

PH2213 : Examples from Chapter 7 : Work and Kinetic Energy

Key Concepts

In the previous chapters, we analyzed motion using various equations of motion (derived from calculus under the assumption that the acceleration is constant) and Newton's Laws. Some of these could be difficult to solve, and often involved careful use of concepts from trigonometry to pull out the needed vector components. They were also limited to constant forces and accelerations.

In this chapter, we morph those earlier equations into a different form, introducing the concept of work and potential energy, which are scalars not vectors, and which allow us to solve many of the same problems more easily, and usually with somewhat less trig. The methods in this chapter also allow us to solve some problems where the acceleration is **not** constant (such as the pendulum, for example).

Key Equations

The **work** done by a force \vec{F} as an object moves from one location to another (represented by a displacement vector of \vec{d}) is defined to be $W = \vec{F} \cdot \vec{d}$. Work can be either positive or negative, depending on the angle between the force and displacement vectors. One definition of dot product that is frequently used in work calculations: $W = |F| |d| \cos \phi$ where ϕ is the angle between the force and displacement.

The **kinetic energy** of an object of mass m moving at a speed v is $K = \frac{1}{2}mv^2$

Work-Energy Theorem : $K_2 = K_1 + \sum W_i$: the initial kinetic energy of an object, plus all the works done on the object by all the forces present, gives us the final kinetic energy of the object.

Varying forces : if a force is present that depends on the position (like a spring), we can still compute the work done by that force as an object moves from position a to position b: $W = \int_a^b F(x)dx$ for example.

That is basically the area under the $F(x)$ vs x curve, which you can still compute without knowing calculus in the case of simple forces (such as springs).

Common Errors

- trig : getting angles for the dot products involved in $\vec{F} \cdot \vec{d} = Fd \cos \phi$. ϕ is the angle between the directions of the two vectors involved; F and d are the **magnitudes** of the two vectors (and are therefore **positive**). Any overall sign comes from the $\cos \phi$ part of that expression.
- work can be negative (work done by kinetic friction is always negative)
- force and work are connected but separate (not even the same units). Force is a vector (and has components); work is a scalar (no components)
- Repeating the previous one: energy is a scalar, not a vector. Work and kinetic energy do not themselves have components.

1. Sliding Block : No friction

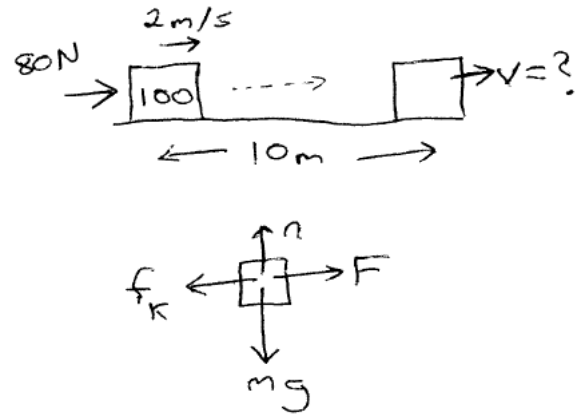
Suppose we have a 100 kg crate being moved across a frictionless floor. It is initially moving to the right at 2 m/s and a person behind it is pushing horizontally with a constant force of 80 N. How fast will the crate be moving after travelling a distance of 10 m?

The block is moving horizontally to the right, so let's call that our +X coordinate direction. The +Y axis will be pointing vertically upward.

This is fairly easy to do using Newton's methods combined with equations of motions from earlier chapters, but here we'll use work and energy, which says that $K_2 = K_1 + \sum W$: if we can account for all the works done on the object, we can add those to its initial kinetic energy to find its final kinetic energy (from which we can find the speed since $K = \frac{1}{2}mv^2$).

What are all the forces present on the block?

- We have the person's pushing force of 80 N to the right
- gravity (the weight of the block) downward
- a normal force (keeping the block from passing through the floor)
- friction (well, not yet - this example has $\mu_k = 0$)



The work done by a force \vec{F} as an object displaces by \vec{d} is $W = \vec{F} \cdot \vec{d}$.

Here, the box is displacing to the right, horizontally. The force of gravity is vertically downward, and the normal force is perpendicular to the surface, so it is vertically upward here. Both of those forces are perpendicular to the displacement vector \vec{d} so neither of them does any work on the crate.

The person's force is 80 N to the right, as the block displaces 10 m to the right: the person did work of $W = \vec{F} \cdot \vec{d} = (80 \text{ N})(10 \text{ m}) \cos \phi$ where ϕ is the angle between the vector force and the vector displacement. Here, those two vectors are in the same direction (both pointing exactly to the right) so $\phi = 0$ and $\cos 0 = 1.0$ so the work done is $W = 800 \text{ J}$.

(If we had friction, which we'll add in the next example, we'd have to compute how much work it did, but $\mu_k = 0$ at this point, so we're done.)

The initial and final kinetic energies are related by: $K_2 = K_1 + \sum W$. The initial kinetic energy was $K_1 = \frac{1}{2}mv^2 = \frac{1}{2}(100 \text{ kg})(2 \text{ m/s})^2 = 200 \text{ J}$ so the final kinetic energy will be $K_2 = 200\text{J} + 800\text{J} = 1000\text{J}$.

$K = \frac{1}{2}mv^2$ so we can relate that to the speed of the block at this point: $1000\text{J} = \frac{1}{2}(100 \text{ kg})v^2$ or $1000 = 50v^2$ and finally $v = 4.47 \text{ m/s}$.

2. Sliding Block : With Friction

(Refer to the figure in the previous problem.)

Here we have the identical problem, but we'll add friction between the block and the floor, with $\mu_k = 0.10$.

What changes? Our work-energy equation remains the same, and the person is still doing the same amount of work as before, but now we have an additional term in our $\sum W$ calculation: the work that friction did on the block as it underwent the given displacement.

The magnitude of the frictional force is given by $f_k = \mu_k n$, and its direction is always to oppose motion, so the vector frictional force here will be acting to the **left** as the box slides to the right. That means that the work done by friction, $W = \vec{f}_k \cdot \vec{d}$ will be $W = f_k d \cos \phi$ and now the angle between the two vectors is 180° , so the cosine term becomes $\cos 180^\circ = -1$. The work done by friction here is $W = -f_k d$. We have the displacement (10 m) but we need to find the magnitude of the frictional force, which means we'll need to find the normal force.

$\sum F_y = 0$ since the box isn't accelerating (or moving at all) in the Y direction. Looking at all the Y components of the forces present: friction is entirely to the left, the person's force is entirely to the right, so neither of those have any Y component. All we have is n upward and mg downward, so $\sum F_y = 0$ becomes $n - mg = 0$ or $n = mg = (100 \text{ kg})(9.81 \text{ m/s}^2) = 981 \text{ N}$. The magnitude of the force of friction then is $f_k = \mu_k n = (0.10)(981 \text{ N}) = 98.1 \text{ N}$. The **work** that friction did is $W = -f_k d = -(98.1 \text{ N})(10 \text{ m}) = -981 \text{ J}$.

Note : the frictional force is always in the direction opposite the motion, so always does **negative** work on an object, slowing it down.

Our overall work-energy equation then is: $K_2 = K_1 + \sum W$. We started with 200 J of kinetic energy (see previous problem), and the person did work of +800 J and we just found that friction did work of -981 J so substituting in these values: $K_2 = (200 \text{ J}) + (800 \text{ J}) + (-981 \text{ J}) = 19 \text{ J}$.

Converting this kinetic energy into the speed of the crate: $K = \frac{1}{2}mv^2$ so $19 = \frac{1}{2}(100)v^2$ or finally $v = 0.62 \text{ m/s}$

Warning : Kinetic Energy can **never** be negative, since it is defined as $K = \frac{1}{2}mv^2$. In this problem, note that friction removed more energy than the person put into the block. If the block had not already been moving (with that initial kinetic energy of 200 J) we would have ended up with a negative value for K_2 , which is **not** possible. This means one of two things: either the initial statement of the problem is wrong (the block cannot have moved the 10 meters that was claimed), or somewhere along the line you made a math mistake. **Never** ignore this type of situation. If you end up with a negative kinetic energy, something is wrong somewhere!

3. Sliding Block : Stopping Distance

Let's change things up a bit. Here, let's say we have a 100 *kg* block moving to the right at 2.0 *m/s* on a floor with $\mu_k = 0.1$ and we **do not** push it. Basically we have the **same figure** as in the previous two examples, **except** we have removed the person pushing on the block.

We want to find how far it will slide before coming to a stop. We have an initial kinetic energy of $K_1 = \frac{1}{2}mv^2 = \frac{1}{2}(100)(2)^2 = 200 \text{ J}$. Friction will be doing negative work as the block slides, removing energy from it. Eventually it comes to a stop, but that means we're looking for the point where $K_2 = 0$. (Stopped = not moving = no kinetic energy.)

In the previous example, we saw that the work done by friction is $W = -f_k d$ and we found that the force of friction was $f_k = 98.1 \text{ N}$.

So work-energy tells us that $K_2 = K_1 + \sum W$ and the only force doing work here is friction, so this becomes: $0 = (200 \text{ J}) + (-f_k d)$ or $f_k d = 200$ or $(98.1)(d) = 200$ and finally $d = 200/98.1 = 2.04 \text{ m}$

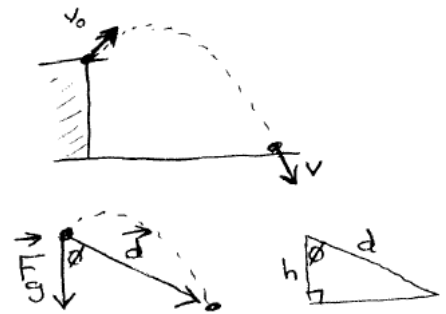
4. Object Thrown at an Angle

From the roof of Hilbun (15 m above the ground below), we throw a ball with an initial speed of 20 m/s at an angle of 30° above the horizontal.

- How fast is it moving at the instant just before it hits the ground?
- Can we say anything about the **angle** it hits the ground with?
- How far did it travel laterally?

We did this problem earlier using 2D equations of motion, which got pretty long and involved solving a quadratic equation. We can bypass all those intermediate steps using work-energy (and in the next chapter we'll see how we can make it even simpler using conservation of energy).

Let's label position 0 as the initial point when we've tossed the ball into the air, starting its trajectory, and position 1 is the instant just before it hits the ground. Then $K_1 = K_0 + \sum W$. The only force doing work here is gravity, $F_g = mg$ acting straight downward. The overall displacement is from point 0 to point 1 and the work done by gravity will be $\vec{F}_g \cdot \vec{d}$ which is $|F_g| |d| \cos \phi$ where ϕ is the angle between those two vectors. Looking at the two figures on the bottom, we see that we can collect the $d \cos \phi$ terms together and note that $d \cos \phi = h$, the height of the building (15 m). We can write the work done by gravity here then as just $W_g = (mg)(h)$.



Our overall work-energy equation $K_1 = K_0 + \sum W$ then becomes: $\frac{1}{2}mv_1^2 = \frac{1}{2}mv_0^2 + mgh$. The mass cancels out, leaving us with $\frac{1}{2}v_1^2 = \frac{1}{2}v_0^2 + gh$ or multiplying the whole equation by '2': $v_1^2 = v_0^2 + 2gh$. (We've seen that before...)

With the numbers here, $v_1^2 = (20)^2 + (2)(9.81)(15) = 400 + 294.3 = 694.3$ or $|v_1| = 26.35$ m/s. This gives us the speed of the ball when it hits the ground. It doesn't tell us anything about the direction though.

Determining The Direction

We do know from earlier chapters that v_x will remain constant throughout the motion since there isn't any acceleration in the X direction. We can find that value: $v_{ox} = v_o \cos 30 = (20$ m/s)(0.866) = 17.32 m/s.

Just before it hits the ground, the ball has some velocity vector that has an overall magnitude of 26.35 m/s (what we found earlier) **but** that vector has X and Y components, and the overall speed and the components are related: $v^2 = v_x^2 + v_y^2$. We just found the X component to be 17.32 m/s, so we can find the Y component: $v_x^2 + v_y^2 = (26.35)^2$ or $(17.32)^2 + v_y^2 = (26.35)^2$. Solving this, we find that $v_y^2 = 394.34$ or $|v_y| = 19.86$ m/s. Since that equation involved the square of v_y , we don't really know its sign but we do know the ball will be falling downward, so we can assume that $v_y = -19.86$ m/s instead of the positive option.

This new information lets us determine how long the ball is in the air. If we look at just the Y direction for a moment, $v_y = v_{oy} + a_y t$. The initial velocity in the Y direction is $v_{oy} = v_o \sin 30 = (20$ m/s)(0.500) = 10.00 m/s. The final velocity (the instant before hitting the ground) was $v_y = -19.86$ m/s, and the acceleration is $a_y = -9.81$ m/s², so $v_y = v_{oy} + a_y t$ becomes $(-19.86) = (10.00) - 9.81(t)$ from which $t = 3.04$ s.

At the ground then, the ball is moving at an angle given by $\tan \theta = v_y/v_x = -19.86/17.32$ or $\phi = -48.9^\circ$.

How far did the ball travel laterally? $x = x_o + v_{ox}t + \frac{1}{2}a_x t^2$ but $v_{ox} = 17.32 \text{ m/s}$ and $a_x = 0$ and we now know that the ball hits the ground at $t = 3.04 \text{ s}$, so this occurs at $x = 0 + (17.32 \text{ m/s})(3.04 \text{ s}) + 0 = 52.7 \text{ m}$.

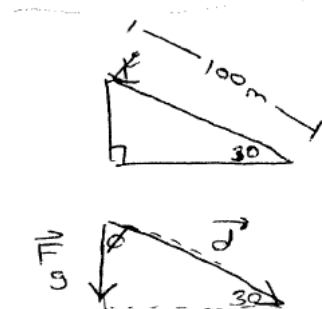
Discussion on Methods

When we did this problem earlier, using 2D equations of motion directly, we computed the time to hit the ground first, using the Y equation of motion (which ended up requiring us to solve a quadratic equation with two solutions, so we had to think about which was the right one). Once we determined the time, we could find the X and Y components of the velocity, and the X position where it hit the ground. So basically we got everything in minute detail about the motion of the object.

If all we care about is the speed with which the ball hits the ground, the methods of this chapter (work and energy) allowed us to bypass all those intermediate steps and jump straight to the final speed of the object.

Once we have that information, if we had to go back and find some of the details, we can do so, and that's what we did in this problem. It's certainly not the most direct way to get that information, but at least it's an option.

5. **Skier on a slope with friction** An 80 kg skier at the top of a 100 meter long, 30° slope, is initially moving down the slope at a speed of 3.0 m/s. How fast is the skier moving when they reach the bottom of this slope? The coefficient of kinetic friction is $\mu_k = 0.06$.

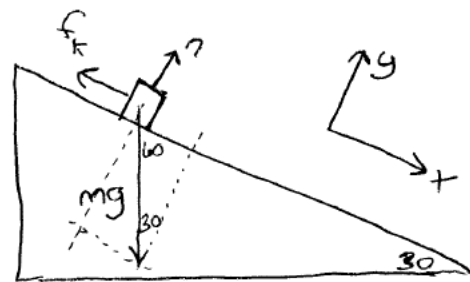


This problem is taken from the sample problems for test 2, where we used Newton's Laws and equations of motion to solve for the speed at the bottom of the ramp, so you may want to refer to that example. We used $g = 9.8 \text{ m/s}^2$ in that version of the solution, so let's do that again here so we should get the identical answer.

Here, we will solve the problem using the work-energy approach, $K_2 = K_1 + \sum W$. We know the initial speed of the skier at the top of the slope, $v_1 = 3 \text{ m/s}$ and we know their mass, so we can find their initial kinetic energy: $K_1 = \frac{1}{2}mv^2 = \frac{1}{2}(80 \text{ kg})(3 \text{ m/s})^2 = 360 \text{ J}$.

We need to compute the work done by all the forces present in the problem. First, we annotate the figure showing all the forces present and their directions:

We have the force of gravity (the weight of the skier) acting straight down, we have friction acting opposite to the motion (so in this case, the vector \vec{f}_k will be pointing up-slope), and we have some normal force preventing the skier from falling through the ground. $f_k = \mu_k n$ so we need to find the magnitude of the normal force. Looking at the figure, we see that in our rotated coordinates, $\sum F_y = 0$. n is already in the +Y direction, but the weight mg is not, so we need to resolve that vector into components. From the figure, we see that we have $mg \cos 30^\circ$ acting in the negative Y direction.



$\sum F_y = 0$ becomes $n - mg \cos 30^\circ = 0$ or $n = mg \cos 30^\circ = (80 \text{ kg})(9.8 \text{ m/s}^2)(0.8660) = 678.96 \text{ N}$. The force of friction is $f_k = \mu_k n = (0.06)(678.96 \text{ N}) = 40.74 \text{ N}$.

Let's compute the work done by each of the forces acting on the skier:

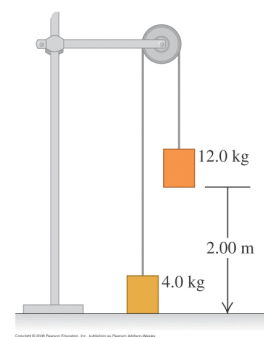
- normal force : n is perpendicular to the displacement, so does no work
- friction : f_k is pointing upslope, and the displacement is 100 m downslope, so the angle between those two vectors is 180° . $W_f = \vec{f}_k \cdot \vec{d} = (40.74 \text{ N})(100 \text{ m}) \cos 180^\circ = -4074 \text{ J}$
- gravity : $m\vec{g}$ is pointing straight down, and the displacement is pointing along the slope. The angle between those two vectors is 60° :
 $W_g = \vec{F}_g \cdot \vec{d} = (mg)(d) \cos 60^\circ = (80 \text{ kg})(9.8 \text{ m/s}^2)(100 \text{ m})(0.5) = 39,200 \text{ J}$.

$$\sum W = 0 + (-4074 \text{ J}) + (39,200 \text{ J}) = 35,126 \text{ J}$$

Finally $K_2 = K_1 + \sum W$ becomes $K_2 = (360 \text{ J}) + (35,126 \text{ J}) = 35,486 \text{ J}$. We can convert this into the speed of the skier from $K_2 = \frac{1}{2}mv_2^2$ or $35,486 = 40v_2^2$ or $v_2^2 = 887.15$ and $v_2 = 29.79 \text{ m/s}$.

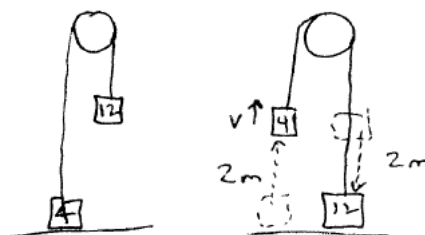
6. Atwood Machine

A system of two paint buckets connected by a lightweight rope is released from rest with the 12 kg bucket initially 2.00 m above the floor. Find the speed of the buckets the instant before the 12 kg one hits the floor. (Ignore friction and the mass of the pulley.)



This is a problem we worked using Newton's Laws earlier (see the sample problems for test 2). We set up coordinates for each object, looked at the forces present on each, and were able to derive the acceleration of the blocks, from which we could find the speed at which the 12 kg block hits the floor (4.43 m/s).

Here, we will use work-energy to find a solution. Let's label things with 'A' representing the 4 kg block, and 'B' will represent quantities associated with the 12 kg block. We'll use the label '1' to represent the initial conditions (the lighter block on the floor, the heavier block temporarily up in the air) and '2' will be the conditions at the instant just before the heavy block hits the floor.



Work-energy for block A: $K_{A2} = K_{A1} + \sum W$. Initially, this block is not moving, so $K_{A1} = 0$. This block will be moving upward a distance of 2 m. What work is being done on this block over that displacement?

- Gravity: \vec{F}_g is acting downward while the block displaces upward so in $W = \vec{F} \cdot \vec{d}$ the angle between these two vectors is 180° so gravity did work of $W = (mg)(d)(-1) = -(4 \text{ kg})(9.81 \text{ m/s}^2)(2 \text{ m}) = -78.48 \text{ J}$.
- Tension: the tension is acting **upward** as the block displaces **upward**, so the angle between the force and displacement vectors here is zero. The work that the tension does on this block then is $W = \vec{F} \cdot \vec{d} = (T)(d) \cos 0 = Td$.

Overall then, our work-energy equation for block A becomes: $K_{A2} = 0 - 78.48 + Td$

Work energy for block B: Repeat the arguments from above, but block B is displacing **downward**. Thus gravity (also aimed down) is doing **positive** work, while the tension is doing **negative** work. The work done by gravity will be $W = \vec{F}_g \cdot \vec{d} = (mg)(d) \cos 0 = mgd = (12 \text{ kg})(9.81 \text{ m/s}^2)(2.0 \text{ m}) = 235.44 \text{ J}$. The work done by tension will be $W = (T)(d) \cos 180 = -Td$. Overall our work-energy equation for block B becomes: $K_{B2} = 0 + 235.44 - Td$

Adding the two boxed work-energy equations together lets us cancel out the terms involving the tension, leaving us with:

$$K_{A2} + K_{B2} = -78.48 + Td + 235.44 - Td = 156.96.$$

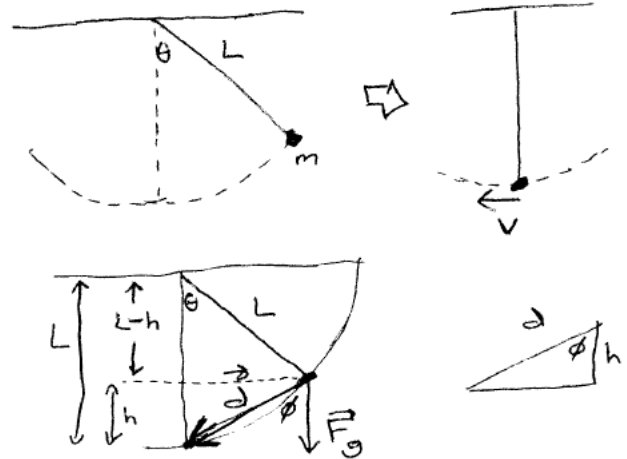
Since the rope connecting the two blocks does not stretch, both blocks will be moving at the same speed v , so: $K_{A2} = \frac{1}{2}M_A v^2 = \frac{1}{2}(4)v^2 = 2v^2$ and $K_{B2} = \frac{1}{2}M_B v^2 = \frac{1}{2}(12)v^2 = 6v^2$.

Making those substitutions: $(2v^2) + (6v^2) = 156.96$ or $8v^2 = 156.96$ which leads to $v = 4.43 \text{ m/s}$.

7. Pendulum : Symbolic

We form a pendulum by hanging an object of mass m from a string of length L that is attached to the ceiling. We pull the object out by an initial angle of θ and let it go (initially at rest). How fast will the object be moving as it passes through the lowest point on its path?

We'll do this entirely symbolically, then in the next example we will apply these results to an example of a swing-set and look at the tension in the rope at the bottom of the swing.



As the mass swings, we have gravity downward and tension in the rope always acting towards the pivot, so it's direction keeps changing, which means that the forces acting here are not constant, and therefore acceleration isn't constant and we can't directly use Newton's Laws or any of our equations of motion here. This is an example of a type of problem that we can still solve using work-energy (or later, conservation of energy).

Let's say that position 1 is the initial location of the mass when it's been pulled out θ from the vertical, and position 2 will be as it passes through the lowest point (the bottom of the circle).

$K_2 = K_1 + \sum W$. The initial kinetic energy is $K_1 = 0$ since it isn't moving there. At the bottom of the swing, $K_2 = \frac{1}{2}mv^2$ where v will be the speed the mass is moving at that point.

How about the works being done here? Tension is always directed towards the center of the circle, so millimeter by millimeter as the object moves along the circle, the tension is **always** perpendicular to each little ds of displacement. Tension thus is doing no work on the mass. Gravity is doing work here. In the lower left figure, we've labelled the force of gravity and the displacement. The work that gravity does will be $W = \vec{F}_g \cdot \vec{d} = (mg)(d) \cos \phi$. Now ϕ is the angle between the two vectors. Looking at the lower right figure, we see that $d \cos \phi$ is just the change in height between the initial and final positions. But on the lower left figure we see that $(L - h) = L \cos \theta$ so rearranging this we can find that $h = L - L \cos \theta$.

Putting this together: $K_2 = K_1 + \sum W$ becomes: $\frac{1}{2}mv^2 = 0 + mgh = mgL(1 - \cos \theta)$.

Both sides of this equation involve m , so we can divide the entire equation by m and remove it:

$\frac{1}{2}v^2 = gL(1 - \cos \theta)$. Multiplying both sides of the equation by 2 and then taking the square root:

$$v = \sqrt{2gL(1 - \cos \theta)}$$

8. Pendulum and Tension

In an episode of Mythbusters, the goal was to get a swing going fast enough that it made a complete 360° loop. Before they even got this far, the chains holding the swing broke. Why?

Refer to the previous page, where we found that if we start off a swing at rest at some angle θ from the vertical, the speed of the swing as it passes through the lowest point will be $v = \sqrt{2gL(1 - \cos\theta)}$.

The swing is moving in a circle, which means there is a centripetal acceleration of $a_c = v^2/r$. Substituting in the expression we just calculated for v and noting that here $r = L$ we have:

$$a_c = v^2/r = \frac{2gL(1 - \cos\theta)}{L} \text{ or } a_c = 2g(1 - \cos\theta)$$

If there is an acceleration, there must be a force providing it. At the very bottom of the circle, we have two forces acting on the swing: its weight downward, and the tension in the cable upward. Newton's laws tell us that $\sum F = ma$ so here: $-mg + T = ma$ or $T = mg + ma$. We just found what the acceleration was, so finally:

$T = mg + m2g(1 - \cos\theta)$. We can rearrange this a bit to result in:

$$T = mg(1 + 2(1 - \cos\theta)) \text{ or finally: } T = mg(3 - 2\cos\theta)$$

In the limit of $\theta = 0$, which means the swing is starting off sitting at the bottom already and therefore won't be going anywhere, we have $T = mg(3 - 2) = mg$ which is just what we expect: the tension in the cable is just balancing out the weight of the swing.

Suppose we start the swing out at a 90° angle though? In that case $\cos 90 = 0$ so the tension at the bottom of the arc will be $T = mg(3 - \cos 90) = 3mg$. That means the tension will be **three** times higher than the not-swinging-at-all case.

If we want the swing to make a complete 360° circle, it has to make it all the way to the **top** of the circle. If we say it just barely makes it to the top then we could think of this as a starting position of $\theta = 180^\circ$. In this case, $\cos 180 = -1$ and the tension at the bottom of the swing becomes $T = mg(3 - 2(-1)) = 5mg$ or **five** times as much tension as the rope handles when the person is just sitting at rest on the swing.

Springs (1) Suppose I have an uncompressed spring with a spring constant of $k = 100 \text{ N/m}$.

(a) How much work do I have to do in order to push the spring in by 10 cm?

Let's set up our coordinates so that the +X direction is the direction we have to push in order to compress the spring. Then the spring will exert a force back against me of $F_s = -kx$ and the force that I have to exert then will be $F_{push} = +kx$. The work that this pushing force does in compressing the spring from $x = 0$ so $x = 0.1 \text{ m}$ will be:

$$W_{push} = \int F_{push} dx = \int_{0.0}^{0.1} kx dx = \frac{1}{2} kx^2 \Big|_{0.0}^{0.1} = \frac{1}{2} (100)(0.1)^2 - 0 = 0.5 \text{ J}.$$

(b) At this moment (the spring compressed by 10 cm), how much force am I exerting?

$$F_{push} = +kx = (100 \text{ N/m})(0.1 \text{ m}) = 10 \text{ N}$$

Springs (2) Suppose I have a horizontal uncompressed spring with a spring constant of $k = 100 \text{ N/m}$. A 2 kg block initially moving at 3 m/s encounters this spring and compresses it until the block comes to a stop. (Assume we don't have any friction, and the floor is horizontal and flat.)

(a) How far in does the spring compress?

From work and kinetic energy, $K_2 = K_1 + \sum W$. Let position 1 be the point where the block has just started touching the spring, and position 2 be the point where the spring has compressed some amount d from its rest length. Then $0 = \frac{1}{2}mv_o^2 + W_{spring} + W_{gravity} + W_{normal} + \dots$. The block is moving horizontally, and gravity and the normal force are vertical here, so those do no work. We don't have any friction, so it's not doing any work either. Only the spring is doing work on the box.

$$W_{spring} = \int_0^d F_{spring} dx = \int_0^d (-kx) dx = -\frac{1}{2}kd^2$$

Putting this into our work-energy equation:

$0 = \frac{1}{2}mv_o^2 - \frac{1}{2}kd^2$ or rearranging terms and multiplying by 2: $kd^2 = mv_o^2$. Substituting in what we know:

$(100)(d)^2 = (2)(3)^2$ from which $d^2 = 0.18$ or $|d| = 0.4243 \text{ m}$. The spring is either compressed or extended by 42 cm: both of those are mathematically right, but physically we know the spring is being compressed in this event.

(b) At the instant where the spring has brought the block to a stop, how much force is the spring exerting on the block?

$F_{spring} = -kx = -(100 \text{ N/m})(0.4243 \text{ m}) = -42.43 \text{ N}$: i.e. it's pushing outward on the block with 42.43 N of force. That means that at this instant, the block is feeling an acceleration of $F = ma$ or $a = F/m = (-42.43)/(2) = -21.2 \text{ m/s}^2$. It may be momentarily at rest, but it has an acceleration of 21.2 m/s^2 pushing away from the spring and it will start moving back in the direction it came from.

(c) How fast will the block be moving when it 'bounces' off the spring?

Let position 2 be the point where the block has encountered the spring and come to a stop (so $K_2 = 0$), and position 3 be the point where the block has bounced back and is no longer in contact with the spring. $K_3 = K_2 + \sum W$. As in the previous part, the only force doing work on the block here is the spring. It's moving from $x = 0.4243 \text{ m}$ to $x = 0$ so that changes the limits of integration: $K_3 = 0 + \int_{0.4243}^0 (-kx dx)$ or $\frac{1}{2}mv^2 = -\frac{1}{2}kx^2 \Big|_{0.4243}^0 = -\frac{1}{2}(100)x^2 \Big|_{0.4243}^0 = +9.00 \text{ J}$. So $\frac{1}{2}(2 \text{ kg})(v)^2 = 9.00$ or $v^2 = 9$ from which $|v| = 3.0 \text{ m/s}$. This just tells us the speed, not the direction, but we know it's flying back away from the spring, so technically $v = -3.0 \text{ m/s}$.

HW Problem 7.84 : What should be the spring constant k of a spring designed to bring a 1300 kg car to rest from a speed of 90 km/h so that the occupants undergo a maximum acceleration of $5g/s$?

We have the car initially moving at the given speed when it first touches the spring. The spring now does negative work on the car, bringing it to a stop over some distance d .

The speed is not given in standard metric units, so we'll need to convert that first: $(90 \text{ km/hr}) \times \frac{1 \text{ hr}}{3600 \text{ s}} \times \frac{1000 \text{ m}}{1 \text{ km}} = 25 \text{ m/s}$

The work done by a spring on the car is $W_s = -\frac{1}{2}kd^2$, so our work-energy equation $K_2 = K_1 + \sum W$ becomes $0 = \frac{1}{2}mv^2 - \frac{1}{2}kd^2$ or $kd^2 = mv^2 = (1300 \text{ kg})(25 \text{ m/s})^2 = 812,500$.

Well that's fine but we have two unknowns still: the spring constant k and the amount the spring has to compress d .

WRONG SOLUTION : We can determine d though: we desire that the car drop from 25 m/s to 0 m/s with an acceleration of 5 g's or $a = -49 \text{ m/s}^2$.

Using our v^2 equation from earlier: $v^2 = v_o^2 + 2ad$ so $(0)^2 = (25)^2 + (2)(-49)(d)$ which leads to $d = 6.38 \text{ m}$. (That's about 20 feet, so this has to be a pretty long spring...)

$kd^2 = 812500$ so $(k)(6.38)^2 = 812500$ or $k = 19,976 \text{ N/m}$ (or to two significant figures, $k = 20,000 \text{ N/m}$).

What was wrong with this? Unfortunately, the force the spring is exerting on the car is **not constant** so the acceleration isn't constant either and we can't use our v^2 equation. When the car has just started touching the spring, there is no force. The more the car compresses the spring, the stronger the force is that the spring exerts on it: $F = -kx$. The spring force keeps increasing the more the spring gets compressed. At the point where the car has come to a stop, the spring is compressed an amount d which means the force it is exerting is $|F| = kd$ which means that right at this instant, the car has an acceleration of magnitude $F = ma$ or $kd = ma$. The car may not be moving at this instant, but its acceleration has reached its maximum value and it's right at this point where we want to reach that limit of $a = 5g$!

CORRECT SOLUTION : At the point where the car has compressed the spring an amount d and has (momentarily) come to a stop, the force on the car is $F_s = kd$ and $F = ma$ so: $kd = ma = (1300 \text{ kg})(5 \times 9.8 \text{ m/s}^2) = 63,700 \text{ N}$.

We have two equations now. Conservation of energy earlier gave us: $kd^2 = 812500$ and now we know that $kd = 63700$ so if we divide the first equation by the second: $\frac{kd^2}{kd} = \frac{812500}{63700}$ or $d = 12.76 \text{ m}$. Since $kd = 63700$, then $(k)(12.76) = 63700$ or $k = 4994 \text{ N/m}$. (Rounded to 2 significant figures, $k = 5000 \text{ N/m}$.)

What happens to the car now? At this point, we have the car sitting against the compressed spring, so the spring will exert a force of $F_s = -kx = (-5000)(12.76) = -63700 \text{ N}$ pushing the car away from the spring. At this instant, the car is feeling an acceleration of $F = ma$ or $a = F/m = (-63700)/(1300) = -49 \text{ m/s}^2$ ($5g$'s again, but in the other direction). Once the car leaves contact with the spring, all the work the car did compressing the spring will be returned to it and the car essentially flies away from the spring with the same 90 km/hr speed it originally had, only now it's moving backwards back out onto the road. Doesn't seem like a very good idea...