PH2213 Fox : Lecture 02
Chapter 2 : Kinematics in One Dimension

- Related book material: chapter 2, sections 1 through 4 (and the first equation of motion from section 5).
- Related examples: see the examples02.pdf file in Canvas; specifically example 1 and the first part of example 2.
- The Mastering Physics online homework system (if you scroll down past the various chapter homeworks) has a set of 'dynamic study modules' which guide you through solving problems. The one relevant to this chapter is called:
  Kinematics Module 05 : 1D Kinematics Dynamic Study Module.

In many fields (physics, oceanography, atmosphere) the interesting things start to happen when things are in motion. Even in civil engineering, it's important that structure not be too rigid or they may fracture and collapse. (On the other hand, you don't want them to be too flexible either - try a search for videos on the Tacoma Narrows Bridge collapse.)

Before jumping into complex 3-D forces and motions, we will build up the **concepts** and **machinery** to analyze such motion with simple 1-D motion.

Read over sections 1 through 4 in the textbook, which introduce the concepts of distance, displacement, velocity (and speed), and acceleration. They do a good job of introducing each of these concepts, so there's no need to repeat that information here.

The Khan Academy website also has a series of good, short videos on this material at:
`https://khanacademy.org/science/physics/one-dimensional-motion`

Ball Drop

Suppose we drop a ball from about 4 feet above the floor and are tasked with analyzing it's motion. We'll use this example to introduce some of the basic concepts and definitions involved in analyzing motion.

First, what coordinate system should we use?

Watching the ball drop, it appears to be moving in a straight line down and up, so let's choose a coordinate inline with that motion.

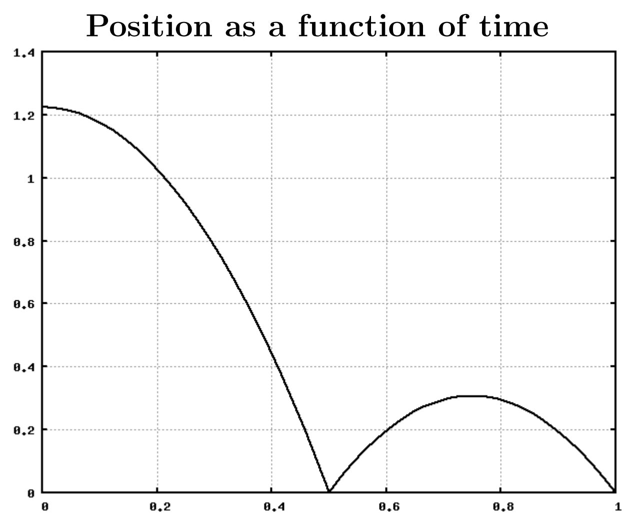
We still have choices though: where should the origin of this axis be? What should the positive direction be? This choice is an integral part of solving the problem.

(Short detour into coordinate systems)

For now, let's choose an axis we'll call X, with an origin on the ground where the ball will bounce, and with the positive direction upward.

We start our time axis when we release the ball.

We record the height of the ball, which we see in the graph here.



Position : where is the object : $x(t)$

Displacement : $\Delta x = x(t_2) - x(t_1)$: basically how did the position coordinate change in a given time interval?

What is the displacement of the ball between $t = 0$ and $t = 0.5$ s?

Velocity

How fast is the ball moving? Seems like a simple question, but need to be a bit picky here:

Velocity : time rate of change of **displacement**

- $v_{avg} = \Delta x / \Delta t$ (so over some time interval)
- $v = dx/dt$ (instantaneous)

What was the ball's average velocity between $t = 0$ and $t = 0.5\ s$?

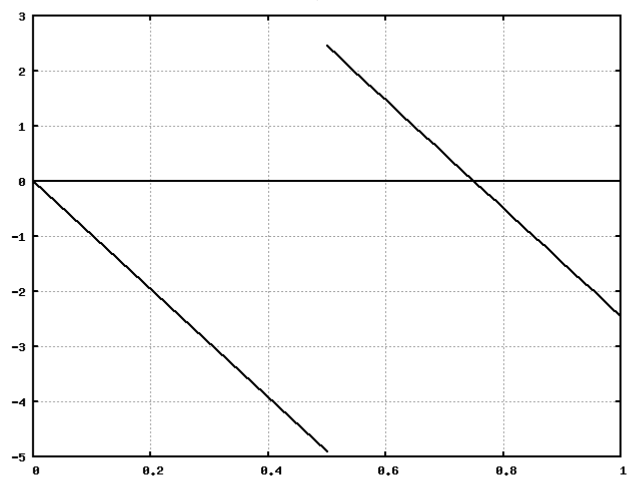
$$v_{avg} = \frac{\Delta x}{\Delta t} = \frac{(0.0\ m) - (1.225\ m)}{(0.5\ s) - (0.0\ s)} = -2.45\ m/s$$

How about between $t = 0.5\ s$ and $t = 1.0\ s$?

$$v_{avg} = \frac{\Delta x}{\Delta t} = \frac{(0.0\ m) - (0.0\ m)}{(1.0\ s) - (0.5\ s)} = 0.0\ m/s$$

The **average** velocity (usually) depends on the particular time interval, so is not often useful. We'll most often be dealing with the **instananeous** velocity of objects:

Instantaneous Velocity as a function of time



The word **speed** is good to avoid because it can have two very different definitions:

- speed as simple $|v|$: the magnitude of the velocity (what I probably mean)
- speed as **distance** travelled divided by the time interval.

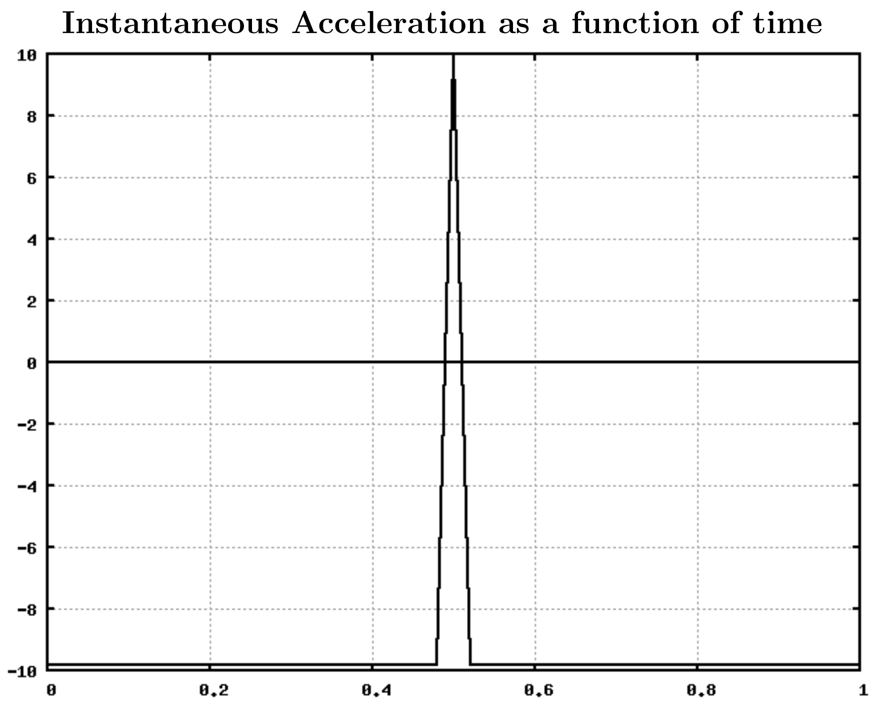
Distance is different from **displacement**. The latter is just how the coordinate has changed: Δx, say. The former is more like the odometer on your car: continuously incrementing regardless of the vehicle's direction of travel. Distance keeps building up, millimeter by millimeter.

Acceleration

The velocity of the ball is changing with time, so let's examine how it's changing.

Acceleration : time rate of change of **velocity**

- $a_{avg} = \Delta v/\Delta t$ (so over some time interval)
- $a = dv/dt$ (instantaneous)



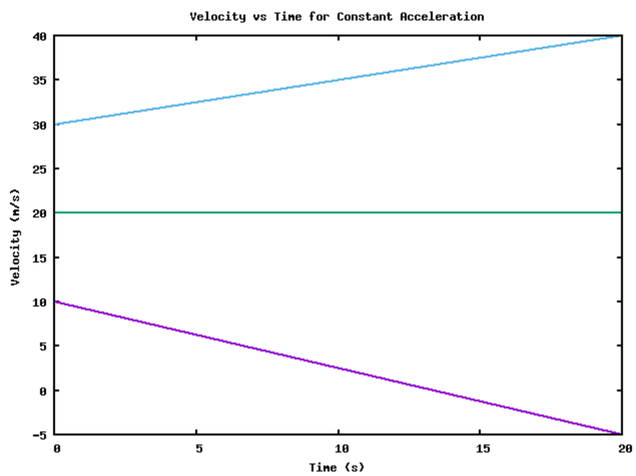
Note that except for the brief moment when the ball is in contact with the floor (i.e. 'bouncing'), the acceleration here is a constant $a = -9.8\ m/s^2$, which is the acceleration any object in free fall will feel near the Earth's surface. (So this graph should have a much narrower and taller 'spike' than the one shown in the figure.)

(We can continue taking derivatives and generate additional quantities. The derivative of acceleration with respect to time is called 'jerk'. Most of what we'll be dealing with for the next few chapters will be situations where the acceleration is constant, so we'll stop here for now...)

$\boxed{\text{Motion with Constant Acceleration}}$

We will see later how acceleration is related to the **forces** being applied to an object. A fairly common situation is where the forces are constant, resulting in a constant acceleration, so it's worth developing some special-purpose equations to handle these cases.

Acceleration is the time derivative of velocity: $a = dv/dt$, so a plot of v as a function of time is seriously constrained. v has to vary in a way that the slope of that curve, dv/dt must remain a constant value. That means that a graph of v vs t can only be a straight line, as seen in the figure to the right.



Any straight line can be written in slope-intercept form. Here we're looking at how v depends on t so the line would be $v = INTERCEPT + SLOPE * t$. The intercept will be the value of v at $t = 0$, and the symbol v_o is commonly used to represent that. The SLOPE is just dv/dt which will be a (the acceleration), so ultimately our velocity equation MUST be of the form $\boxed{v(t) = v_o + at}$.

The blue line at the top has a positive slope, representing a positive acceleration (velocity is increasing with time).

The green line in the middle has a zero slope, representing an acceleration of zero (velocity is constant over time).

The purple line at the bottom has a negative slope, representing a negative acceleration (velocity decreasing with time).

The key point here is that **if the acceleration is constant**, the velocity, initial velocity, acceleration and time MUST fit that equation: $v(t) = v_o + at$. It's a **constraint** on the object's motion.

$\boxed{\text{Example: accelerating vehicle}}$

Let's say that it took my old truck 20 seconds to accelerate from 20 mph to 60 mph, and let's assume that the accelerating over this interval is constant (it actually isn't constant in the case of real vehicles, but for now let's assume that it is).

(a) What acceleration does this represent?

(b) How far did the truck travel during this interval?

We want to write a $v(t)$ equation describing this motion, so we'll need v_o and the acceleration a.

If we start our stop-watch when the truck is travelling at 20 mph, we'll have $v_o = 20\ mph$. Let's convert this to standard metric units though, which means we need to convert from miles to meters, and from hours to seconds.

From the inside cover of the book, we know that one miles is equal to 1.609 km, and we know there are 1000 m in one kilometer. We also know that one hour is 60 minutes, and each minute has 60 seconds. This all gives us a series of units-conversion factors that we can use:

$$20\ mph = \frac{20\ miles}{1\ hour} \times \frac{1.609\ km}{1\ mile} \times \frac{1000\ m}{1\ km} \times \frac{1\ hour}{60\ min} \times \frac{1\ min}{60\ sec} = 8.93888..\ m/s.$$
$$60\ mph = \frac{60\ miles}{1\ hour} \times \frac{1.609\ km}{1\ mile} \times \frac{1000\ m}{1\ km} \times \frac{1\ hour}{60\ min} \times \frac{1\ min}{60\ sec} = 26.81666...\ m/s.$$

Note how the various fractions are organized - each one cancelling a previous unit and replacing it with a new one.

This was the initial velocity (for the 20 second interval we're focused on), so here $\boxed{v_o = 8.94\ m/s}$.

The acceleration is the time derivative of the velocity, i.e. the slope of the $v(t)$ graph. Since we're dealing with a line, the slope will have the same value all across the line so we can just use the end points and use the $slope = rise/run$ method: $a = \frac{\Delta v}{\Delta t} = \frac{(26.82-8.94)\ m/s}{(20-0)\ s} = 0.89\ m/s^2$.

That's a pretty feeble acceleration, and most vehicles are capable of at least a few m/s^2 of acceleration. According to wikipedia, the production car with the highest acceleration (so far) was the 2015 Porsche 918 Spyder, with an acceleration of $a = 12.8\ m/s^2$. (Note that real vehicles don't maintain a constant acceleration, so over the 2.1 sec it took this car to get from 0 to 60 mph it's acceleration would vary a bit above and below that value.)

If we expand our search to include ludicrously expensive cars where only a few are built, the current record appears to be held by the (electric) Aspark Owl, which goes from 0 to 60 mph in 1.69 s, representing an acceleration of 15.87 m/s^2, which is about 18 times what my old truck was capable of (and is over 200 times as expensive...).

To finish off with my truck then, it's velocity function would be $v(t) = 8.94 + 0.89t$ (with time in seconds, yielding a velocity in m/s).

Technically, if we wanted to carefully keep track of units, that should be written as: $v(t) = (8.94\ m/s) + (0.89\ m/s^2) * t$.

How far did my truck travel during this 20 second interval?

What information do we have? We know the truck was initially travelling at $v_o = 8.94\ m/s$ and accelerated up to $v = 26.82\ m/s$ over a time interval of 20 s.

Given only the equations we have so far, how can we determine the displacement Δx? The only equation we have that involves Δx is $v_{avg} = \Delta x/\Delta t$, so we can write $\Delta x = v_{avg}\Delta t$. We know the time (20 seconds) but what about the average velocity? Looking at the $v(t)$ graph we see that the velocity is varying **linearly**. In this situation, the average of all those points is actually equal to the average of the two endpoints: $\boxed{v_{avg} = \frac{1}{2}(v_o + v)}$. (NOTE that this generally only true when the function is a line like this, so we can **only** use this when the acceleration is **constant**.)

Here then, $v_{avg} = (0.5)(8.94 + 26.82)\ m/s = 17.88\ m/s$.

And finally: $\Delta x = v_{avg}\Delta t = (17.88\ m/s)(20\ s) = 357.6\ m$.

If we leave things symbolic, we can derive an equation for position directly, which we'll do on the next page:

Position Equation of Motion

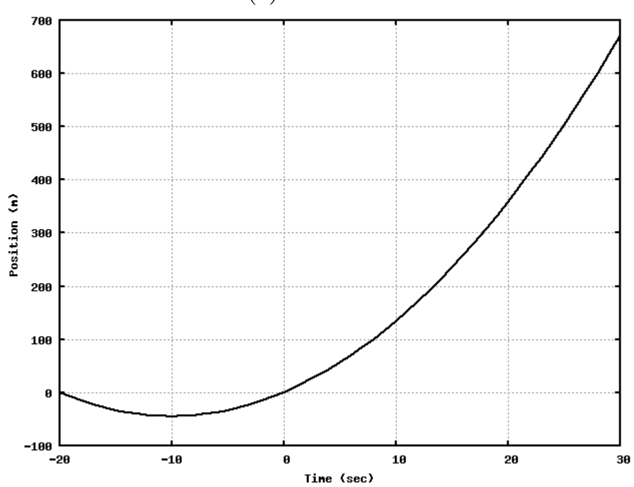
- Start with $\Delta x = v_{avg}\Delta t$
- $\Delta x = x - x_o$ and we're starting our stopwatch at $t = 0$ so $\Delta t = t$
- $v_{avg} = \frac{v_o+v}{2}$ but $v = v_o + at$ so we can write $v_{avg} = \frac{v_o+v_o+at}{2}$ or simply $\boxed{v_{avg} = v_o + \frac{1}{2}at}$
- $v_{avg} = v_o + \frac{1}{2}at$ and $\Delta x = v_{avg}\Delta t$ so:
- $x - x_o = (v_o + \frac{1}{2}at) * t$ or $\boxed{x = x_o + v_o t + \frac{1}{2}at^2}$

Position equation of motion for the truck: $x_o = 0$, $v_o = 8.94\ m/s$, and $a = 0.894\ m/s^2$ so:

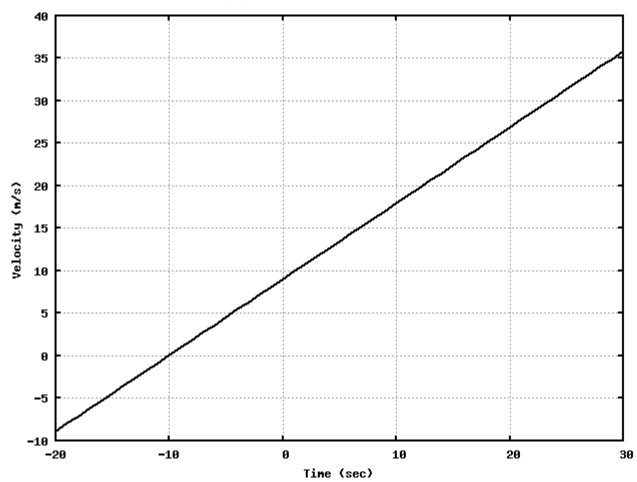
$x(t) = 0 + 8.94t + (0.5)(0.894)t^2$ or $x(t) = 0 + 8.94t + 0.447t^2$

These functions are defined for any value of t, so it's tempting to graph them, which I did here from $t = -20\ s$ to $t = +30\ s$. I purposely went a bit outside of the 0 to 20 second time interval where we had data.

Position : $x(t) = 0 + 8.94t + 0.447t^2$



Velocity : $v(t) = v_o + at = 8.94 + 0.894t$



The problem here is that we only know that $a = 0.894\ m/s^2$ during the time window from $t = 0$ to $t = 20\ s$. Prior to reaching the onramp at $t = 0$, we have no information about the truck's motion. It most likely was moving at a constant speed of 20 mph before starting this acceleration phase, so before $t = 0$ and acceleration was almost certainly **not** the same. And after the truck reaches highway speed at $t = 20\ s$, is probably didn't continue accelerating, but rather just maintained a constant 60 mph velocity.

That means the only part of these graphs is valid: the segment between $t = 0$ and $t = 20\ s$. We'll encounter this situation multiple times throughout the course, where we take some given data and generate equations for the object but those equations may not be valid (and in most cases are definitely not valid) outside of a certain window.

The equations of motion we've derived so far mostly involve the time (t) variable, but we can combine them to create another useful equation that doesn't involve the time.

- $v_{avg} = \Delta x / \Delta t$, which we can rearrange into: $\Delta x = t * v_{avg}$
- Now, $v_{avg} = \frac{v_o + v}{2}$, which I'll rearrange to write as: $\boxed{v_{avg} = \frac{v + v_o}{2}}$
- $v = v_o + at$, so let's rearrange that: $\boxed{t = \frac{v - v_o}{a}}$
- Going back to the first item above, we'll replace both terms using the boxed expressions:
$$\Delta x = t * v_{avg} = \left(\frac{v - v_o}{a}\right)\left(\frac{v + v_o}{2}\right)$$
- Combining the terms on the right: $\Delta x = \frac{v^2 - v_o^2}{2a}$
- One final rearrangement yields: $\boxed{v^2 = v_o^2 + 2a\Delta x}$

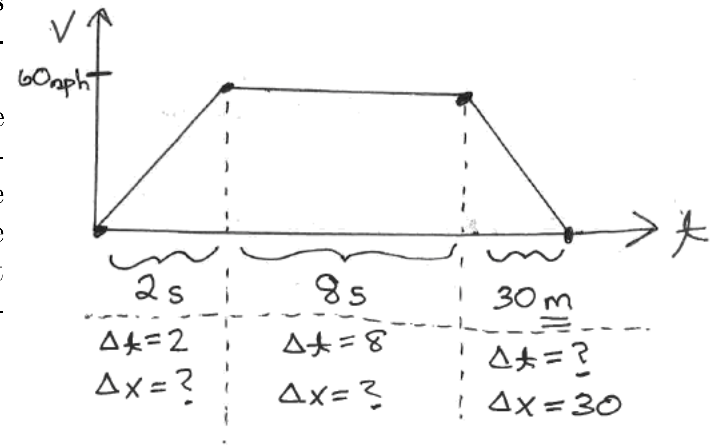
Our toolkit for analyzing motion now consists of:

Definitions (1-D motion)	
Displacement	$\mathbf{\Delta x}$
time interval	$\mathbf{\Delta t}$
Average velocity	$\mathbf{v_{avg} = \Delta x / \Delta t}$
Instantaneous velocity	$\mathbf{v = dx/dt}$
Average acceleration	$\mathbf{a_{avg} = \Delta v / \Delta t}$
Instantaneous acceleration	$\mathbf{a = dv/dt}$

One-Dimensional Equations of Motion   **$\Longrightarrow$ *ONLY* if constant acceleration $\Longleftarrow$**
$\mathbf{v = v_o + at}$
$\mathbf{v_{avg} = \frac{1}{2}(v_o + v) = v_o + \frac{1}{2}at}$
$\mathbf{x = x_o + v_{avg} * t}$
$\mathbf{x = x_o + \frac{1}{2}(v_o + v)t}$
$\mathbf{x = x_o + v_o t + \frac{1}{2}at^2}$
$\mathbf{v^2 = v_o^2 + 2a\Delta x}$

Example : Suppose a particular high performance car accelerates from rest to **60 MPH** in **2.0 sec**. It continues at that velocity along a straight road for **8 seconds**, then slams on the brakes and comes to a stop in **30 meters**.
(a) What was the car's **average velocity** from start to stop?

Here's a sketch of what we know. We have **times** for the first two segments of the scenario, but **distance** for the third segment.
The average velocity is $v_{avg} = \Delta x/\Delta t$, so if we want to find the average velocity over the entire scenario, we'll need to find the total distance travelled, and the total time involved. We have three different accelerations going on here, but can break the problem into three segments, analyze each one, then combine the results.
Converting to metric:
$60\frac{miles}{hr} \times \frac{1609\ m}{1\ mile} \times \frac{1\ hr}{3600\ s} = 26.82\ m/s$



Segment 1 : 0 to 26.82 m/s in 2 seconds. How far did the car travel?

One option is to find the acceleration and use the $x(t)$ equation of motion: $v = v_o + at$ so here $26.82 = 0+(a)(2)$ from which $a = 13.41\ m/s^2$. Then: $x = x_o + v_o t + \frac{1}{2}at^2 = 0+0+(0.5)(13.41)(2)^2 = 26.82\ m$.

Another option: $\Delta x = v_{avg}\Delta t$ and during this segment, $v_{avg} = \frac{v_o+v}{2} = \frac{0+26.82}{2} = 13.41\ m/s$, so $\Delta x = v_{avg}\Delta t = (13.41\ m/s)(2\ s) = 26.82\ m$.

Segment 2 : constant 26.82 m/s for 8 seconds. How far did it travel?

On this segment $a = 0$, so the velocity doesn't change. $v_{avg} = \frac{v_o+v}{2} = \frac{26.82+26.82}{2} = 26.82\ m/s$ and $\Delta x = v_{avg}\Delta t = (26.82\ m/s)(8\ s) = 214.56\ m$

Segment 3 : car drops from 26.82 m/s to 0 m/s in 30 m. Find the time involved.

One option: find the acceleration, then use the velocity equation of motion to find the time:
$v^2 = v_o^2 + 2a\Delta x$ so here: $(0)^2 = (26.82)^2 + (2)(a)(30)$ from which $a = -11.989\ m/s^2$. Then: $v = v_o + at$ so $0 = 26.82 - 11.989t$ from which $t = 2.24\ s$ (for this segment).

Another option: $v_{avg} = \frac{v_o+v}{2} = \frac{26.82+0}{2} = 13.41\ m/s$, then $\Delta x = v_{avg}\Delta t$ so $30 = (13.41)(\Delta t)$ from which $\Delta t = 2.24\ s$.

Combining the three segments now

- Total distance travelled is $\Delta x = 26.82 + 214.56 + 30 = 271.38\ m$
- Total time involved: $\Delta t = 2 + 8 + 2.24 = 12.24\ s$
- Overall average velocity: $v_{avg} = \Delta x/\Delta t = (271.38\ m)/(12.24\ s) = 22.17\ m/s$

Looking at the sketch, the car spent most of its time at the cruising speed of 26.82 m/s, so we'd expect the overall average velocity (22.17 m/s) to be just a little lower than that, which it was.