PH2213 : Examples from Chapter 10 : Rotational Motion

Key Concepts

Methods in this chapter give us tools for analyzing rotational motion. They are basically the rotational analogs of what we did in earlier chapters on linear motion.

Translational motion involved vector position $\vec{r}$, velocity $\vec{v}$ and acceleration $\vec{a}$ and we derived numerous 'equations of motion' relating various combinations of these (and the time t). Similarly, the complete description of angular motion also involves vectors. An angle of 30^o is meaningless without knowing what axis that angle is a rotation about. So the axis (which could be pointing anywhere, and is thus a 3-D vector) is part of the concept of angular position $\vec{\theta}$, angular velocity $\vec{\omega}$ and angular acceleration $\vec{\alpha}$. This can get **very** complicated, so we restrict ourselves to rotations that are confined to a plane and usually define the axis of rotation to be the Z axis.

Key Equations

For an object rotating in the XY plane (i.e. rotating about the Z axis), the angle, angular speed, and angular acceleration are defined to be about the Z axis with the convention that the positive direction is counter-clockwise about that axis.

NOTE: the standard unit for doing calculations involving rotation is the **radian**. One complete rotation is $2\pi\ rad$ so $2\pi\ rad = 360^o$ which allows to convert between them as needed.

Angular equivalents to some common 1-D equations of motion:

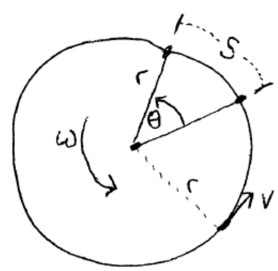
$\omega = d\theta/dt$

$\alpha = d\omega/dt$

$\theta = \theta_o + \omega_o t + \frac{1}{2}\alpha t^2$

$\omega = \omega_o + \alpha t$

$\omega^2 = \omega_o^2 + 2\alpha(\Delta\theta)$



Relating Linear and Angular Kinematics

$v = r\omega$

$a_{tan} = dv/dt = r d\omega/dt = r\alpha$

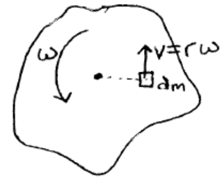
$a_{rad} = v^2/r = r\omega^2$

Common Errors

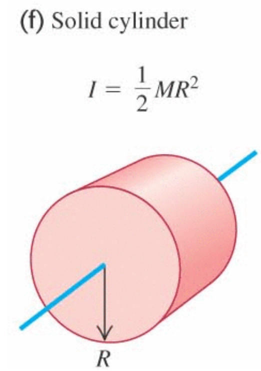
- for a collection of objects, $I = \sum m_i r_i^2$ where r_i is the **radius** of the circle that object will move in when the overall collection rotates, so it's the **perpendicular** distance from the object to the axis of rotation
- units: all the rotational equations of motions imply units of radians, (rad/s, rad/s^2, etc)
- conversions between period, RPM, RPS, rad/s
- determining ϕ : angle between $\vec{r}$ and $\vec{F}$
- sign of torque

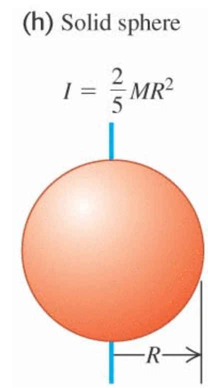
Moment of Inertia and rotational kinetic energy

If an object is rotating, every piece of it is rotating at the same ω even though each piece has different linear speeds. We can accumulate all the kinetic energies in each moving piece though and show that: $K = \frac{1}{2}I\omega^2$ where: $I = \sum m_i r_i^2$ for collections of point masses, or $I = \int r^2 dm$ for solid objects.



See page 260 for the moments of inertia of some objects that have common geometric shapes.





Parallel Axis Theorem : If we compute the value of I for some object rotating about an axis through it's center of mass and now desire to rotate it about some other axis - but one that is parallel to the original one and just shifted some distance d to the side, we don't need to recompute I from scratch. The new moment of inertia I is related to the original one I_{cm} by: $I = I_{cm} + Md^2$ where M is the total mass of the object.

Gravitational Potential Energy : for an extended object, we can add up the U_g of each little dm element of the object and show that $U_g = Mgy_{cm}$: that is, it is the same as if all the mass were located at the center of mass of the object.

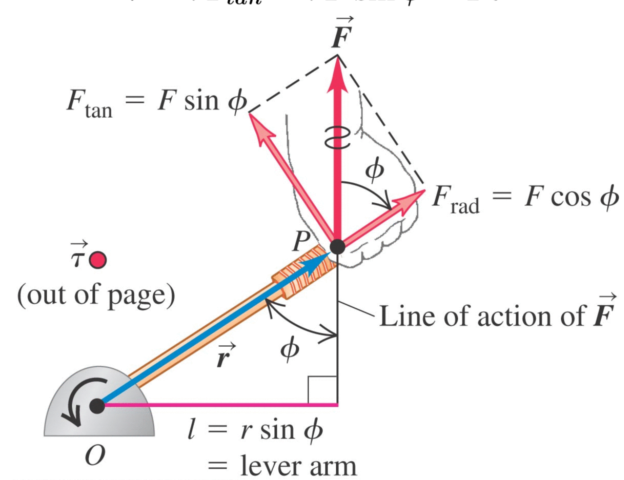
Torque : A force $\vec{F}$ applied at a location $\vec{r}$ from the axis of rotation O produces a torque of $\vec{\tau} = \vec{r} \times \vec{F}$. The magnitude will be $|\vec{\tau}| = rF\sin\phi$; direction or sign from right-hand rule. (See figure at right for alternate ways to compute.)

Equilibrium: $\sum \vec{\tau} = 0$

Rotational Versions of Newton's Laws, energy, and power: $\sum \vec{\tau} = I\vec{\alpha}$ $\quad W = \vec{\tau}\cdot\vec{\theta}$ $\quad P = \vec{\tau}\cdot\vec{\omega}$

Torque due to weight of an extended object: acts as if all the mass were located at the CM of the object.

Three ways to calculate torque
$$\boldsymbol{\tau = rF_{tan} = rF\sin\phi = Fl}$$



1. **Some Random Short Examples**

   - What is the angular speed of the earth rotating about its axis?

     The earth makes one complete rotation in about 24 hours, so $\omega = \Delta\theta/\Delta t = (2\pi\ rad)/(1\ day) = (2\pi\ rad)/(86400\ s) = 7.272 \times 10^{-5}\ rad/s$

   - If an object makes 1 revolution per second ($1\ RPS$), what angular speed does that represent?

     One complete revolution means the object has turned through and angle of $2\pi\ rad$ so $\omega = \Delta\theta/\Delta t = (2\pi\ rad)/(1\ s) = 2\pi\ rad/s$

     This gives us a conversion factor: $1\ RPS\ =\ 2\pi\ rad/s$

   - If an object makes 1 revolution per minute ($1\ RPM$), what angular speed does that represent?

     One complete revolution means the object has turned through and angle of $2\pi\ rad$ so $\omega = \Delta\theta/\Delta t = (2\pi\ rad)/(60\ s) = \pi/30\ rad/s$ or about $0.1047\ rad/s$

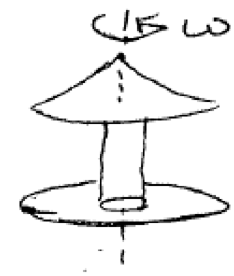
     This gives us the conversion factors: $1\ RPM\ =\ 0.1047\ rad/s$ or $1\ rad/s = 9.549\ RPM$

   - A common way of describing rotational motion is to use the **period** of rotation T. (Be careful not to confuse this with the use of the symbol T for tension...). So T would present the number of seconds for an object to make one complete rotation, which is an angle of $2\pi\ rad$ so $\omega = \Delta\theta/\Delta t = 2\pi/T$ or: $\omega = 2\pi/T$ and $T = 2\pi/\omega$.

     T represents the seconds per rotation. The **frequency** f is the rotations per second and is the inverse of T: i.e. $f = 1/T$ or $T = 1/f$.

   The relationships in the last three examples give us a way of converting ω into something that has more physical meaning.

2. **Merry-go-round : Basic Definitions**



Suppose we have a merry-go-round that makes one revolution in 10 seconds. Find the angular speed, and the linear speed and radial acceleration at various points out from the center.

The **period** of rotation of this merry-go-round is:
$T = 10\ s$, which means that the angular speed is:
$\omega = 2\pi/T = 0.6283\ rad/s$.

(a) Standing 1 meter from the central axis of rotation ($r = 1\ m$)

The linear speed at this point will be $v = r\omega = (1\ m)(0.6283\ rad/s) = 0.6283\ m/s$. A person standing at this point is moving in a circle of radius $1\ m$, moving at a tangential speed of $0.6283\ m/s$, which implies that the person is undergoing a radial acceleration of $a_c = v^2/r = (0.6283\ m/s)^2/(1\ m) = 0.3948\ m/s^2$. Since $v = r\omega$, we can also write this as $a_c = (r\omega)^2/r$ or just $a_c = r\omega^2$. Using that equation, at $r = 1\ m$ we have $a_c = (1.0\ m)(0.6283\ rad/s)^2 = 0.3948\ m/s^2$ (same result of course).

(b) Standing 3 meters from the central axis of rotation ($r = 3\ m$)

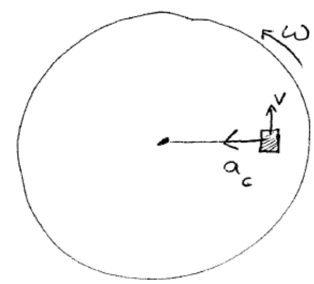
Let's say the outer radius of the merry-go-round is 3 meters and the person is standing right out there on the edge.

The linear speed at this point is now $v = r\omega = (3\ m)(0.6283\ rad/s) = 1.885\ m/s$. This person is moving in a circle of radius $r = 3\ m$, moving at a tangential speed of $1.885\ m/s$, so their radial acceleration is $a_c = v^2/r = (1.885\ m/s)^2/(3\ m) = 1.184\ m/s^2$. Alternately, since $a_c = r\omega^2$ and the ω value is the same at every point on the merry-go-round, we see that a_c is directly proportional to r. Compared to part (a), the person is standing 3 times further out from the center, so should have an acceleration exactly 3 times what we got in part (a) or $3 \times 0.3948\ m/s^2$ which is $1.184\ m/s^2$.

If there is an acceleration, there must be a force to provide it. In this case it would be the static friction between the person's shoes and the floor of the merry-go-round, which we'll investigate in the next problem.

3. **Merry-go-round and Friction**

Suppose it's raining and the floor of the merry-go-round is wet. We want people to be able to stand on the (rotating) floor without slipping. We saw in the previous example that the farther out an object is located, the higher the angular acceleration, but that means there must be a force available to provide that acceleration. How fast can we run the merry-go-round in order to guarantee that nobody slips and falls?



Suppose the coefficient of static friction between typical shoes and the (wet) floor of the merry-go-round is 0.4

We saw in the previous problem that a_c goes up linearly with r. A point near the edge has a higher acceleration than a point near the center, so will need more force to stay in place, but the amount of force available is limited so eventually people could start slipping.

Here, we have a person standing at some distance r from the central axis of rotation. They are moving in a circle, so are undergoing a radial acceleration of $a_c = v^2/r$ or more conveniently here $a_c = r\omega^2$.

The force needed to create this acceleration will be $F = ma = ma_c = mr\omega^2$. This force must be coming from the static friction between the shoes and the floor. That has a max value, though, of $f_{s,max} = \mu_s n$. What is the normal force here? Looking in the vertical direction, we have the person of mass m standing on a horizontal floor, so $\sum F_y = 0$ implies that $n - mg = 0$ or $n = mg$. Thus $f_{s,max} = \mu_s n = \mu_s mg$.

We determined that the force needs to be $F = mr\omega^2$ and now we know how much force we have available, so setting those equal: $\boxed{\mu_s mg = mr\omega^2}$. Conveniently, the mass cancels out leaving: $\omega^2 = \mu_s g/r$, which we can write as $\omega = \sqrt{\mu_s g/r}$. We are interested in the **period** of the merry-go-round and $T = 2\pi/\omega$ so finally $\boxed{T = 2\pi\sqrt{r/(\mu_s g)}}$

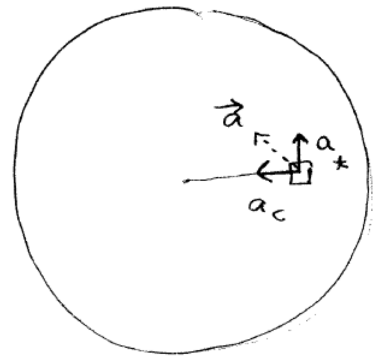
Note what this is telling us: as the person stands closer to the outer edge of the floor, that increases the value of r which increases the 'safe' period of rotation T. For people near the center (small r), T can be very small (i.e. the MGR can spin like crazy and they won't slip), but for people near the edge, T needs to be longer (spinning more slowly). The person at the outer edge is what's limiting the period: they'll be the first to start slipping.

If the radius of our merry-go-round is $r = 3\ m$, that means our T is limited to be no smaller than $T = 2\pi\sqrt{r/(\mu_s g)} = 2\pi\sqrt{\frac{3.0}{0.4\times 9.81}} = 5.49\ s$ (corresponding to $\omega = 2\pi/T = 1.144\ rad/s$) As long as the merry-go-round takes at least 5.49 s to make one complete revolution, nobody should slip, no matter where they're standing.

What **is** the radial acceleration at the outer edge at this value of ω? $a_{rad} = r\omega^2 = (3)(1.144)^2 = 3.93\ m/s^2$. If the person has a mass of 80 kg, that means the force needed to make the person go in this circle will be $F = ma = (80)(3.93) = 314\ N$. How much force can static friction provide? $f_{s,max} = \mu_s n = \mu_s mg = (0.4)(80)(9.81) = 314\ N$, which is just exactly enough.

4. **Merry-go-round (C)**

Suppose we have the same merry-go-round in the previous problem and we want to run it at the just-barely-safe speed we found in the previous problem, representing a period of $T = 5.49\ s$. Initially it's at rest, and we turn it on and start spinning it up. Suppose it takes 5 seconds to get up to full speed. Will people still be able to stand or will they slip and fall?



In the previous problem, we found that we could spin the merry-go-round with a period of 5.49 s without anyone slipping, even right at the outer edge.

That's fine, but we need to spin the merry-go-round up from rest initially. In the previous problems, we saw that a given angular speed ω translated into a radial acceleration but in the scenario here, we have introduced a **tangential** acceleration as well.

If the MGR starts from rest and (angularly) accelerates uniformly, we can write the angular speed as $\omega = \omega_o + \alpha t$. Here, we're starting from rest so $\omega_o = 0$ and we're spinning up to $\omega = 2\pi/T = (2)(\pi)/(5.49) = 1.144\ rad/s$ in a time interval of 5 seconds, so: $(1.144) = (0) + (\alpha)(5)$ and $\alpha = 0.2288\ rad/s^2$.

An angular acceleration of α is related to the tangential acceleration a_{tan} by: $a_{tan} = r\alpha$ so that means in this case we have $a_{tan} = (3.0\ m)(0.2288\ rad/s^2) = 0.6864\ m/s^2$.

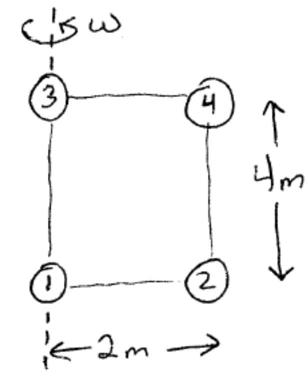
We have two components of (linear) acceleration now: a tangential acceleration of 0.6864 m/s^2 **and** a radial acceleration that depends on ω : $a_{rad} = v^2/r = r\omega^2$. Initially, $\omega = 0$ so we only have the tangential acceleration we just calculated, but as the MGR spins up, that radial component gets larger and larger. Once the MGR has fully spun up, $\omega = 1.144\ rad/s^2$ which represents a (linear) radial acceleration of $a_{rad} = r\omega^2 = (3.0\ m)(1.144\ rad/s^2) = 3.93\ m/s^2$.

These two perpendicular components represent an overall acceleration of magnitude:
$a = \sqrt{a_{rad}^2 + a_{tan}^2} = \sqrt{(3.93)^2 + (0.6864)^2} = 3.99\ m/s^2$. Using the same 80 kg person in the previous example, this acceleration requires a force of $F = ma = (80\ kg)(3.99\ m/s^2) = 319\ N$ to maintain, but we also found that we only have 314 N of static friction available, so the person will slip.

During this spin-up phase (and similarly during 'braking' when the MGR slows down to a stop), the acceleration of the person is actually a bit larger than it is when the MGR is turning around in a circle normally. In the previous problem, we found that static friction can only provide enough force to allow the person to have an acceleration of 3.93 m/s^2, so during the spin-up and braking intervals, the person standing right on the edge can still slip.

5. **Moment of Inertia and Energy in Rotating Objects: (A)**

Suppose we have an object (a mobile, maybe) formed from four point masses located at the corners of a rectangle as shown in the figure. The rods connecting the four masses are assumed to be massless. Define the origin of coordinates to be the location of ball 1, with the +X axis pointing to the right, and the +Y axis pointing towards the top of the paper (a) Find the location of the center of mass, and (b) find the moment of inertia of this object if we want to rotate it about the Y axis? (I.e., the axis shown by the dotted line in the figure, which runs through the 3 kg and 1 kg masses.)



Recall, the center of mass for a collection of objects is defined as: $X_{cm} = (\sum m_i x_i)/M$ and $Y_{cm} = (\sum m_i y_i)/M$, where $M = \sum m_i$

The moment of inertia for a set of point masses is given by $I = \sum m_i r_i^2$, where m_i is the mass of each object, and r_i is the radius of the circle that object will traverse as it rotates. Basically, r_i is the perpendicular distance from the object and the rotation axis.

Let's say ball 1 is the one with a mass of 1 kg, ball 2 is the one with a mass of 2 kg, ball 3 has a mass of 3 kg, and ball 4 has a mass of 4 kg.

Then ball 1 is located right on the axis of rotation, so $r_1 = 0$. Ditto for ball 3. Ball 2 is located 2 meters away from the axis (ditto ball 4), so $r_2 = r_4 = 2\ m$.

It's convenient to organize the calculations in a table:

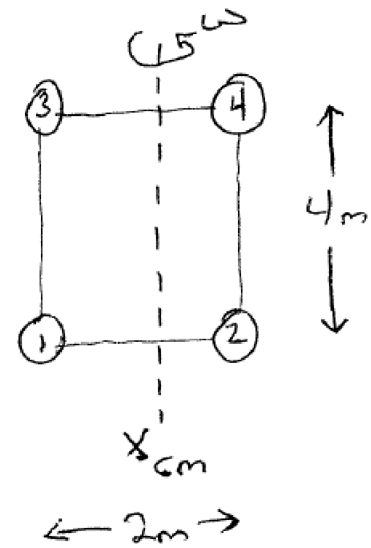
i	m_i	x_i	y_i	$m_i x_i$	$m_i y_i$	r_i	r_i^2	$m_i r_i^2$
1	1.0	0	0	0	0	0	0	0
2	2.0	2	0	4	0	2	4	8
3	3.0	0	4	0	12	0	0	0
4	4.0	2	4	8	16	2	4	16
	$\sum m_i = 10$			$\sum m_i x_i = 12$	$\sum m_i y_i = 28$			$\sum m_i r_i^2 = 24$

Looking at the first row, we have a mass of 1 kg located at $x = 0, y = 0$. The next column is the mass multiplied by the x coordinate, followed by the mass multiplied by the y coordinate (representing terms we will need in computing the center of mass). The next column is the distance of this point from the axis of rotation, followed by the square of that distance and finally the mass times that squared distance (representing the terms we will need in computing the moment of inertia). To fill in the r_i column, we need to draw a line directly from each point mass to the axis about which it will be rotating - i.e. a line that is perpendicular to that axis, since when the mass rotates about that axis, that is the radius of the circle that mass will be moving in.

The last row is the sums we'll need:

$X_{cm} = (\sum m_i x_i)/(\sum m_i) = 12/10 = 1.20\ m$
$Y_{cm} = (\sum m_i y_i)/(\sum m_i) = 28/10 = 2.80\ m$
$I = \sum m_i r_i^2 = 24\ kg\ m^2$.

6. **Moment of Inertia and Energy in Rotating Objects: (B)**



(Review the previous problem first.) Suppose we have this same mobile but now we want to compute its moment of inertia about an axis through the X center of mass, parallel to the Y axis. In the previous problem, we found that $X_{cm} = 1.2\ m$.

Direct Approach : The moment of inertia for a collection of point masses is defined to be $I = \sum m_i r_i^2$ where r is the distance from each object **to the axis** about which it is now rotating. The coordinates and masses are all the same as before, but we're rotating about a **different** axis now, so all the r_i values are different. For example, ball 1 is located at $x = 0$ but the axis we're rotating about is at $x = 1.2\ m$ so the distance from that ball to the axis is now $r_1 = 1.2\ m$. (Ditto for number 3). Ball 2 is located at $x = 2$ and the axis is at $x = 1.2$ which means it's now spinning around in a circle about that axis with a radius of $0.8\ m$. (Ditto for ball 4.)

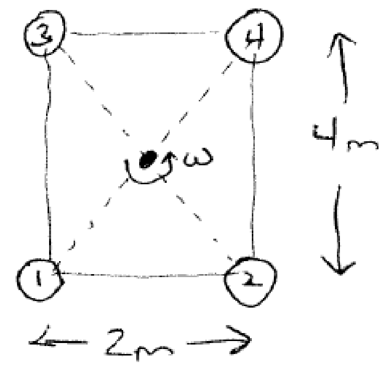
i	m_i	x_i	y_i	r_i	r_i^2	$m_i r_i^2$
1	1.0	0	0	1.2	1.44	1.44
2	2.0	2	0	0.8	0.64	1.28
3	3.0	0	4	1.2	1.44	4.32
4	4.0	2	4	0.8	0.64	2.56
	$\sum m_i = 10$					$\sum m_i r_i^2 = 9.6$

Using the Parallel Axis Theorem : We can also short-cut this calculation using the parallel-axis theorem: $I = I_{cm} + Md^2$, which states that if we have the moment of inertia about an axis through the center of mass and we want to recompute it about some other axis that's shifted parallel to that original axis by a distance d we can do so easily. What we're calculating in **this** problem is the moment of inertia about an axis through through the X center of mass, and what we have in the previous problem is the moment of inertia about an axis that is $d = 1.2\ m$ away, so:

$(24\ kg\ m^2) = I_{cm} + (10\ kg)(1.2\ m)^2$ or $I_{cm} = 24 - 14.4 = 9.6\ kg\ m^2$ (same as we just computed using the definition of I).

7. **Moment of Inertia and Energy in Rotating Objects: (C)**

Let's take the same mobile we've been using, but this time we're going to rotate it about an axis that is **perpendicular** to the plane of the object, going right through the geometric center. Basically the mobile is now laying flat in the X-Y plane (parallel to the ground, say). We stick a vertical rod through the middle and spin it about that axis.



The axis of rotation now is coming perpendicularly up out of the page, intersecting the object at the dot shown right in the middle of the rectangle.

The moment of inertia involves terms $m_i r_i^2$ so we'll need to determine how far each ball is from the rotation axis (the dot in the middle of the rectangle). This is a rectangle, so each of the corners is the same distance from that center point. The distance from point 1 to the midpoint is exactly half the distance between opposite corners of the rectangle. The distance between points 1 and 4, for example, is $d = \sqrt{2^2 + 4^2} = \sqrt{20} = 2\sqrt{5}$ so $r = \frac{1}{2}d = \sqrt{5}\ m$. All the points are located that same distance from the axis of rotation.

i	m_i	x_i	y_i	r_i	r_i^2	$m_i r_i^2$
1	1.0	0	0	$\sqrt{5}$	5.00	5.00
2	2.0	2	0	$\sqrt{5}$	5.00	10.0
3	3.0	0	4	$\sqrt{5}$	5.00	15.0
4	4.0	2	4	$\sqrt{5}$	5.00	20.0
	$\sum m_i = 10$					$\sum m_i r_i^2 = 50.0$

So here, $I = 50.0\ kg\ m^2$.

Work and Energy in Rotating Objects

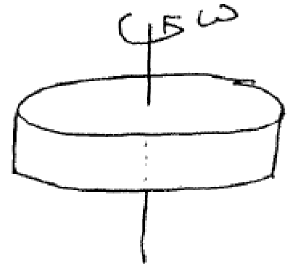
From these three examples, we see that the moment of inertia of an object depends on what axis we choose to rotate the object about. The energy in a rotating object is $K = \frac{1}{2}I\omega^2$ so something with a larger moment of inertia will be storing more energy, and therefore we have to do more work to get that object rotating. Looking at the equation, for objects rotating at the same speed ω, we see that K is directly proportional to I.

Rotating about the center of mass produces the smallest moment of inertia. In example B, we had $I = 9.6\ kg\ m^2$ so this one would be relatively easy to get spinning.

Example A had a moment of inertia of $I = 24\ kg\ m^2$ or over twice as much, so we'd have to do twice as much work to spin this one up to the same speed.

Finally example C, rotating about the axis perpendicular to the plane of the mobile, had an even larger moment of inertia and would require even more work to spin up.

8. **Flywheel (A)**



A rotating object represents a kinetic energy of $K = \frac{1}{2}I\omega^2$ so we can use this to 'store' energy. How fast would a flywheel need to turn to store enough energy to power a house for a day?

Looking at the equation for rotational kinetic energy, $K = \frac{1}{2}I\omega^2$, we see that the higher the moment of inertia and the faster it spins, the more energy it can store. Also, note that ω appears squared, so rotating it twice as fast will store 2^2 or 4 times as much energy.

A common flywheel design is a flattened disk (cylinder) rotating about an axis perpendicular to its center, as shown in the figure. To fit in a typical home, how big could this thing be? Let's assume our cylinder is about a meter across ($R = 0.5\ m$) and has a mass of 240 kg (i.e. has a weight of about 530 *pounds*). The moment of inertia for such a shape is $I = \frac{1}{2}MR^2$ so here $I = (0.5)(240\ kg)(0.5\ m)^2 = 30\ kg\ m^2$.

How much energy do I need to store in it? A typical home uses energy at a rate of roughly 1000 W or 1000 J/s. There are 86,400 seconds in a day, so it looks like we need to store $8.64 \times 10^7\ J$ of energy.

So: $K = 8.64 \times 10^7 = \frac{1}{2}I\omega^2$ becomes: $8.64 \times 10^7 = (0.5)(30)(\omega)^2$ from which $\omega = 2400\ rad/s$.

Putting that in more meaningful terms, the frequency (how many revolutions per second) of the cylinder's rotation is $f = \omega/(2\pi) = 382\ rev/s$. So this large, heavy cylinder has to make 382 rotations every second, or $382 \times 60 = 22,920\ rev/min$. According to Wikipedia, old-school metal flywheels can reach a few thousand RPM's, and high-tech carbon fiber materials can reach $60,000\ RPM$, so this is at least plausible.

The speed of a point on the outer edge of the cylinder will be $v = r\omega = (0.5\ m)(2400\ rad/s) = 1200\ m/s$ which is about four times the speed of sound in air, so the cylinder would probably need to be rotating in a vacuum.

Real-world Application

Around July 2011, Beacon Power, in Stephentown, New York installed the world's largest commercial flywheel energy storage device, consisting of multiple flywheels. (A few months later, it filed for bankruptcy. Hm.) According to the article, it stores 5 MW-hr of energy, which it can release at a rate of 20 MW for 15 minutes. 5 MW-hr of energy represents $(5 \times 10^6\ W\ hr) \times \frac{3600\ s}{1\ hr} = 1.8 \times 10^{10}\ W\ s$ but Watts is Joules/sec so this is $1.8 \times 10^{10}\ J$ of energy. This is about 200 times as much energy as the 'home' flywheel we did in this problem stored, so this giant flywheel 'farm' could provide power for 200 homes for a full day (or all of Starkville for a few minutes anyway).

9. **Flywheel (B)**

Suppose we want to attach the flywheel in the previous example to a motor to spin it up. If the motor generates 100 $ft\ lb$ of torque (a small car engine), how long would it take to spin up the flywheel to the $\omega = 2400\ rad/s$ speed we found in the previous example?

The torque will cause the object to angularly accelerate at some α, which will take some time t to bring the flywheel from rest up to the desired final ω.

$\tau = I\alpha$ and we already found that $I = 30\ kg\ m^2$.

We don't have the torque in the proper units here though, so converting:
$\tau = 100\ ft\ lb \times \frac{1\ m}{3.281\ ft} \times \frac{4.448\ N}{1\ lb} = 136\ N\ m.$

$\tau = I\alpha$ then becomes: $136 = 30\alpha$ or (since everything has been converted to standard metric units) $\alpha = 4.533\ rad/s^2$.

How long will it take to spin the flywheel up from rest?

$\omega = \omega_o + \alpha t$ so $2400 = 0 + (4.533)(t)$ from which $t = 529\ sec$ or roughly 9 minutes.

$\boxed{\text{Power output}}$: how much power would the motor be putting out during this spinup? We have to be careful how this question is asked. The **average power** would be the work done divided by the time interval. From the previous flywheel example, we found that when spun-up, it's storing $8.64 \times 10^7\ J$ of energy, so the motor will have to do that much work in 529 s, representing an average power of: $P_{avg} = work/time = (8.64 \times 10^7\ J)/(529\ s) = 163,200\ watts$ which is about 219 hp. That seems too high to be reasonable for a small gas engine. It gets worse though.

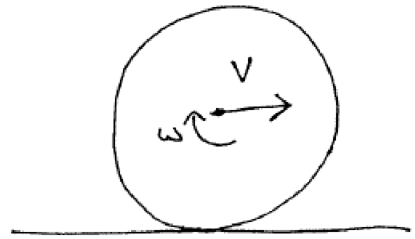
The **instanteous** power output in the case of rotational motion can be written as $P = \tau\omega$ so the motor is putting out essentially no power at the start, but at the end when we have $\omega = 2400\ rad/s$, then $P = \tau\omega = (136\ N\ m)(2400\ rad/s) = 326,400\ watts$ or exactly twice the average power we just found. This is 437 hp which is crazy high for a small motor.

With real engines, the torque and power output are not constant but vary with the angular speed (i.e. the RPM's) of the motor, usually both will start off rising with RPM and then they'll start dropping off. There's usually a certain RPM where the motor is putting out it's maximum torque, and a different RPM where it's putting out it's maximum power. The torque relates to the acceleration, and the power relates to the speed of the vehicle.

(Throughout this example, we presumed a constant angular acceleration, but in a real-world version of our flywheel scenario, it would almost certainly not be.)

10. **Rolling Pool Ball (A)**

Suppose we have a pool ball rolling across a floor, without slipping. Compare the translational kinetic energy (the K represented by the object bodily moving off to the right), to the rotational kinetic energy.
Let's say the ball has a mass of 0.17 kg, a radius of 2 cm, and is moving to the right at $v = 2\ m/s$.



It's rolling, which means it has an angular speed ω. Is this an independent variable (another unknown) or is it related to the translational speed v? If we move along with the ball at 2 m/s, it looks to us like the ball is staying in one place, just rotating about it's center, as the floor glides off to the **left** at 2 m/s. The ball isn't slipping which means that where the ball touches the floor, the tangential speed right at the edge also has to be 2 m/s. But the tangential and angular speeds are related by $v = r\omega$ so that means the ball here has an angular speed of $\omega = v/r = (2\ m/s)/(0.02\ m) = 100\ rad/s$. (From the first sample problem in the examples for chapter 9, we showed that $1\ rad/s\ =\ 9.549\ RPM$ so that means here the ball is rotating at 954.9 revolutions per minute as it rolls across the floor.)

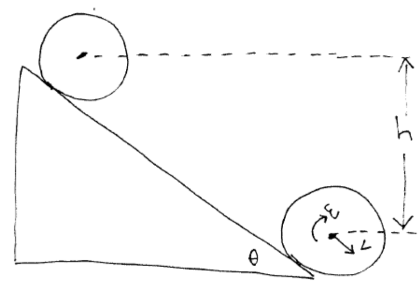
The translational kinetic energy is $K_{trans} = \frac{1}{2}mv_{cm}^2$ and using the values for our pool ball: $K_{trans} = \frac{1}{2}(0.17\ kg)(2\ m/s)^2 = 0.34\ J$.

The rotational kinetic energy is $K_{rot} = \frac{1}{2}I_{cm}\omega^2$, so we'll need to compute the moment of inertia here. The ball is rotating about its center of mass, and I_{cm} for a solid sphere is $I_{cm} = \frac{2}{5}MR^2$. For this pool ball, $I_{cm} = (0.4)(0.17\ kg)(0.02\ m)^2 = 2.72 \times 10^{-5}\ kg\ m^2$. We earlier found that $\omega = 100\ rad/s$ so the rotational kinetic energy is $K_{rot} = \frac{1}{2}(2.72 \times 10^{-5})(100)^2 = 0.136\ J$.

The total kinetic energy in this moving and rolling pool ball, then, is $K = K_{trans} + K_{rot} = 0.340J + 0.136J = 0.476\ J$. About 71 percent of the total energy is in it's translational motion across the table, and about 29 percent in it's rotational motion.

11. **Rolling Pool Ball (B)**

Let's modify the previous problem and have the pool ball rolling down an incline, starting at rest at the top. This time we'll do it symbolically first, and compare the speed at the bottom to our 'old' way of doing this where we assumed the object was just a point mass sliding down a frictionless incline (we'll talk more about that in the solution below).



Point Mass : First, refer back to earlier examples from work-energy or conservation of energy where we looked at objects sliding down inclines. If we call the bottom of the incline our $y = 0$ reference level, then the object of mass M is starting at some height h above the floor. This represents some gravitational potential energy of $U_g = Mgh$.

When the object reaches the bottom, all this energy has been converted into the kinetic energy of the moving mass, so $(K+U_g+U_{el})_2 = (K+U_g+U_{el})_1 + W_{other}$ becomes: $(\frac{1}{2}Mv^2+0+0) = (0+Mgh+0)+0$ or $\frac{1}{2}Mv^2 = Mgh$. Dividing the equation by M, which appears in each term: $v^2 = 2gh$ or just $v = \sqrt{2gh}$. That's the result for a point mass sliding down the ramp with no friction.

Rolling Sphere : Now let's take into account the energy in the rolling ball. Note that we have to add friction now, so that the ball actually will spin as it rolls down but the ball is not **sliding** at all: there is no kinetic friction removing work from the ball as it moves down this incline. Our conservation of energy equation has a new term now: K represents two terms, the translational kinetic energy from earlier chapters, plus now the rotational kinetic energy. Our conservation of energy equation now becomes: $\frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 = Mgh$. We have to be a bit careful/picky at this point. The h in this equation represents the height change of the center of mass of the ball. So in this equation, $y = 0$ represents the level of the **center** of the ball at the bottom, and $y = h$ is the level of the center of the ball at the top of the incline.

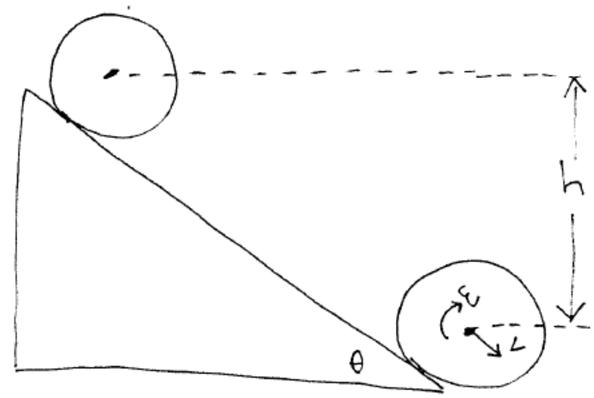
For a uniform solid sphere, $I = \frac{2}{5}MR^2$ so $K_{rot} = \frac{1}{2}I\omega^2$ becomes: $K_{rot} = \frac{1}{2}(\ \frac{2}{5}MR^2\ )\omega^2$ or $K_{rot} = \frac{1}{5}M(R\omega)^2$. But $R\omega = v$, the translational speed of the center of mass down the incline, so we can write this as $K_{rot} = \frac{1}{5}Mv^2$.

Combining this with the translation kinetic energy, the total kinetic energy of the moving and rotating pool ball at the bottom of the incline will be: $K = \frac{1}{2}Mv^2 + \frac{1}{5}Mv^2$ or $K = \frac{7}{10}Mv^2$. Conservation of energy then becomes: $\frac{7}{10}Mv^2 = Mgh$ or $v^2 = \frac{10}{7}gh$ and finally $v = \sqrt{\frac{10}{7}gh}$. This will be slightly less than the v we got for a point mass just sliding down a frictionless incline above. **Note** : this equation for speed v doesn't depend on either the mass of the sphere, or it's radius! So any uniform solid sphere would get to the bottom having this same speed.

If we take a 2 meter long board and raise it to make a 30^o incline, that makes $h = (2\ m)\sin 30 = 1.0\ m$. A point mass sliding down this frictionless ramp would have a speed at the bottom of $v = \sqrt{2gh} = \sqrt{(2)(9.81)(1)} = 4.43\ m/s$.

A ball rolling down this same incline would have a speed at the bottom of $v = \sqrt{\frac{10}{7}gh} = \sqrt{(1.42857)(9.81)(1.00)} = 3.74\ m/s$, which is about 15 percent slower.

12. **Hula-hoop** We've seen in the previous examples that when an object can rotate kinetic energy has two places to go: the overall translation of the object, and it's rotation. The larger the moment of inertia, I, the larger the fraction of energy ends up going into the K_{rot} term, leaving less for K_{trans} which means the object is moving more slowly. What is the 'worst-case' scenario for this rolling motion? That would be when the moment of inertia is as large as possible, which means we want all the mass to be as far from the center-of-mass as possible.



The limiting case is where we put **all** the mass out at some radius R from the axis of rotation, which means we basically have the shape of a hula hoop. (An empty tin-can with the ends cut off would also have this 'maximum' moment of inertia.)

The diameter of a hula hoop is almost exactly one meter, so $R = 0.5\ m$. It has a mass of about $1\ kg$. If we can get the hula hoop to stay upright and roll down an incline, how fast will it be moving at the bottom?

There's no need to reproduce all of the previous derivation here but to summarize, the center of mass of the hoop will be moving with a speed v, and it is rotating about that center of mass with an angular speed $\omega = v/R$. The rotational kinetic energy is $K_{rot} = \frac{1}{2}I\omega^2$ but for this geometry, **all** the mass is located out at the edge so $I = MR^2$ and $K_{rot} = \frac{1}{2}(MR^2)\omega^2 = \frac{1}{2}M(R\omega)^2 = \frac{1}{2}Mv^2$.

That means that the rotational kinetic energy for this rolling hoop is just as large as the translational kinetic energy as it moves bodily down the ramp.

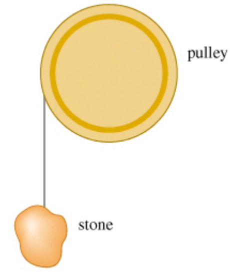
The total kinetic energy at the bottom is $K_{trans} + K_{rot} = \frac{1}{2}Mv^2 + \frac{1}{2}Mv^2 = Mv^2$.

From conservation of energy, all the initial gravitational potential energy has been turned into this kinetic energy, so $Mgh = Mv^2$ or $v = \sqrt{gh}$. Note again here, like in the previous problem, that in the end the **mass** of the hoop and it's **radius** ended up **not mattering**.

Using the same 2 meter board pitched up at a 30 degree angle from the previous problem, we have $h = 1.0\ m$ so $v = \sqrt{(9.81)(1.0)} = 3.13\ m/s$ (slower than the speed we got for the rolling ball in the previous example).

13. **Stone and Pulley (a)**

A frictionless pulley has the shape of a uniform solid disk of mass 2.40 kg and radius 10 cm. A 1.50 kg stone is attached to a very light wire that is wrapped around the rim of the pulley, and the system is released from rest, with the stone located 2 m above the floor. How fast will be stone be moving when it hits the floor? at that point?



As the stone starts to fall, the wire moves with it and since it doesn't slip against the pulley, the pulley will start rotating. If the stone is moving at a speed v downward, points in the wire are also moving at that same speed. Specifically points on the wire that are still wrapped around the cylinder are moving with that same speed v which implies that points around the edge of the cylinder are also moving with that speed. Looking at the cylinder, the speed of a point on the edge is related to the angular speed of the disk through $v = r\omega$ so this connects the speed of the stone to the angular speed of the disk.

We can solve this using conservation of energy. Using the floor as our reference level for U_g, initially we have some gravitational potential energy in the stone. As it falls, that energy is being converted into the kinetic energy of the falling stone **and** the kinetic energy of the rotating disk: $(K + U_g + U_{el})_2 = (K + U_g + U_{el})_1 + W_{other}$. We have no springs, so $U_{el} = 0$ on both sides. Looking at the system as a whole, the tension is basically an internal force so does no work (see the various Atwood machine examples from earlier). At the initial position (1) we have some gravitational potential energy in the rock. At the final position (2), we just have kinetic energy but in two forms: the falling rock, and the rotating disk:

$(\frac{1}{2}M_{stone}v^2 + \frac{1}{2}I\omega^2 + 0 + 0) = (M_{stone}gh + 0 + 0) + 0$

We have the mass of the stone (1.5 kg), and we know $v = R\omega$ where $R = 10$ cm is the radius of the disk. Let's focus on the rotational energy term for a minute: The moment of inertia of the disk is $I = \frac{1}{2}M_{disk}R^2$ which means that the kinetic energy of the spinning disk will be $K_{rot} = \frac{1}{2}I\omega^2 = \frac{1}{2}(\frac{1}{2}M_{disk}R^2)\omega^2$ which we can rearrange into: $\frac{1}{4}M_{disk}(R\omega)^2$ but $R\omega = v$, the speed of the falling rock, so finally the rotational energy of the disk is $K_{rot} = \frac{1}{4}M_{disk}v^2$.

We can thus rewrite our conservation of energy as:

$\frac{1}{2}M_{stone}v^2 + \frac{1}{4}M_{disk}v^2 = M_{stone}gh$ or finally: $(\frac{1}{2}M_{stone} + \frac{1}{4}M_{disk})v^2 = M_{stone}gh$.

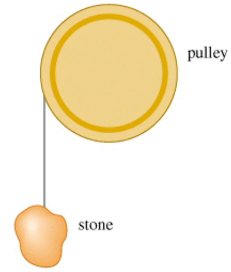
That's about as far as we can go symbolically, so we'll go ahead and substitute in the numeric values were have now:

$(\frac{1}{2}(1.5) + \frac{1}{4}(2.4))v^2 = (1.5)(9.81)(2.0)$ or $1.35v^2 = 29.43$ and finally $v = 4.67$ m/s.

Note: if there were no disk, or the disk had zero moment of inertia (which is what we assumed in prior chapters), we'd end up with just $\frac{1}{2}M_{stone}v^2 = M_{stone}gh$ or $v = \sqrt{2gh} = 6.26$ m/s, so actually including the rotating disk in the (more realistic version of the) problem results in the stone falling more slowly.

14. **Stone and Pulley (b)**

A frictionless pulley has the shape of a uniform solid disk of mass 2.40 kg and radius 10 cm. A 1.50 kg stone is attached to a very light wire that is wrapped around the rim of the pulley, and the system is released from rest, with the stone located 2 m above the floor. How fast will be stone be moving when it hits the floor? at that point?



Let's do the exact same problem, but use torque this time.

As the stone starts to fall, the wire moves with it and since it doesn't slip against the pulley, the pulley will start rotating. If the stone is moving at a speed v downward, points in the wire are also moving at that same speed. Specifically points on the wire that are still wrapped around the cylinder are moving with that same speed v which implies that points around the edge of the cylinder are also moving with that speed. This connects the speed of the stone and the angular speed of the pulley: $v = r\omega$ and also connects the acceleration or the stone and the angular acceleration of the pulley: $a = r\alpha$.

Let's chose a coordinate system such that a and α are both positive, which means for the stone, our positive axis is pointing downward.

Focusing on the stone: we have gravity downward (which is our positive direction now) and tension upward. Applying Newton's laws to the stone: $M_{stone}g - T = M_{stone}a$ or $(1.5)(9.8) - T = 1.5a$ or finally: $14.7 - T = 1.5a$. That's two unknowns, so let's look at the pulley now.

Focusing on the pulley: $\tau = I\alpha$: the torque is being created by the tension in the cable, and it's being applied at the edge so $\tau = Tr = (T)(0.1)$ This is a CCW torque, so will induce a positive angular acceleration.

$I = \frac{1}{2}M_{pulley}R^2 = \frac{1}{2}(2.4)(0.1)^2 = 0.012\ kg\ m^2$ so putting this together: $\tau = I\alpha$ becomes: $0.1T = 0.012\alpha$.

Did we make any progress? Actually yes, since $a = r\alpha$ so $\alpha = a/r = a/0.1 = 10a$ so making that substitution: $0.1T = (0.012)(10)(a)$ or $T = 1.2a$

Finally we have two equations and two unknowns: $14.7 - T = 1.5a$ and $T = 1.2a$. Replacing T in the first equation:

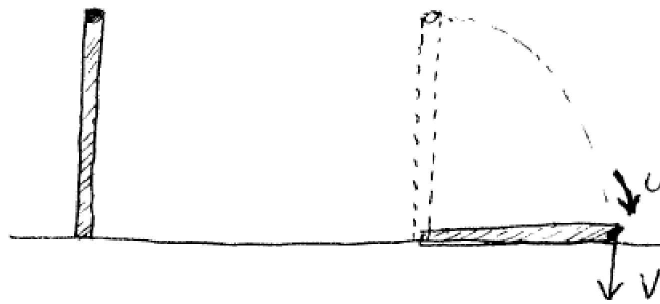
$14.7 - 1.2a = 1.5a$ or $14.7 = 2.7a$ and finally $a = 5.444\ m/s^2$.

How fast is the stone moving when it hits the floor, 2 meters below where it was released from rest?

$v^2 = v_o^2 + 2a\Delta y$ (remember our positive axis is pointing downward) so: $v^2 = 0 + (2)(5.444)(2.0) = 21.78$ or $|v| = 4.67\ m/s$ which is the same result we got using conservation of energy earlier.

15. **Falling Pole**

Suppose we have a 4 m long pole of mass 120 kg, with the mass evenly distributed (i.e. the same density throughout). The pole is fixed at the bottom so it can rotate about that point. If it falls over, how fast will a point at the top end of the pole be moving when it hits the ground?



The pole just standing there is technically stable, but the slightest puff of air will cause it to start falling. For simplicity, let's say the initial angular speed is essentially zero.

As the pole rotates about the point on the ground, it is rotating with some (varying) rotational speed ω. Consider a point at the top end of the pole. A point out there is moving with a linear speed of $v = r\omega$ or here $v = L\omega$. We can use conservation of energy to solve this. Using the level of the ground as $y = 0$, we see we have some gravitational potential energy in the pole initially. Just before it hits the ground, the pole has converted all that U_g into rotational kinetic energy.

Here we basically have a thin cylinder of mass M and length L rotating about one of its ends. From table 9.2, the moment of inertia of such an object is $I = \frac{1}{3}ML^2$, or here: $I = \frac{1}{3}(120\ kg)(4\ m)^2 = 640\ kg\ m^2$.

For an extended object, recall that the gravitational potential energy is just $U_g = Mgy_{cm}$ (i.e. it acts as if all the mass were concentrated at the center of mass). For our pole, the Y center of mass will be at the midpoint, or 2.0 m above the ground.

So: we initially have $U_g = MgY_{cm} = (120\ kg)(9.81\ m/s^2)(2.0\ m) = 2354.4\ J$.

Just before we hit the ground, all this energy is in the form of $K_{rot} = \frac{1}{2}I\omega^2$ so $(2354.4) = (0.5)(640)(\omega)^2$ or $\omega = 2.71\ rad/s$ at the instant just before the pole hits the ground. The point on the top of the pole will be moving with a speed of $v = r\omega = (4.0\ m)(2.71\ rad/s) = 10.8\ m/s$.

Let's compare that to how fast a mass dropped vertically downward from 4 m would be moving just before hitting the ground? In that case, it would have an initial U_g of mgh and at the ground all that energy is converted to kinetic, so $mgh = \frac{1}{2}mv^2$ or $v = \sqrt{2gh} = \sqrt{(2)(9.81)(4)} = 8.85\ m/s$.

The end of the falling pole actually hits the ground faster than an object falling straight down from that height.

16. **Circular Saw**

A flat circular saw blade has a mass of 0.5 kg and a radius of 10 cm. The motor in the saw spins this blade from rest up to 2000 RPM in 2 sec.

(a) How much energy is in the rotating saw blade?

(b) How much power, on average, was the motor expending during the spin-up?

(c) How much torque did the motor have to generate (assume constant angular acceleration)?

(a) We have a flattened cylindrical shape rotating about an axis perpendicular through its center, so looking at the table of moments of inertia, I for this shape will be $I = \frac{1}{2}MR^2$. Here, $M = 0.5\ kg$ and $R = 10\ cm = 0.1\ m$ so $I = \frac{1}{2}(0.5\ kg)(0.1\ m)^2 = 2.5 \times 10^{-3}\ kg\ m^2$.

When the blade is completely spun up, it's rotating at 2000 RPM or 2000 complete resolutions per minute. Rotational kinetic energy is $K_{rot} = \frac{1}{2}I\omega^2$ so we need ω here: the angular speed in radians per second.

Each revolution is equivalent to 2π radians, so $2000\ RPM = 2000 \frac{rev}{min} \times \frac{2\pi\ rad}{1\ rev} \times \frac{1\ min}{60\ sec} = 209.4\ rad/s$.

$K_{rot} = \frac{1}{2}I\omega^2 = \frac{1}{2}(2.5 \times 10^{-3}\ kg\ m^2)(209.4\ rad/s)^2 = 54.83\ kg\ m^2/s^2 = 54.8\ J$.

(b) Power is the rate at which work is being done: $P_{avg} = \Delta W/\Delta t$. The power the motor is putting out is going into increasing the kinetic energy (rotational speed) of the blade. The blade went from rest to storing 54.8 J of energy in 2 seconds, so the motor was doing work at an average rate of $P_{avg} = (54.8\ J)/(2\ s) = 27.4\ Watts$ (pretty feeble for a circular saw...)

(c) $\tau = I\alpha$ and we already found the moment of inertia, so what is the angular acceleration here? The blade spins up from $\omega = 0$ to $\omega = 209.4\ rad/s$ in two seconds so: $\omega = \omega_o + \alpha t$ becomes: $209.4\ rad/s = 0 + (\alpha)(2\ s)$ or $\alpha = 104.7\ rad/s^2$.

$\tau = I\alpha = (0.0025\ kg\ m^2)(104.7\ rad/s^2) = 0.262\ N\ m$.