PH2213 Fox : Lecture 03
Chapter 2 : Kinematics in One Dimension

Our toolkit for analyzing motion now consists of:

Definitions (1-D motion)	
Displacement	$\mathbf{\Delta x}$
time interval	$\mathbf{\Delta t}$
Average velocity	$\mathbf{v_{avg} = \Delta x/\Delta t}$
Instantaneous velocity	$\mathbf{v = dx/dt}$
Average acceleration	$\mathbf{a_{avg} = \Delta v/\Delta t}$
Instantaneous acceleration	$\mathbf{a = dv/dt}$

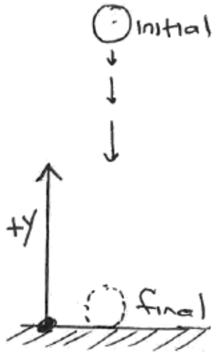
One-Dimensional Equations of Motion $\Longrightarrow$ ***ONLY* if constant acceleration** $\Longleftarrow$
$\mathbf{v = v_o + at}$
$\mathbf{v_{avg} = \frac{1}{2}(v_o + v) = v_o + \frac{1}{2}at}$
$\mathbf{x = x_o + v_{avg} * t}$
$\mathbf{x = x_o + \frac{1}{2}(v_o + v)t}$
$\mathbf{x = x_o + v_o t + \frac{1}{2}at^2}$
$\mathbf{v^2 = v_o^2 + 2a\Delta x}$

Let's look at one common type of motion where acceleration is constant:

Free-fall motion : the object is **only** moving under the influence of gravity, which means it has an acceleration directed **downward** towards the earth (or moon, or other object). The **magnitude** of the acceleration due to gravity at the surface of the **earth** is approximately $|\mathbf{a}| = \mathbf{g} = \mathbf{9.80\ m/s^2}$.

- The entity **g** is always a **positive constant**.
- The **acceleration** may be either $\mathbf{a = +9.8\ m/s^2}$ or $\mathbf{a = -9.8\ m/s^2}$ depending on **your** choice of coordinate system.

Example : A ball is released (at rest) from a height of 2 meters above the floor. (a) How much time does it take for the ball to hit the floor? (b) How fast is the ball moving at that point?



Solution 1 In this version, we'll solve for the TIME first, then use that to aid in finding the velocity.

Let's use a **coordinate system** where the origin is on the floor below the ball, with the positive direction being upward, as shown in the figure on the right.
We need to associate the information we have with the related variables in our equations of motion but can't do that until we've defined our coordinate system.

This is particularly important in the case of the **acceleration**. IF we have a coordinate system pointing upward, we know that if we throw a ball upward it will gradually slow down. That initial large positive (upward) velocity is getting smaller and smaller, so a graph of $v(t)$ has a downward (i.e. negative) slope, meaning the acceleration due to gravity in this case is negative. That's entirely due to our choice of coordinates though. If we release a ball from rest in a coordinate system that's pointing downward, it's velocity will get take on higher and higher positive values: that graph of $v(t)$ will have a positive slope (and therefore a positive acceleration).

In our upward-pointing coordinate system, we have:

- initial position: $y_o = +2\ m$
- initial velocity: $v_o = 0\ m/s$
- acceleration: $a = -9.8\ m/s^2$: see discussion above; with our choice of coordinates (positive upward), the acceleration is down towards the earth, so a is negative, in our coordinate system
- final position: $y = 0\ m$

Perusing our available equations of motion, the only viable option to directly determine the **time** based on what we do know is: $y(t) = y_o + v_o t + \frac{1}{2}at^2$

Inserting what we know this becomes: $y(t) = 2 + (0)(t) + \frac{1}{2}(-9.8)t^2$ or: $\boxed{y(t) = 2 - 4.9t^2}$

This gives us the position of the ball as a function of time. Mathematically, this equations runs over all possible times from negative to positive infinity, but only a small segment of this graph represents the actual motion of the ball: from $t = 0$ when it was released, until some time later when the ball first hits the floor. **(See next page for figures.)**

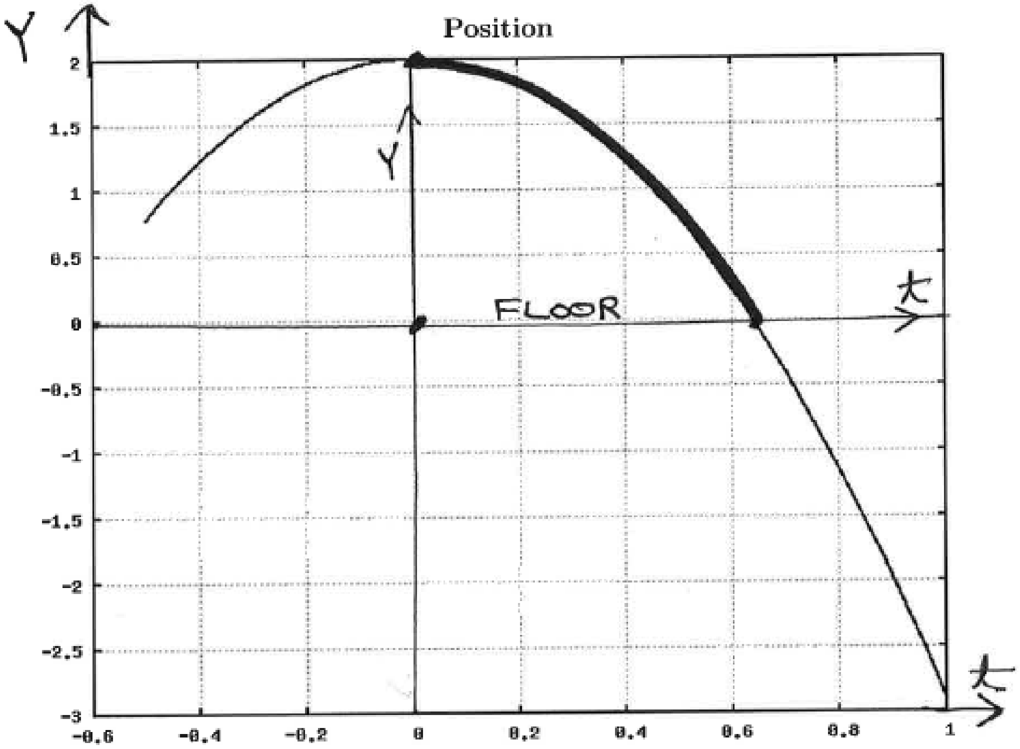
When the ball hits the floor, it will have a y value of zero so we can use that information to determine the time: $y(t) = 2 - 4.9t^2$ so $0 = 2 - 4.9t^2$ or $4.9t^2 = 2$

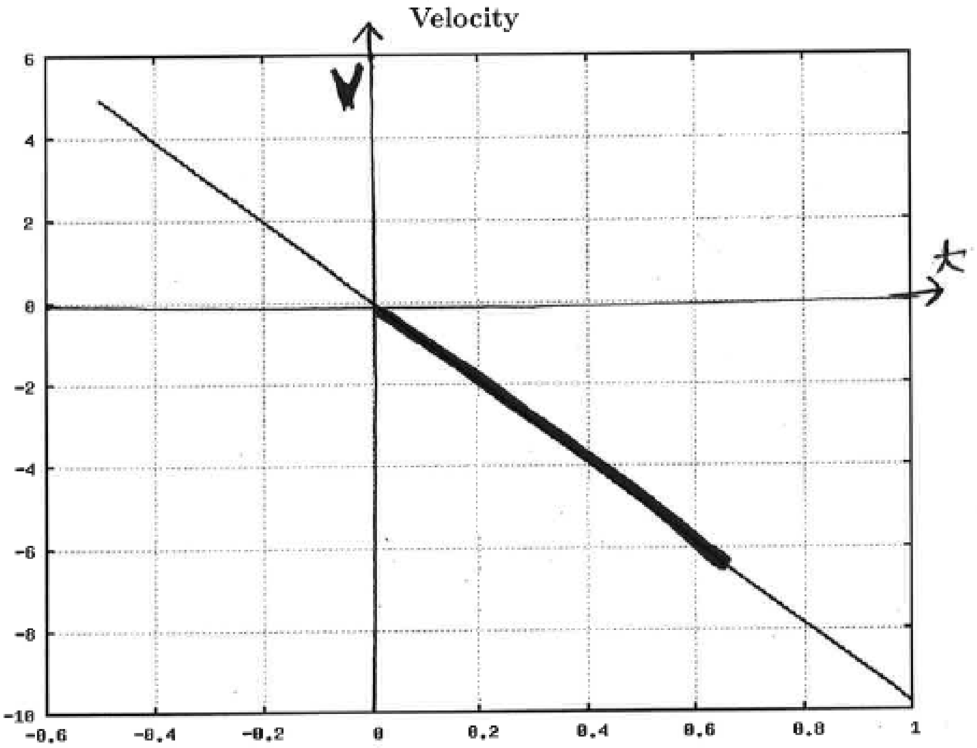
The solution then is $t = \pm\sqrt{(2/4.9)} = \pm 0.63888\ sec.$

We know the ball will hit the ground **after** it was released at $t = 0$, so the actual solution must be $\boxed{t = +0.63888\ s}$.

Velocity: $v(t) = v_o + at$ so here we have $v(t) = 0 + (-9.8)t$ or just $v(t) = -9.8t$. Evaluating that at $t = 0.63888\ s$, we have $\boxed{v = -6.261\ m/s}$.

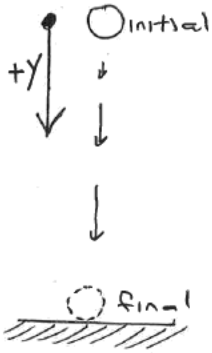
Figures for version 1 of the solution (with positive upward)





Note: the ball was released at $t = 0$ and it hit floor at $t = 0.6389\ s$ so only that segment of the curve is valid (and is highlighted with a thicker line).

Solution 2 In this version, we'll solve for the VELOCITY first, then use that to aid in finding the time.



This time, let's use a **coordinate system** where the origin is located at the ball's initial position, and let's have the positive direction being **downward**. This is actually a common choice also: placing the origin where the problem starts, and have positive be in the direction the thing is initially moving. Here, the ball will start falling downward once released, so we'll make 'down' our positive direction this time.

The ball is going faster and faster once we release it; it's moving downward (our positive direction), so it's velocity is taking on larger and larger positive values as time increases. The slope of this graph is positive, giving us a positive acceleration this time: $a = +9.8\ m/s^2$.

With this coordinate system choice (which had to be made first), label all the variables we know or desire: $v_o = 0$, $y_o = 0\ m$, and $y_{final} = +2\ m$. Our unknowns are t and v at the floor.

One solution : find VELOCITY first : given the variables we have and what we're looking for (v at the floor in this case), our only choice really is:

$$v^2 = v_o^2 + 2a\Delta y.$$

$\Delta y = y_{final} - y_{initial} = 2 - 0 = 2\ m$

Putting this together: $v^2 = (0)^2 + (2)(9.8)(2) = 39.2$ from which $v = \sqrt{39.4} = \pm 6.261\ m/s$.

We know the ball is travelling downward when it hits the floor, and we've defined 'down' to be our positive direction, so that resolves the sign ambiguity, leaving us with $\boxed{v = +6.261\ m/s}$.

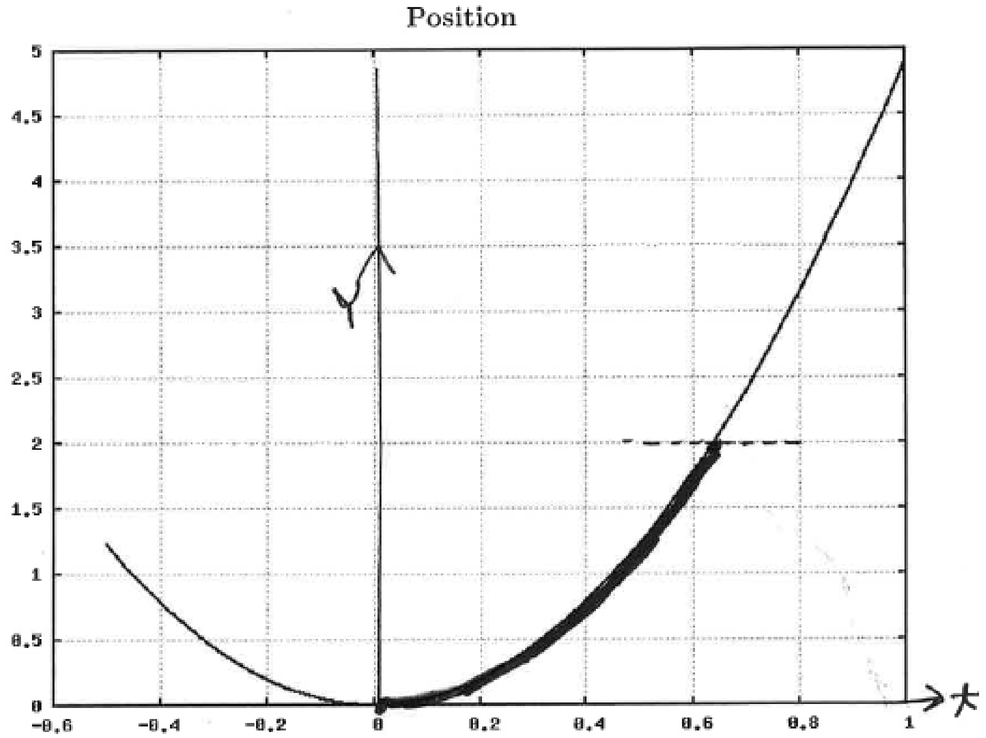
We can now use that information to determine the time when the ball hits the floor: $v = v_o + at$ so $6.262 = 0 + 9.8t$ from which $\boxed{t = +0.63888\ s}$.

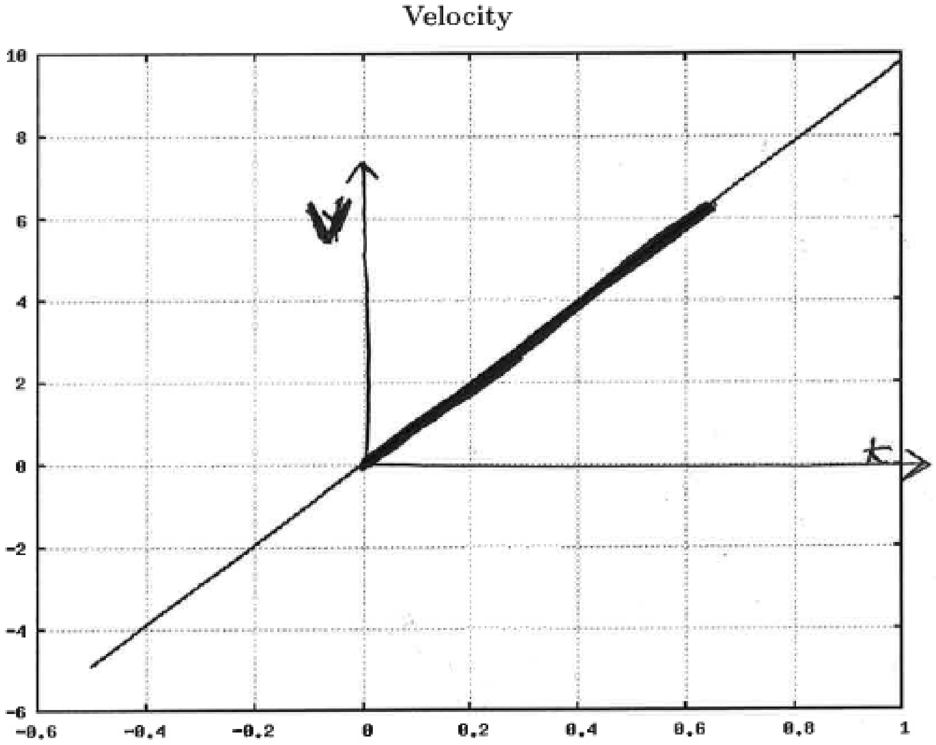
Note we got the same TIME either way; our velocity had the same magnitude each way, but a different sign depending on our coordinate system.

Another Solution : find TIME first : we could determine the time first, then use that to find the velocity. With the coordinate system we chose here, the ball is starting at rest, at $y = 0$. The acceleration is downward and that's our positive direction now, so $a = +9.8\ m/s^2$. Our position equation of motion $y = y_o + v_o t + \frac{1}{2}at^2$ then becomes just: $y = 4.9t^2$. The ball hits the ground at $y = 2$, so that implies a time of: $2 = 4.9t^2$ from which $t = \pm 0.63888\ s$. The ball hits the ground AFTER being released, so the solution must be $t = +0.63888\ s$.

Then $v = v_o + at = 0 + 9.8t = (9.8\ m/s^2)(0.63888\ s) = +6.261\ m/s$. (Remember, we're using a coordinate system with positive downward, so that positive velocity means the ball is travelling downward when it hits the floor.)

Figures for version 2 of the solution : positive downward



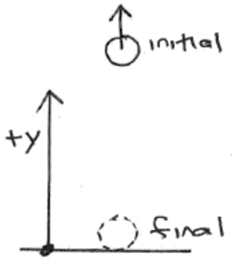


Note: the ball was released at $t = 0$ and it hit floor at $t = 0.6389\ s$ so only that segment of the curve is valid (and is highlighted with a thicker line).

Let's try a slightly different problem. If we **throw** the ball **vertically upward** initially, it will continue to travel upwards for a while before reaching some maximum height (called the **apogee**) where it stops and then falls back down towards the ground.

Example : A ball is tossed upward at 4 m/s from a height of 2 meters above the floor. (a) How much time does it take for the ball to hit the floor? (b) How fast is the ball moving at that point? (c) Find the maximum height above the floor that the ball reaches.

Coordinates : Let's use the same coordinate system we used in the first solution earlier: the origin will be on the ground below the ball, and we'll have **positive** pointing **upward**.
In this coordinate system, our initial variables will be: $y_o = +2$, $v_o = +4\ m/s$, $a = -9.8\ m/s^2$, and when the ball finally hits the ground, it will have a y coordinate of $y_{final} = 0\ m$.



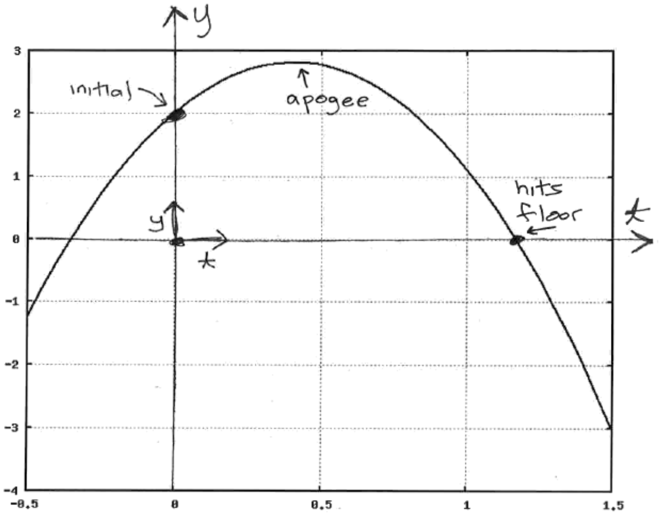
Solving for the time first:

$$y = y_o + v_o t + \frac{1}{2}at^2 = 2 + 4t - 4.9t^2.$$

The figure on the right shows this parabola. Setting $y = 0$ we can find the time when the ball hits the ground but there will (mathematically) be two times when this happens.

$$0 = 2 + 4t - 4.9t^2.$$

The quadratic formula gives us two solutions: $t = 1.16628...\ s$ and $t = -0.349966..\ s$. The actual physical trajectory of the ball is only valid from $t = 0$ (when it was released) until some later time when it hits the ground, so that leads to the solution being: $\boxed{t = 1.16628\ s}$.



We can find the velocity at that time using our velocity equation of motion: $v = v_o + at = 4 - 9.8t = 4 - 9.8(1.16628)$ so $\boxed{v = -7.42967..\ m/s}$.

NOTE: we can approach this problem by finding v first, just like we did in the previous example.

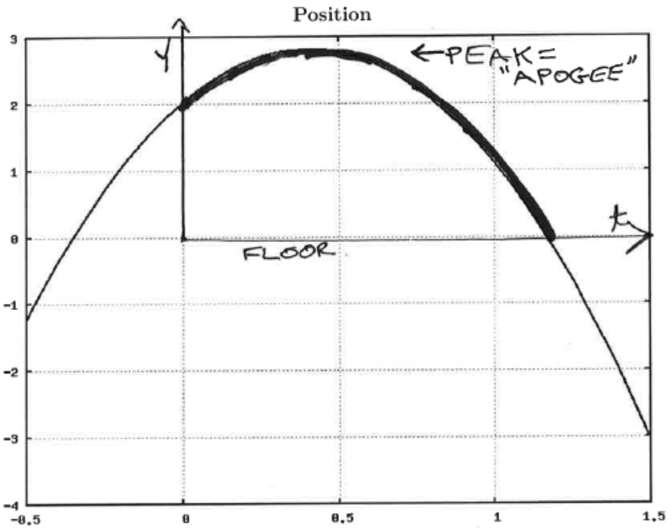
$v^2 = v_o^2 + 2a\Delta y$ and again $\Delta y = y_{final} - y_{initial}$ so here $\Delta y = 0 - 2 = -2\ m$. With our positive-upward coordinate system, the change in the ball's Y coordinate from initial to final positions is $-2\ m$.

Here then $v^2 = v_o^2 + 2a\Delta y$ becomes: $v^2 = (4)^2 + (2)(-9.8)(-2) = 16 + 39.2 = 55.2$ so $|v| = 7.4297\ m/s$. $\boxed{v = \pm 7.4297\ m/s}$ and which is it?

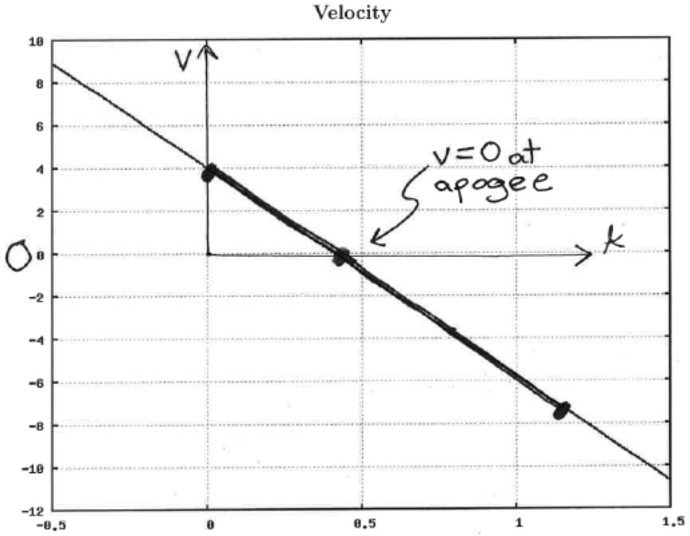
The ball is moving downward at that point, and we've defined positive to be upward, so we must have: $\boxed{v = -7.4297\ m/s}$.

TIME: $v = v_o + at$ so $v = 4 - 9.8t$ and here $-7.4287 = 4 - 9.8t$. Rearrange: $9.8t = 4 + 7.4287 = 11.4287$ from which $t = 1.1663\ s$.

Figures for this example



Position



Velocity

$\boxed{\text{How high up in the air did the ball go?}}$

Looking at the sketch of the parabola representing this motion above, we see that the ball initially rises, reaches a maximum height, then falls back down. Where does that peak occur?

ONE option: calculus. We're looking for the point on the $y(t)$ curve where we have a max/min, so that occurs when $dy/dt = 0$, so could differentiate our $y(t)$ equation. BUT dy/dt is just the velocity, so $\boxed{\text{that point is where } v(t) = 0}$:

$v(t) = 4 - 9.8t$ so setting $v = 0$ at that point: $0 = 4 - 9.8t$, which implies $\boxed{t = 0.4082\ s}$. That is the TIME when the ball reaches its apogee (highest point).

Evaluating $y(t)$ at that point to determine it's position: $y(t) = 2 + 4t - 4.9t^2 = 2 + (4)(0.4082) - 4.9(0.4082)^2 = 2.816\ m$ (or about 82 cm above where it started).

We can also find the height directly without needing to find the time first.

Let's apply the equation: $v^2 = v_o^2 + 2a\Delta y$ between the starting point and the point where the ball has reached it's apogee. At that point, $v = 0$ so:

$(0)^2 = (4)^2 + (2)(-9.8)\Delta y$ or $0 = 16 - 19.6\Delta y$ from which $\Delta y = +0.816\ m$.

$\Delta y = y - y_o = y - 2.0$ though, so $y = 2 + 0.816 = 2.816\ m$ (same as before, of course).

> There are almost always **multiple paths** through the **equations of motion** to take data we have and infer other quantities. **Part of the solution to any problem though is defining what coordinate system you are using, and then making sure everything is consistent with that choice (especially signs).**

Let's exercise our equations of motion where two objects (with potentially different accelerations) are involved.

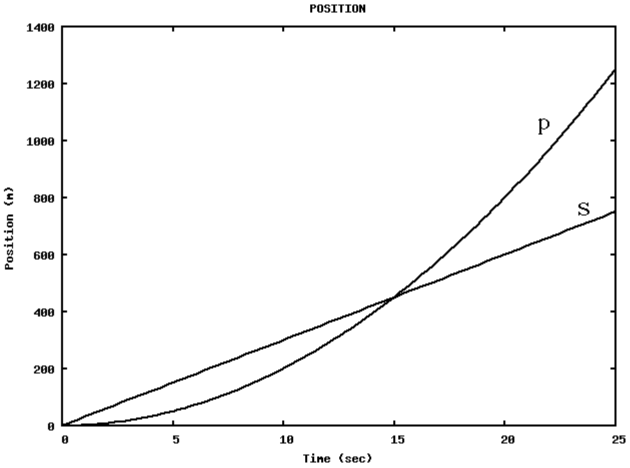
Example 1 : A speeding car is travelling down a straight road at 30 m/s. A police car on the side of the road sees the speeder coming and at the instant the speeder passes the police car, the police car starts accelerating after the speeder with an acceleration of 4 m/s^2. The police car eventually catches up to the speeder: when and where does that happen?

We have two objects here, each with their own set of equations of motion. Each has a constant acceleration (zero for the speeder, 4 m/s^2 for the police car). One way of solving this would be to use a single coordinate system to describe each vehicle and then their meeting spot would correspond to the point where they both have the same X coordinate.

Let's define our coordinate system so that $x = 0$ where the police car is initially siting at rest. And we'll let $t = 0$ be the instant the speeder passes the police car. We'll let the $+X$ direction be along the road in the direction the speeder is travelling.

The generic position equation for an object (when acceleration is constant) is given by $x = x_o + v_o t + \frac{1}{2}at^2$.

To simplify writing all these equations, let's make sure everything is in standard metric units and then drop writing them. Then: $\boxed{x_s = 30t}$ gives the position of the speeder as a function of time. For the police car, they're starting at $x = 0$ at rest and accelerating at 4 m/s^2 so this vehicle's equation of motion would be $x_p = 0 + 0 + \frac{1}{2}(4\ m/s^2)t^2$ or $\boxed{x_p = 2t^2}$.
The graph here shows the position of the two objects as a function of time.



Position

When the police car catches up to the speeder, they'll both have the same value of x, so setting the equations equal to one another: $x_p = x_s$ or $\boxed{2t^2 = 30t}$.

We can write this as $2(t - 15)(t) = 0$ which has **two** solutions: $t = 0$ sec and $t = 15$ sec.

If we plug $t = 0$ into the equations of motion, we get $x_s = x_p = 0$ and this solution corresponds to the point where the speeder is just passing by the stationary police car at the beginning of the problem. (Meaning it's not the solution we're looking for.)

The $t = 15$ solution then is the later time when the police car has caught up to the speeder.

At this time, $x_p = 2t^2 = 2(15)^2 = 450\ meters$ and $x_s = 30t = (30)(15) = 450\ meters$ (did both of them to make sure they actually did come out to the same value).

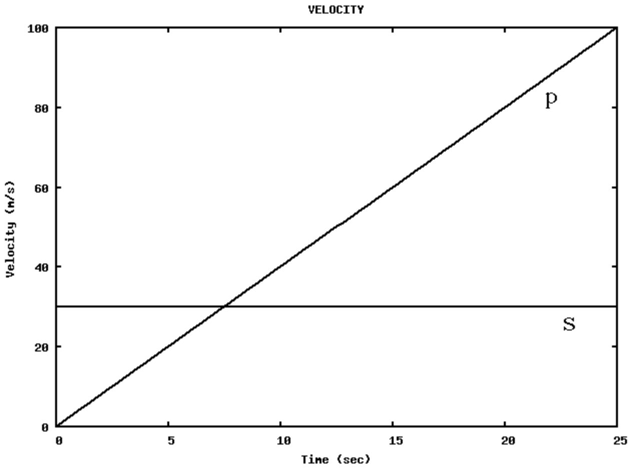
How fast are the vehicles moving at this time?

The generic equation of motion for velocity is $v(t) = v_o + at$.

The speeder is moving at a constant of 30 m/s, so has an acceleration of zero. The speeder's velocity equation of motion would be $v_s(t) = 30$.

The police car is accelerating, so $v = v_o + at$ becomes $v_p(t) = 0 + 4t$ and evaluating this at the meeting time: $v_p = (4)(15) = 60\ m/s$ (a bit over 130 miles/hour).

This graph below shows the velocities of the two vehicles as functions of time.

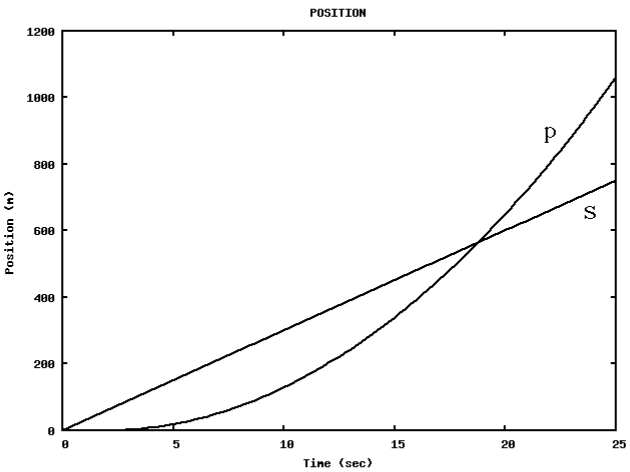


Velocity

Example 2 : Let's look at a more realistic version of the same scenario. Suppose the police car doesn't start accelerating right away: instead, let's say it takes **2 seconds** for the driver to react before starting to accelerate after the speeder.
The police car will still eventually catch up to the speeder, but when and where does that happen now?

Basically, the police car's 'graph' of position and velocity is the same as it was before, just **delayed** by 2 seconds. That is, the graph of it's equations is shifted to the right by 2 seconds.

We need to be able to write an equation for the position of each vehicle as a function of time and then set them equal to one another. How do we do that for the police car though? For two seconds, they're just sitting at rest and only then do they start accelerating?



Position

One approach is to shift our coordinate system so that $t = 0$ represents when the police car starts accelerating, but that means we need to adjust the speeder's equations of motion to reflect this. See **example 9** in the **examples02.pdf** file on Canvas for this approach.

Another approach to these 'delay' type problems is to adjust the original equation using a trick you may have seen before. If you're comfortable with the process, this can sometimes simplify the solution.

If we have some function $x(t)$ and we want to 'shift' it's shape to the right by 2 seconds, all we have to do is take the original $x(t)$ function and symbolically replace t with $t-2$ everywhere. (Similarly, shifting a function to the left means replacing the independent variable everywhere in the function by $t+(shift)$ instead of $t-(shift)$.)

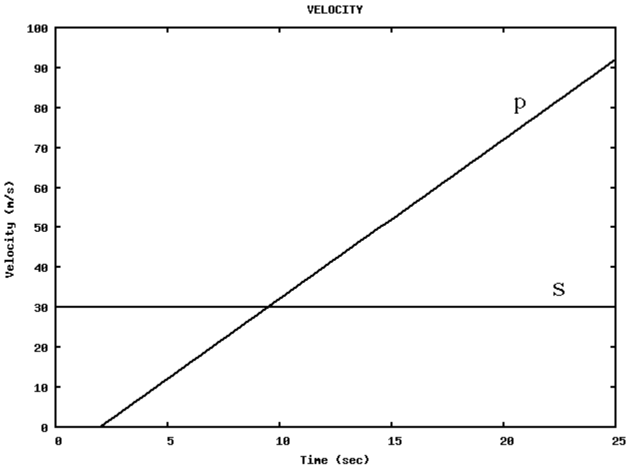
In the original problem, the police car's equation of motion was $x_p(t) = 2t^2$ so we can shift that parabola 2 seconds to the right by replacing t with $t-2$:
$$x_p(t) = 2(t-2)^2 = 2(t^2 - 4t + 4) = 2t^2 - 8t + 8.$$

Now $x_s = x_p$ becomes $30t = 2t^2 - 8t + 8$ or after collecting terms: $2t^2 - 38t + 8 = 0$ which has solutions $t = 0.2129\ sec$ or $t = 18.787\ sec$, so it took a few seconds longer for the police car to catch up to the speeder.

Warning : Be careful interpreting times here though. That 18.787 sec time starts at the point where the speeder passes the police car's original location. From the police car's point of view, they don't start moving until $t = 2$, so they were accelerating for $18.787 - 2 = 16.787\ sec$ (still longer than the 15 seconds it took them to catch up to the speeder before though).

How about their velocities this time?

With constant acceleration, $v = v_o + at$ in general so for the speeder we still have $v_s = 30 + (0)t$ or just $v_s = 30\ m/s$. (Replacing t with $t-2$ doesn't affect this equatio since t doesn't explicitely appear on the RHS anywhere.)
For the police car, our original equation was $v_p = 0 + 4t$ and when we shift this equation 2 seconds to the right, our **new** velocity equation would be $v_p = 0 + (4)(t-2) = 4t - 8$ (m/s).
This equation is only valid for $t >= 2$ of course; before when $v_p = 0$.

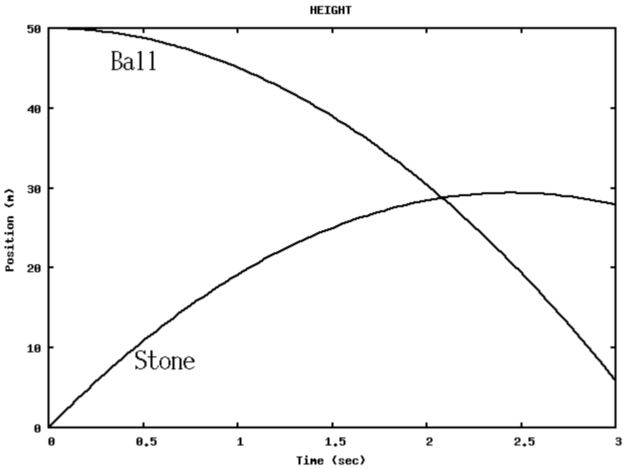


Velocity

At $t = 18.787\ s$ then, the police car has a velocity of $v_p = 4(18.787) - 8 = 67.15\ m/s$ or about 150 $miles/hr$. It took them longer to catch up this time, so their constant 4 m/s^2 acceleration has built up to a higher 'final' velocity than before.

Example 3 : Basically chapter 2 homework problem 71. A ball is dropped from the top of a 50 m high cliff. **At the same time**, a carefully aimed stone is thrown straight up from the bottom of the cliff at a speed of 24 m/s. The stone and ball collide part way up. (a) How far above the base of the cliff does this happen? (b) What velocities do the ball and stone have when they collide?

COORDINATES: Let's use a Y axis with positive upward and with $y = 0$ at the base of the cliff.



Position

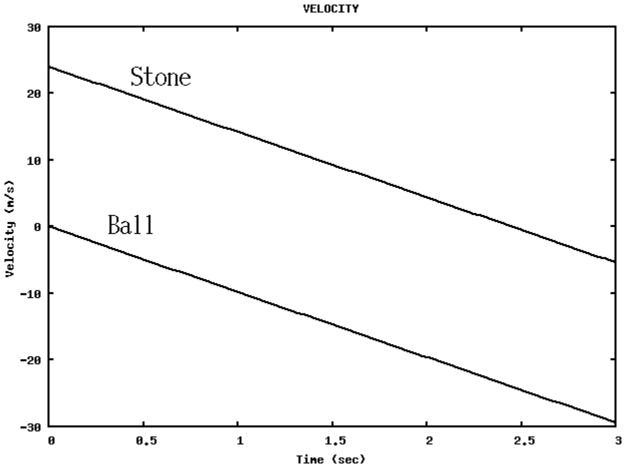
In these coordinates, the ball (A) starts at $y_o = 50\ m$, at rest so $v_o = 0$, and it's accelerating downward under the influence of gravity so $a = -9.8\ m/s^2$. The generic equation of motion for the ball: $y(t) = y_o + v_o t + \frac{1}{2}at^2$ becomes: $\boxed{y_A = 50 - 4.9t^2}$
The stone (B) starts at ground level, moving upward initially at 24 m/s, but it's also accelerating downward under the influence of gravity, so it's acceleration is also $a = -9.8\ m/s^2$. The generic equation of motion applied to the stone then becomes: $\boxed{y_B = 0 + 24t - 4.9t^2}$

The objects will be at the same y coordinate when they meet, so setting $y_A = y_B$ yields:

$$50 - 4.9t^2 = 24t - 4.9t^2 \text{ or } 50 = 24t \text{ from which } \boxed{t = 2.0833\ s}.$$

Where are the objects at this meeting point? (They have to be at the same y coordinate, so I'll calculate both as a check.)

- Ball : $y_A = 50 - 4.9t^2 = 50 - 4.9(2.0833)^2 = 28.73\ m$
- Stone : $y_B = 24t - 4.9t^2 = (24)(2.0833) - (4.9)(2.0833)^2 = 28.73\ m$



Velocity

How fast are the objects moving when they meet?

- Ball : $v_A = v_o + at = 0 - 9.8t = 0 - (9.8)(2.0833) = -20.42\ m/s$
- Stone : $v_B = v_o + at = 24 - 9.8t = +3.58\ m/s$ (still upward, but much more slowly than initially)

Note that their **velocities** are quite different when they collide: the ball is moving downward when they meet ($v < 0$) but the stone is still moving upward ($v > 0$) at that point.

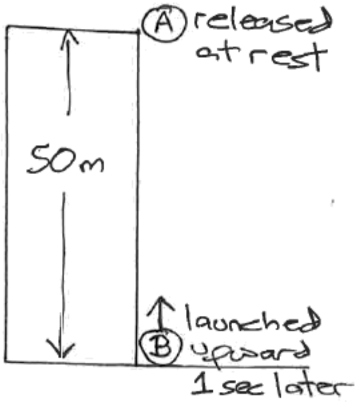
Example 4 : A ball (object A in the figure) is dropped from the top of a 50 m cliff. **ONE SECOND LATER**, a stone (object B) is launched vertically upward from the ground directly below at 24 m/s in an attempt to 'intercept' the ball before it hits the ground. Is it successful? Does the stone manage to strike the ball before it reaches the ground? What velocities do the objects have when they meet?

Like before, we'll use a coordinate system with $+Y$ vertically **upward** (with it's origin located at ground level), and starting $t = 0$ when the ball at the top of the building was released.
As before, the ball (A) will have a position equation of motion of $y_A(t) = 50 - 4.9t^2$
For the stone (B) launched upward from ground level, we basically have the same situation as before BUT delayed by 1 second, so let's use the shift-trick. Our original equation for the stone was $y_B = 24t - 4.9t^2$ so delaying this graph by one second (by replacing t with $t - 1$ everywhere on the RHS):

$$y_B = 24(t - 1) - 4.9(t - 1)^2$$

Expanding that out and recollecting terms:

$$y_B = -28.9 + 33.8t - 4.9t^2$$



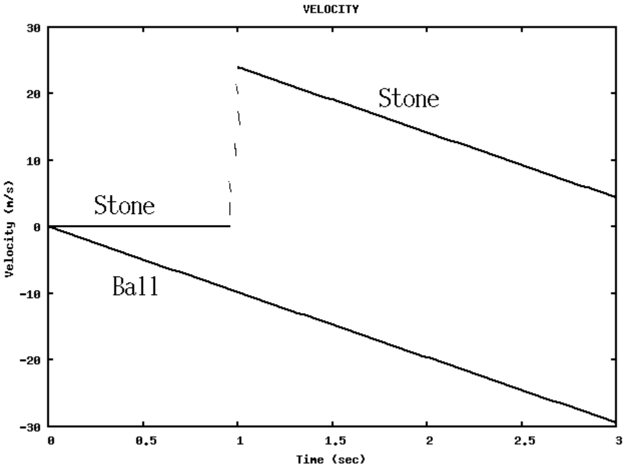
Setting these equal to one another: $50 - 4.9t^2 = -28.9 + 33.8t - 4.9t^2$ (exact at this point)

Cancelling the common $4.9t^2$ term from each side: $50 = -28.9 + 33.8t$ which yields $t = 2.334$ *sec*.

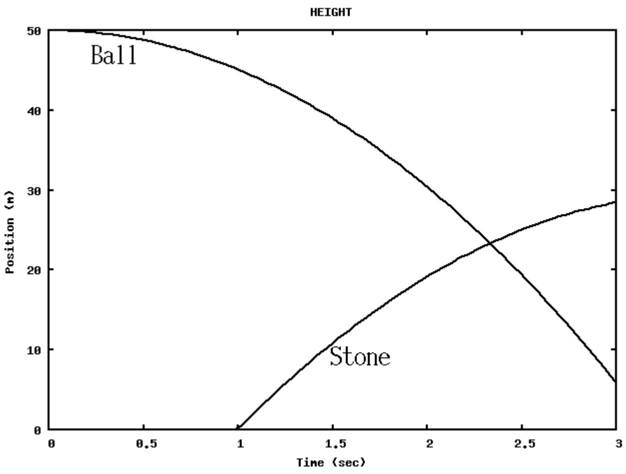
Evaluating each position equation:

- $y_A = 50 - 4.9t^2 = 50 - (4.9)(2.334)^2 = 23.31\ m$
- $y_B = -28.9 + 33.8t - 4.9t^2 = -28.9 + 33.8(2.334) - 4.9(2.334)^2 = 23.30\ m$

(I rounded off the time t value to 4 significant figures, so should expect slight differences in the fourth significant figure of these results...)



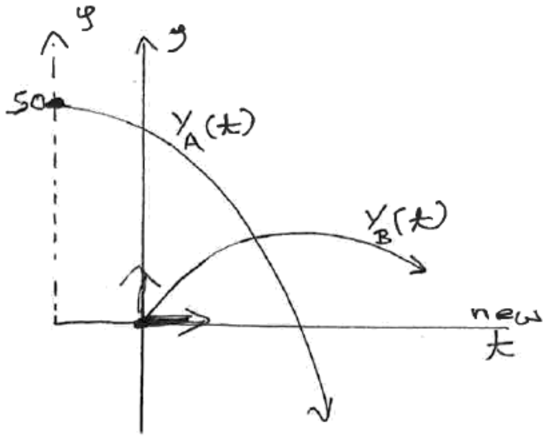
Velocity



Position

Let's attack this problem a different way.

We need acceleration to be constant to use our equations of motion. What if we **wait** for one second and let $t = 0$ be the time when the **stone** (B) is launched from the ground this time? (Instead of defining $t = 0$ as the point when the ball was dropped.)



That makes the stone's equation of motion simple: at this new $t = 0$, the stone is located at $y_o = 0$, is moving upward with a velocity of $v_o = +24\ m/s$ and has an acceleration of $a = -9.8\ m/s^2$ so: $y_B = 0 + 24t - 4.9t^2$.
What about the ball though? At this **new** time that we're now calling $t = 0$, **the ball has been falling for a full second**, so it's no longer at $y = 50\ m$ and its velocity is no longer 0.

Where will object A be one second after it was released? Going back to it's original equation of motion ($y_A(t) = 50 - 4.9t^2$) we find that at the old $t = 1$ it would be located at $50 - 4.9(1)^2 = 45.1\ m$. That will be it's new y_o value in our new, delayed time coordinate. And what is it's velocity at this point? $v_A(t) = v_o + at = 0 - 9.8t$ so at $t = 1$ it will have a velocity of $v = -9.8(1) = -9.8\ m/s$.

Moving to our NEW time coordinate, those will now be the **initial** (what we're now calling $t = 0$) position and velocity values for object A.

Putting this together, our generic position equation of motion: $y(t) = y_o + v_o t + \frac{1}{2}at^2$ becomes:
$y_A(t) = 45.1 - 9.8t - 4.9t^2$.

We now finally have two equations that describe the motion of the two objects (in our new time coordinate that started at $t = 0$ when object B was launched, one second after object A starting falling): $y_A(t) = 45.1 - 9.8t - 4.9t^2$ and $y_B(t) = 24t - 4.9t^2$

When will they collide? Setting $y_A = y_B$ we have: $45.1 - 9.8t - 4.9t^2 = 24t - 4.9t^2$

At least here we get a break. Note that we have the identical t^2 terms on both sides of this equation so can cancel those out, leaving: $45.1 - 9.8t = 24t$
Rearranging: $45.1 = 9.8t + 24t = 33.8t$ or $t = 45.1/33.8 = 1.334\ s$.
The two objects (mathematically) will 'meet' exactly 1.334 *seconds* after object B was launched (which is 2.334 *seconds* after object A was dropped, as we found in the first solution).

Where do the objects meet? : It's possible the two parabolas might mathematically intersect at some negative y value (i.e. below ground) which means that B didn't reach A in time. Evaluating our two expressions for $y(t)$ at $t = 1.334\ s$ we find that $y_A(1.334) = y_B(1.334) = +23.3\ m$ (to 3 significant figures anyway), so B successfully reached A that far above the ground.

What are their velocities when they meet? : Sticking with our new time coordinate, we have:
Ball: $v_A = v_o + at = -9.8 - 9.8t = -9.8 - 9.8(1.334) = -22.9\ m/s$
Stone: $v_B = v_o + at = 24 - 9.8t = 24 - 9.8(1.334) = +10.9\ m/s$

The ball (A) is moving downward rapidly, and the stone (B) has slowed down quite a bit from it's initial launch velocity but is still moving upward when they collide.

ADDENDUM : A ball is dropped from the top of a 50 m building. Simultaneously, a little model rocket is launched vertically upward from ground level to try and intercept the ball. The rocket starts at rest, but accelerates upward at 12 m/s^2. Where and when does the rocket intercept the ball?

We'll use a coordinate system with Y vertically upward and with $Y = 0$ down at ground level.

The generic equation of motion for the ball, $y(t) = y_o + v_o t + \frac{1}{2}at^2$ then becomes $y_{ball} = 50 - 4.9t^2$ (as we've seen multiple times already, so I won't rejustify it here...).

For the rocket, it's starting at rest ($v_o = 0$) down at $y_o = 0$ but then it accelerates upward at $a = +12\ m/s^2$ (in our positive-upward coordinate system). It's equation of motion then is $y_{rocket} = 0 + 0 + \frac{1}{2}(12)t^2 = 6t^2$.

The objects meet when $y_{ball} = y_{rocket}$ so here $50-4.9t^2 = 6t^2$ which we can rearrange into $50 = 10.9t^2$, yielding $t = \pm 2.142\ sec$. They meet after the ball is released, so $\boxed{t = +2.142\ s}$.

Where are the objects at that time? $y_{ball} = 50 - 4.9(2.142)^2 = +27.52\ m$, and as a check: $y_{rocket} = 6t^2 = 6(2.142)^2 = +27.53\ m$. (Note I rounded off the time to 4 significant figures, so should expect the two answers to agree to 3 signficant figures, but probably not to 4.)

(You should try converting this into a delay version, where the ball is allowed to fall for a second before the rocket is launched. Don't forget to use the 'trick' for shifting a function. Here we want to DELAY the rocket's launch, so just take the original (simple) equation of motion for the rocket and replace the t symbol with $t-1$ to create the 1 second delay.)

Addendum: Additional 1-D motion example

Example : A race car, starting from rest, accelerates uniformly and passes the 1/4-mile marker exactly 12 seconds later. Determine (a) the car's acceleration, and (b) it's velocity when it passed by the marker.

What information do we have here?

- It started from rest, so $v_o = 0$
- It's displacement over this interval is 1/4 of a mile; converting that to metric: $0.25\ mile \times \frac{1609\ m}{1\ mile} = 402.25\ m$. Using a coordinate system that starts at $x = 0$ when the car starts accelerating, we have $x_o = 0$ and $x = 402.25\ m$ at $t = 12\ s$, giving us a displacement of $\Delta x = 402.25\ m$.

One Approach : finding acceleration first : Looking through the equations of motion, what equation(s) do we have where we know everything except for the one thing we are seeking. I see only one really: $x = x_o + v_o t + \frac{1}{2}at^2$, which we can use to determine the acceleration. Substituting in the parameters we know, this becomes: $402.25 = 0 + (0)(12) + (0.5)(a)(12)^2$ or $402.25 = 72a$ from which $\boxed{a = 5.5868\ m/s^2}$.

Knowing the acceleration, we can use other equations, such as $v = v_o + at$ to determine the velocity when it reaches the marker: $v = (0) + (5.5868\ m/s^2)(12\ s) = 67.04\ m/s$ (150.0 miles/hr).

Now that we know the acceleration, we can use that to find the velocity. One option: $v^2 = v_o^2 + 2a\Delta x$ since we now know everything there except for the final velocity: $v^2 = (0)^2 + (2)(5.5868\, m/s^2)(402.25\, m) = 4494.58\ m^2/s^2$. Taking the square root of both sides, $|v| = 67.04\ m/s$. (NOTE that technically the math here isn't revealing the sign for v since both $+67.04\ m/s$ and $-67.04\ m/s$ are solutions. We'll see more of this later: how the pure math gives us more than one solutions and we'll need to look back at the scenario to determine which one is 'right.')

Another Approach : finding final velocity first : We know that $v_{avg} = \Delta x/\Delta t$ and $v_{avg} = \frac{1}{2}(v_o + v)$ so that gives us a way. The vehicle covers $\Delta x = 402.25\ m$ in exactly $\Delta t = 12\ s$ so $v_{avg} = (402.25\ m)/(12\ s) = 33.5208\ m/s$.

That's the **average** velocity during this interval, but when the acceleration is constant, v_{avg} is also equal to the average of the starting and ending velocities: $v_{avg} = \frac{1}{2}(v_o + v)$ so here $33.5208\ m/s = (0.5)(0 + v_{final})$ from which $v_{final} = 67.04\ m/s$. (Same as we had before.)

And now that we have the final velocity, we have multiple options to find the acceleration. $v = v_o + at$ would work since we have the initial and final velocities and the time interval. $v^2 = v_o^2 + 2a\Delta x$ works fine too since we have the initial and final velocities and the displacement.

The key point here is that there is **often** more than one path to the solution. All of these equations of motion are just different mathematical ways of saying $a = constant$.

Appendix : Calculus-based Derivations

Some students have already seen integral calculus, and we can derive the equations of motion that way too:

Equations of Motion

If the acceleration is constant, that means that $dv/dt = a = constant$.

Rearranging: $dv = adt$ so integrating: $\int_{v_o}^{v} dv = \int_0^t adt$ so $v - v_o = at$ or $\boxed{v = v_o + at}$.

Since $v = dx/dt$, we can write: $dx = vdt$, with $v = v_o + at$ so integrating: $\int_{x_o}^{x} dx = \int_0^t (v_o + at)dt$ which yields: $x - x_o = v_o t + \frac{1}{2}at^2$ or $\boxed{x = x_o + v_o t + \frac{1}{2}at^2}$.

Average Velocity

The (time) average of any continuous function (in this case $v(t)$) between $t = 0$ and some later time t can be written as:

$$v_{avg} = \frac{\int\limits_0^t v(t)dt}{t}$$

IF the acceleration is constant, then $v(t) = v_o + at$ so we can do the integral:

$v_{avg} = \frac{\int\limits_0^t (v_o + at)dt}{t}$ or:

$v_{avg} = \frac{v_o t + \frac{1}{2}at^2}{t}$. Cancelling a common t:

$v_{avg} = v_o + \frac{1}{2}at$ (which is one of the forms we arrived at via the hand-waving argument in class).

We can write that as: $v_{avg} = \frac{2v_o + at}{2}$ or $v_{avg} = \frac{v_o + (v_o + at)}{2}$ but that is just $v_{avg} = \frac{v_o + v}{2}$.

So when the acceleration is constant, the average velocity over an interval is just the average of the velocities at the two endpoints.