From earlier,  $F_1 + F_2 = 98.1 \text{ N}$  so we have two equations and two unknowns. We can rearrange the first equation into the form:  $0.2F_2 = 0.9F_1$  or multiplying by 5:  $F_2 = 4.5F_1$ .

Substituting this into  $F_1 + F_2 = 98.1$  leads to  $F_1 + 4.5F_1 = 98.1$  or  $5.5F_1 = 98.1$  and finally  $F_1 = 17.84$  and  $F_2 = 98.1 - F_1 = 80.26 \text{ N}$ .

### Summary

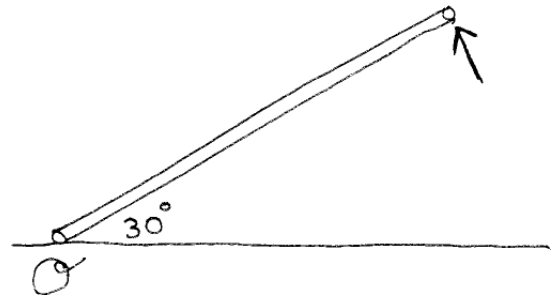
That's the same result we got before using a different pivot point  $O$ . The choice of that point is up to you: you could literally choose any point in the figure but in practice the solution is simpler if you choose a point at the center of mass, or at one of the contact points.

Also: we have two unknowns here, the forces that each of the supports are exerting on the board. We need two equations to solve that. The ones we chose above were (a) the fact that the sum of all the forces in the Y direction had to be zero, and (b) the sum of the torques about some point had to be zero. We could instead also solve this by taking two torque equations: the sum of the torques about the left pivot must be zero, and the sum of the torques about the right pivot must be zero. Try that. It leads to a slightly more complicated set of two equations and two unknowns, but still leads to the same solution.

## 6. Equilibrium : Lifting Pole Revisited

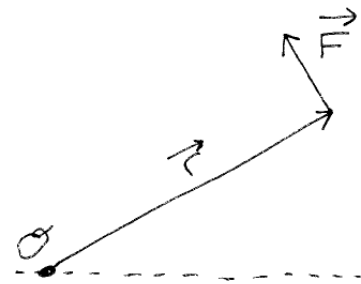
Suppose we have a long pole laying on the ground and we start lifting it from one end. We desire to lift it with a constant angular speed. How much force does the person have to provide, and how does this force change with angle as we lift the pole?

We are lifting a  $50\text{ kg}$  pole that is  $4\text{ m}$  long by pushing up on one end, while the other end remains in place on the ground at the point marked  $O$ . At the point shown in the figure, the pole is making a  $30^\circ$  angle with the ground, and we're pushing with some force  $F$  being applied to the end of the pole, and exactly perpendicular to it. What force must the person apply to cause the pole to move with a constant angular speed?



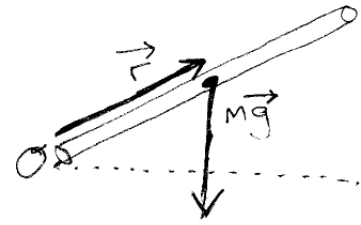
Recalling the previous version of this problem, the pole itself is exerting a torque about  $O$  since we can think of it as a  $50\text{ kg}$  point mass with a weight of  $mg = 490.5\text{ N}$  acting straight downward at the center of gravity of the pole (i.e. the exact center). The person lifting the pole also represents a torque, so we have two torques to calculate here. If the net torque is positive, the pole will have a positive angular acceleration and will be lifting away from the ground, but if the net torque is negative, the angular acceleration will be negative (clockwise) and the pole will fall towards the ground. In this case we want the net torque to be **zero** so that the pole just moves at a constant angular speed and does **not** accelerate.

**Torque due to person :** this one is the easiest since the force is being applied perpendicular to the pole. It's a tangential force, then, which makes the angle between  $\vec{r}$  and  $\vec{F}$  exactly  $90^\circ$  so  $|\tau| = rF \sin \phi = (4\text{ m})(F) \sin 90 = 4F$  (We'll make sure everything is in standard metric units and just drop them. The units for the torque will be newton-meters.) Looking at the right-hand rule between the directions of  $\vec{r}$  and  $\vec{F}$ , the torque will be coming up out of the plane of the page, which makes it a positive or counter-clockwise torque, so  $\tau = +4F$ .

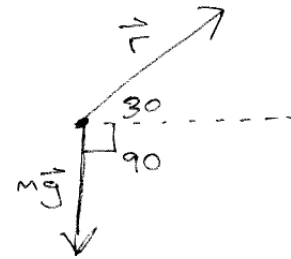


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**Torque due to the weight of the pole :** here we have the weight of the pole  $W = mg = (50 \text{ kg})(9.81 \text{ m/s}^2) = 490.5 \text{ N}$  acting downward at a point midway between the two ends of the pole, so  $|\tau| = rF \sin \phi = (2.0 \text{ m})(490.5 \text{ N}) \sin \phi$



What is the angle between the vector force  $\vec{F}$  and the vector  $\vec{r}$  which runs from the axis of rotation at  $O$  out to the point where the force is being applied: the midpoint of the pole? In this figure we put the two vectors tail-to-tail and see that if the pole is at an angle  $\theta$  up from the vertical that the angle  $\phi$  between the two vectors (the thing we need to compute the cross product) is  $\phi = 90 + \theta$  or here  $\phi = 90^\circ + 30^\circ = 120^\circ$ , but take note of the boxed equation since we're going to use it later to find the force the person must push with as the pole changes its angle with the ground.



Thus  $|\tau| = (2.0 \text{ m})(490.5 \text{ N}) \sin 120^\circ = 849.6 \text{ N m}$ . Looking at that lower figure and using the right-hand rule, starting in the direction of  $\vec{r}$  and curling towards the direction of  $\vec{F}$  we see that the direction of this cross product is **into** the page, which makes this a **negative** torque: i.e. a torque in the clockwise direction. You can also see this by just thinking of how the pole would move if it was just sitting there at a  $30^\circ$  angle with no other forces acting on it: it's own weight would cause it to fall, which means motion in the clockwise direction (which is the negative direction). So this torque technically is  $\tau = -849.6 \text{ N m}$ .

The total torque being applied about the point  $O$  then will be  $\sum \tau = (+4F) + (-849.6)$ . This is where this problem differs from the previous version since this time we want the angular acceleration to be zero, which means the sum of the torques must be zero. The force needed then will be  $0 = 4F - 849.6$  or  $F = 212.4 \text{ N}$ .

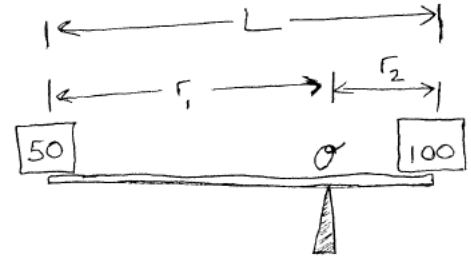
**Other Angles :** Does the person need to exert this force always as the pole rises from the ground to the vertical direction? Going back to our computations of the two torques involves, the person's torque was  $\tau_{person} = +(4 \text{ m})(F) \sin 90 = +4F$ . The torque from the weight of the pole was  $\tau_{weight} = -(2 \text{ m})(490.5 \text{ N}) \sin \phi = -981 \sin \phi$  but  $\phi = 90 + \theta$  and from trig,  $\sin(90 + \theta) = \cos \theta$  so we can write the torque caused by the weight of the pole as:  $\tau_{weight} = -981 \cos \theta$ .

The sum of these two torques must be zero to raise the pole at a constant angular speed, so:  $4F - 981 \cos \theta = 0$  or  $F = 245.25 \cos \theta$ .

What is this telling us? At the beginning of this effort, the pole is laying flat on the ground, so  $\theta = 0$ . At that position, the person needs to push (or pull) on the end the pole with a force of  $F = 245.24 \cos 0 = 245.24 \text{ N}$  (which is exactly half the weight of the pole). As the pole rises, the angle increases which means that the cosine terms is getting smaller, so **less** force is needed. As the pole gets near the vertical direction, the angle  $\theta$  is moving towards  $90^\circ$  but  $\cos 90 = 0$  so less and less force is needed as the pole rises, eventually dropping to zero when the pole is perfectly vertical.

## 7. Equilibrium : Off-Center Seesaw (A)

Let's look at another situation where we have multiple torque's present. We'll start with a massless board, on which we have placed two boxes: 50 kg at one end, and 100 kg at the other end. The board is 2 m long and it's sitting on a little triangular wedge (a 'fulcrum') on the ground. Think of this like a see-saw but where it's being balanced somewhere off-center. We want to determine how to place the board on this fulcrum so that it just sits there and doesn't rotate either way.



Let's do this one symbolically. We'll have a mass  $M_1$  on the left side, and  $M_2$  on the right side. The distance from the fulcrum (the balance point) to the left mass will be  $r_1$  and the distance from the fulcrum to the mass on the right side will be  $r_2$ . (Clearly  $r_1 + r_2 = L$ , the overall length of the board.)

Torque due to  $M_2$  : at this point, we have a force of  $M_2g$  acting downward, so  $|\tau_2| = rF \sin \phi$  becomes  $|\tau_2| = r_2 M_2 g \sin 90 = r_2 M_2 g$ . What sign should this torque have? You could use the right hand rule, but it's probably simpler to just visualize what would happen if this were the only torque present: that end of the board would fall downward, causing the board to be rotating clockwise, but that's the negative direction, so  $\tau_2 = -r_2 M_2 g$ .

Torque due to  $M_1$  : basically same discussion as above but this torque would cause the board to spin counterclockwise so  $\tau_1 = +r_1 M_1 g$ .

There is another force acting on the board: the force the fulcrum is exerting upward on it, but this force is occurring right at the point of rotation.  $|\tau| = rF \sin \phi$  where  $r$  is the distance from the rotation point to where the force is being applied, but in the case of the fulcrum's upward force on the board,  $r = 0$  so it's not contributing anything to the torque.

Finally we want the board to just sit there without rotating, so we need  $\sum \tau = 0$  which implies that  $\sum \tau = r_1 M_1 g - r_2 M_2 g = 0$  or finally  $r_1 M_1 = r_2 M_2$ . If  $M_2$  is the heavier box (which it is in this case), then  $r_2$  needs to be smaller than  $r_1$  : that is, the balance point is closer to the heavier mass, and the exact location can be found simply since the product of  $m$  and  $r$  has to be the same for each box.

For our specific example,  $M_1 = 50 \text{ kg}$ ,  $M_2 = 100 \text{ kg}$  and the overall length of the board is  $r_1 + r_2 = L = 2 \text{ m}$ .

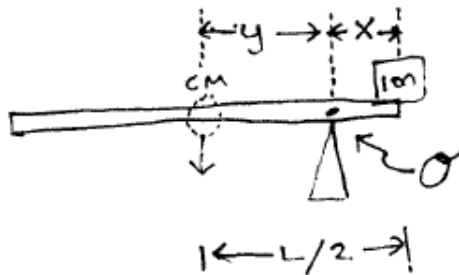
This gives us two equations and two unknowns:

$(r_1)(50) - (r_2)(100) = 0$  and  $r_1 + r_2 = 2$ . We can rewrite the first equation as  $r_1 = \frac{100}{50} r_2$  or  $r_1 = 2r_2$ . Substituting this expression for  $r_1$  into the second equations leads to:  $(2r_2) + r_2 = 2$  or finally  $r_2 = \frac{2}{3} \text{ m}$ , making  $r_1 = 2r_2 = \frac{4}{3} \text{ m}$ .

For completeness, we can look at the other half of equilibrium: the fact that the sum of the forces has to be zero also. Specifically here,  $\sum F_i = 0$  in the Y (vertical) direction. (There aren't any forces in other directions, so Y is the only direction we need to think about.) Here we have the weight of the two boxes acting downward **on the board**, so the fulcrum must be exerting a force **upward** on the board that's just enough to cancel those other forces out:  $\sum F_y = 0 = -(50 \text{ kg})(9.81 \text{ m/s}^2) - (100 \text{ kg})(9.81 \text{ m/s}^2) + F$  or  $F = 1471.5 \text{ N}$ . From Newton's third law, the board will be exerting an equal and opposite force downward on the fulcrum so it will need to be designed to handle that much force.

### 8. Equilibrium : Off-Center Seesaw (B)

Let's modify our seesaw example and say that the board has a mass of  $20\text{ kg}$  and is  $2\text{ m}$  long. We've placed the  $100\text{ kg}$  mass on one end and now we want to figure out where the balance point is when we **do not** have a mass on the other end of the board.



The weight on the right end of the board is creating some torque but now that the board has weight, it can exert a torque about the pivot point as well. We just have to set things up here so that those two torques can cancel each other out.

Let  $x$  be the distance from the pivot (fulcrum) to the mass  $M$ .

The mass  $M$  by itself will be causing a clockwise torque, so  $|\tau| = rF \sin \phi$  will become  $\tau = -xMg$  (It's weight is straight down which is at right angles to the  $\vec{r}$  vector from the pivot to the point where the force is acting, so the  $\sin \phi$  term became  $\sin 90 = 1$ .)

As long as the center of mass of the board is over on the left side of the pivot, it will be trying to cause the board to rotate counterclockwise, which makes this torque into a positive value. Let's use a lowercase  $m$  to represent the mass of the board. Then it's exerting a force of  $mg$  at a distance  $y$  from the pivot. We just argued this torque will be positive, so  $|\tau| = rF \sin \phi$  becomes  $\tau = +mgy$ . (Note: this  $y$  has nothing to do with the vertical coordinate axis; it's just the variable we're using to represent the distance from the center of mass of the board to the pivot.)

The sum of these two torques has to be zero, or the board will rotate, so  $\sum \tau = 0$  becomes  $\boxed{-Mgx + mgy = 0}$ .

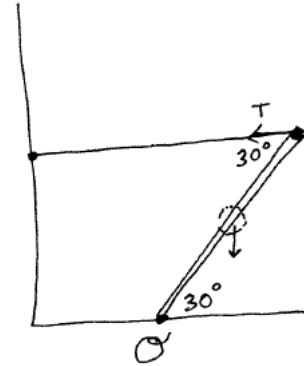
How can we relate  $x$  and  $y$  so we don't have to deal with two variables? The board has an overall length of  $L$ . The center of mass will be right at the midpoint of the board (marked as  $L/2$  in the figure). But now we see that  $x + y = L/2$  so  $y = \frac{L}{2} - x$ . Substituting that into our torque-balance equation:  $-Mgx + mg(\frac{L}{2} - x) = 0$  or  $-Mgx + \frac{1}{2}mgL - mgx = 0$ . Moving the two negative terms to the other side of the equation:  $\frac{1}{2}mgL = mgx + Mgx = (m + M)gx$ . Dividing out the common value of  $g$  and solving for  $x$ :  $\boxed{x = \frac{1}{2}L \frac{m}{m+M}}$ .

For our specific problem,  $m = 20\text{ kg}$ ,  $M = 100\text{ kg}$ , and  $L = 2\text{ m}$  so  $x = (0.5)(2.0) \frac{20}{20+100} = 0.167\text{ m}$ .

The point where things will balance is very close to the edge of the board where the heavy box is sitting.

### 9. Equilibrium : Tension Revisited

Suppose we have a 5 m long 100 kg pole that is leaning over at a 30° angle from the horizontal. We're holding it in place using a (horizontal) cable that runs from the top of the pole to a nearby building. How much tension is in the cable? What are the x and y components of the force that the bottom end of the pole is exerting on the ground?



The obvious pivot point to use in our torque calculations is the point where the pole is in contact with the ground. If that cable breaks, the pole will start falling, rotating about the point on the ground.

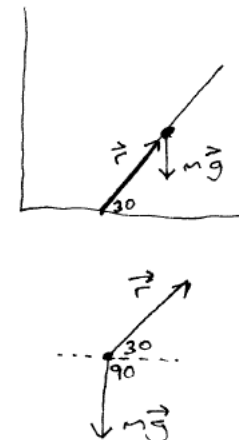
What are all the forces acting on the pole? (a) the tension in the cable, (b) the weight of the pole, (c) some force the ground is exerting to keep the pole from sliding sideways or pushing through the ground.

Since we've chosen point  $O$  to be our pivot point, the 'ground force' will not be exerting any torque about this axis (since  $r = 0$  for this force). The cable and weight of the pole are generating torques about that point though.

**Torque due to tension in the cable :** The left figure shows the  $\vec{r}$  and  $\vec{F}$  we're addressing here. The right figure shows how to determine the angle. Propagating the 30 deg angle around, we see that the angle between the  $r$  and  $F$  vectors is  $90 + 60 = 150$  deg. Also, the directions here are such that the tension in the cable is trying to cause the pole to rotate counterclockwise, which makes  $\tau$  positive. Finally:  $|\tau| = rF \sin \phi$  becomes  $\tau = +(5\text{ m})(T) \sin 150 = +2.5T$ .



**Torque due to weight of pole :** as in the previous examples, the weight acts as if it were concentrated at the center of mass, so we have the  $mg = (100\text{ kg})(9.81\text{ m/s}^2) = 981\text{ N}$  weight acting downward at a distance of  $L/2$  or  $2.5\text{ m}$  from the pivot point. Looking at the lower figure, we see the angle between  $\vec{r}$  and  $\vec{F}$  in this case is  $120^\circ$ . Also, this force wants to cause the pole to rotate clockwise, which makes it a negative torque. Finally:  $|\tau| = rF \sin \phi$  becomes  $\tau = -(2.5\text{ m})(981\text{ N}) \sin 120 = -2124\text{ N m}$ .



The sum of the torques has to be zero so the pole doesn't rotate, so:  $\sum \tau = 0$  becomes  $0 = 2.5T - 2124$  or  $T = 850\text{ N}$ .

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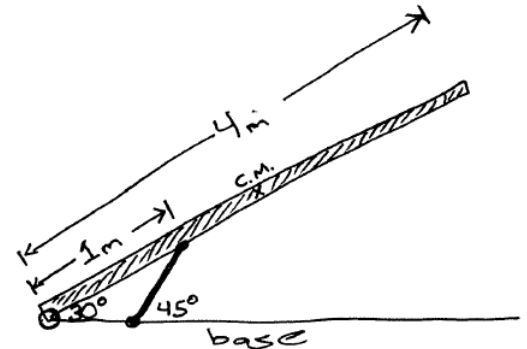
## Ground Force

The sum of all the (vector) forces acting on the pole has to add to zero. We have the tension of  $850\text{ N}$  acting to the left, the weight of  $981\text{ N}$  acting downward, and some force being exerted by the ground.  $\sum \vec{F}_i = 0$  becomes:  $-850\hat{i} - 981\hat{j} + \vec{F} = 0$  or  $\vec{F} = +850\hat{i} + 981\hat{j}$ . This is the force the **ground** is exerting on the **pole**. The original problem asked for the force the **pole** is exerting on the **ground** but by Newton's third law, that force would be the exact negative of what we just calculated.

Let's look at that ground force for a minute. The vertical component is basically the normal force between the pole and the ground: the ground is pushing upward with  $981\text{ N}$  to keep the pole from moving into the ground. The horizontal component of  $850\text{ N}$  (to the right) might be the force of static friction that's keeping the pole from sliding to the left.  $f_{s,max} = \mu_s n$  though, so here that means the force of static friction is close to the normal force, which implies that  $\mu_s$  has to be fairly large (here about 0.87) to accomplish this.

10. **Equilibrium : Bucket-Lift / Cherry-Picker (\*)** Last year, Hilbun was retrofitted to install a fire-escape stairwell. The workers used a bucket-lift to do some of this work. Here we look at the forces involved in a simplified version of this device. (Note: too complicated for a test or final, but a hint of what you'll see in Statics class.)

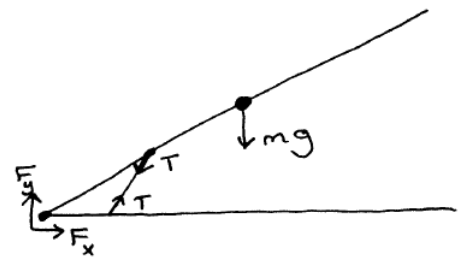
We'll model the bucket-lift as a long arm that extends from the base to the desired work location. The weight of the arm is creating a large torque, so a counter force in the form a hydraulic ram is present. Looking at the bucket-lift in action, let's say the arm has a length of  $4\text{ m}$  and a mass of  $500\text{ kg}$ . The hydraulic ram we'll model as a rod as shown in the figure. It is connected to the long arm about  $1\text{ meter}$  from the axis  $O$  shown in the figure.



Assume all the connection points in the figure are pivots - meaning that the parts can swivel there (i.e. they're not welded or connected in a way that will create a torque just due to the connection itself: we leave that extra complication to your Statics class).

**What are the forces acting on the arm**, so that we can compute the torques present? We have:

- the weight of the arm, acting at the center of mass of the arm
- some force where the arm is attached to the base, whose components we'll call  $F_x$  and  $F_y$
- the force that the hydraulic ram is exerting to keep the arm from falling.



When we have complicated structures with multiple elements, we usually don't know up front whether each element will have positive or negative tensions within, so we usually start the problem off by assuming all those forces are tensions, like we did with the earlier problems with lamps suspended by multiple wires. By calling this a tension, that means the force acts inward, as if it were a rubber-band. We will set up all the equations assuming the force is in that direction and once we solve them, if we find that one of our tensions is a negative number, well that just means that element was actually under compression and is 'pushing back' outward.

For this particular problem, we can be pretty certain that the force this little element is exerting is outward - it is pushing the arm up to counteract the force of gravity pulling it down - but we'll stick with the convention and let the final equations confirm this.

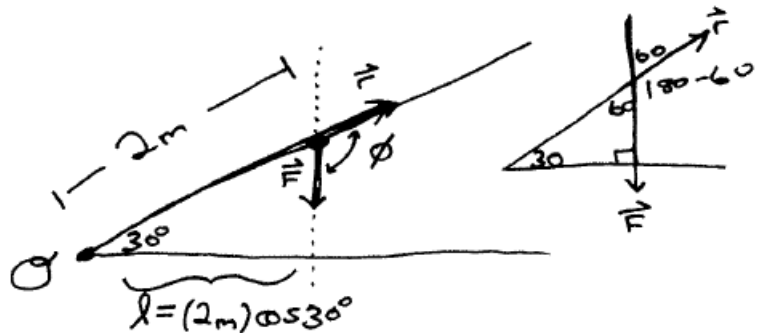
$\sum \vec{\tau} = 0$  about any axis we want to choose. The common practice is to select a 'rotation point' that is either the center of mass of the object, or a point where one or more of the forces is acting. If we choose the point on the left where the arm is attached to the base, that means the **torque** created by  $F_x$  and  $F_y$  is zero (because that force is occurring right at that point, so in our  $|\tau| = rF \sin \phi$  calculation,  $r = 0$  right there).

Choosing our rotation point to be right there essentially let's us ignore those two unknowns, leaving us just the single unknown  $T$  (tension in the supporting rod).

Let's compute the torques about that point now:

### Torque due to the arm's weight

$|\tau| = rF \sin \phi$  is one way we can calculate it. The weight acts at the center of mass of the 4 meter long arm, so  $r = 2.0 \text{ m}$ . The force we are analyzing here is the weight of the arm, so  $F = mg = (500 \text{ kg})(9.8 \text{ m/s}^2) = 4900 \text{ N}$ . What is the angle between those two vectors though?

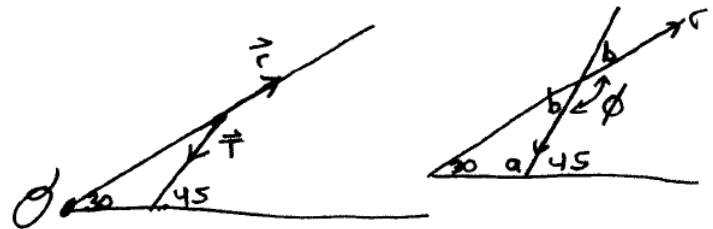


We'll need to propagate angles around in the figure to find out. The figure on the right shows that  $\phi = 120^\circ$  so finally  $|\tau| = (2 \text{ m})(4900 \text{ N}) \sin 120 = 8487 \text{ N m}$ . Using the RHR or just looking at the force and where it is being applied, this force will cause the arm to rotate clockwise, so this is a negative torque:  $\tau = -8487 \text{ N m}$ .

**Note** : Another option is  $|\tau| = Fl$  where  $l$  is the lever-arm, which is actually easy to find here. Putting a dotted line through  $\vec{F}$  creates the **line of action** and a line drawn from  $O$  perpendicular to that line produces the **lever arm** which here is clearly  $l = (2.0 \text{ m}) \cos 30 = 1.732 \text{ m}$  so  $|\tau| = (4900 \text{ N})(1.732 \text{ m}) = 8287 \text{ N m}$  (same as before).

### Torque due to the hydraulic ram

$|\tau| = rF \sin \phi$  is one way we can calculate it. Here, the force is  $\vec{T}$  which we're (by convention) assuming is in the direction shown. It is being applied 1 m from the rotation point we chose, so  $r = 1 \text{ m}$ . What is the angle between the directions of  $\vec{r}$  and  $\vec{F}$  though?



Looking at the right half of the figure,  $a + 45 = 180$  so  $a = 180 - 45 = 135^\circ$ . The angles in a triangle add to 180, so  $30 + a + b = 180$  or  $b = 180 - 30 - 135 = 15^\circ$ . Looking at where the two lines cross:  $b + \phi = 180$  so we can finally find that  $\phi = 180 - 15 = 165^\circ$ . Whew.

Finally then,  $|\tau| = rF \sin \phi = (1.0 \text{ m})(T) \sin 165 = 0.2588T$ . How about the sign? We have to be careful here since remember we've defined this vector to be pointing out from the arm, so this is again a negative torque:  $\tau = -0.2588T$ .

**Putting all this together finally** :  $\sum \tau = 0$  becomes  $-8487 - 0.2588T = 0$  or  $T = -8487/0.2588 = -32,790 \text{ N}$ .

That confirms our initial guess that this force is not really a tension at all, but a compression: this little ram is pushing outward, not pulling inward.

Note the magnitude of the force the ram is having to exert. The weight of the arm was  $4900 \text{ N}$  but this little support element is having to exert a force that is 6.7 times larger.

**Summary** : there are a lot of steps here (especially the propagation of angles), so this is probably too lengthy to be on the final, but is a good illustration of the application of  $\sum \vec{\tau} = 0$  to solve such a problem, and gives a hint of how to approach such problems in Statics.