PH2213 Fox : Lecture 04
Chapter 3 : Kinematics in 2 or 3 Dimensions; Vectors

If you haven't seen vectors before, please be sure to look over the first few sections of Chapter 3 in the textbook.

We'll go through their introduction quickly today and then start applying them to simple motion problems.

Scalars vs Vectors

The various measurements we deal with can be broken into two broad categories: scalars and vectors:

```
Scalar          Vector (scalar with direction)
--------------  ------------------------------------
mass
weight
length
distance        displacement (length in a direction)
temperature
speed           velocity (wind, ocean currents)
```

Symbols:

Scalars are represented with variables like: x,y,z,m,t,T,d,... (that is, some letter).

Vectors are represented by a letter with the vector symbol over top of it: $\vec{r}$, $\vec{F}$, $\vec{v}$

(Another common notation used in some textbooks is to just use a bold font for a vector variable: **r**, for example.)

> We've already encountered an important vector: g, which I can now write more properly as $\vec{g}$, which has a **magnitude** of 9.80 m/s^2 and a **direction** of 'down' (pointing towards the center of the Earth, more or less), making it definitely a vector quantity.

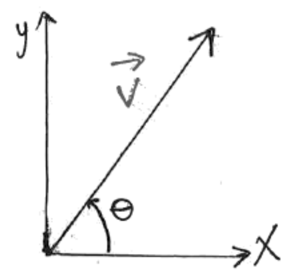
At it's core, a vector is an abstract mathematical construct that has a **magnitude** and a **direction**.

Technically that's all it is: it isn't attached to any particular location, but we'll often do so when we use vectors in practical, real-world situations.

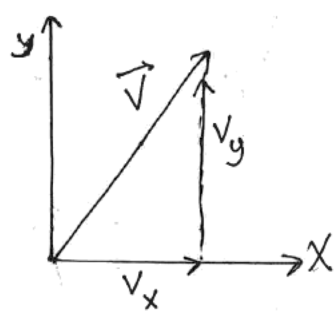
Conventionally, the **magnitude** of a vector is denoted by using the same symbol without the vector sign above the symbol:

$$\boxed{v = |\vec{v}|}$$

Which leads to the **polar coordinates** representation of the vector:
$\vec{v} = (v, \theta)$
This vector might represent a velocity of 20 m/s at an angle of 60^o relative to the $+X$ axis, so we could write it as $\vec{v} = (20\ m/s, 60^o)$.

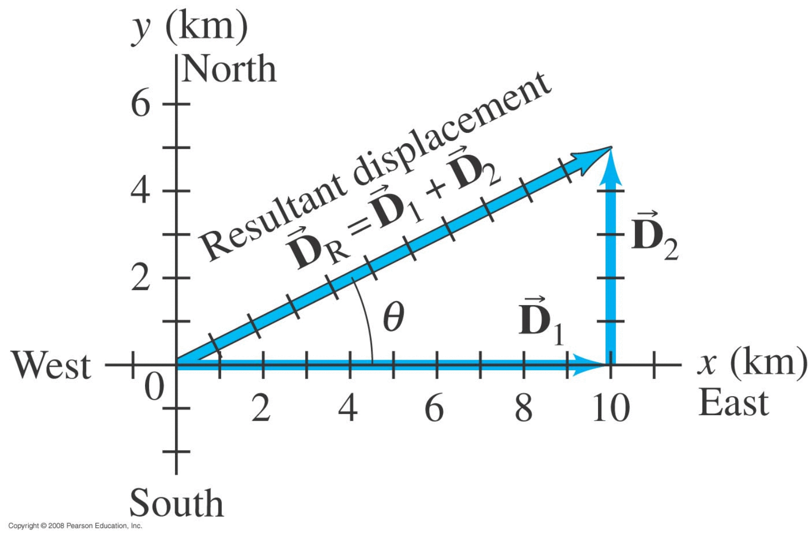


We'll see more of this in later slides, but mostly in this course we'll be using a **Cartesian** representation for vectors, where we break down the vector into 'components' (basically projecting the vector onto each of our coordinate directions).



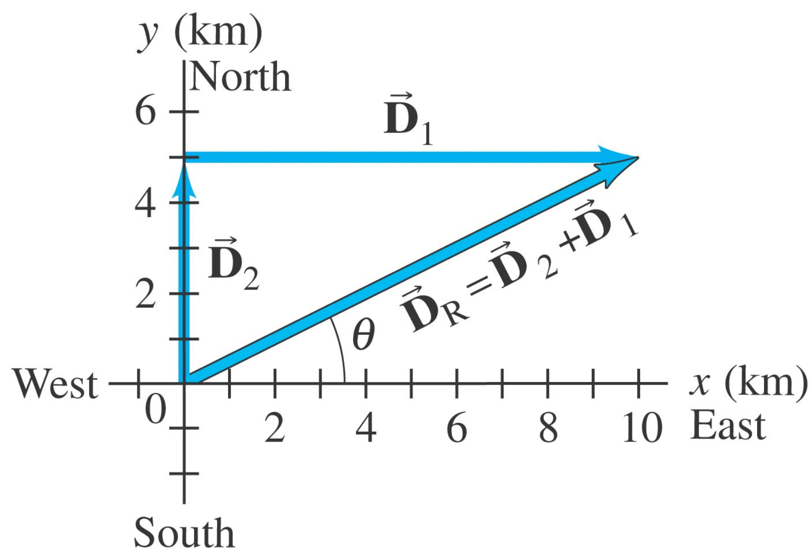
Vector Addition

Suppose we have two displacement vectors: $\vec{D}_1 = (10\, km\, to\, the\, East)$ and $\vec{D}_2 = (5\, km\, to\, the\, North)$. If we pace off $\vec{D}_1$ (i.e. we walk 10 km to the East) and then pace off $\vec{D}_2$ (i.e. we then walk 5 km to the North), then our total trip is, in effect, the **sum** of these two segments. The result of 'adding' these two vectors is another vector that would represent a single straight path from the starting to the ending point.



Resultant vector: $\vec{D}_1 + \vec{D}_2 = \vec{D}_R$

If we do these two segments in the opposite order, we still end up at the same point though:

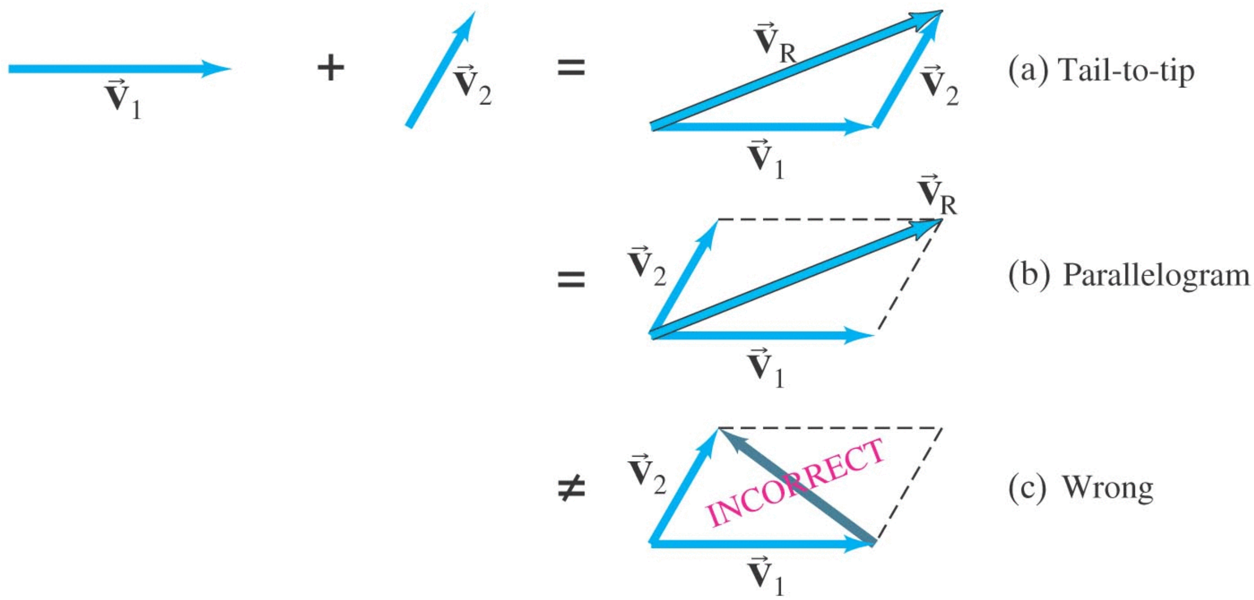


Resultant vector: $\vec{D}_2 + \vec{D}_1 = \vec{D}_R$

Since the order didn't matter, we see that **Vector Addition is Commutative** : $\vec{A} + \vec{B} = \vec{B} + \vec{A}$

Parallelogram Method for adding vectors

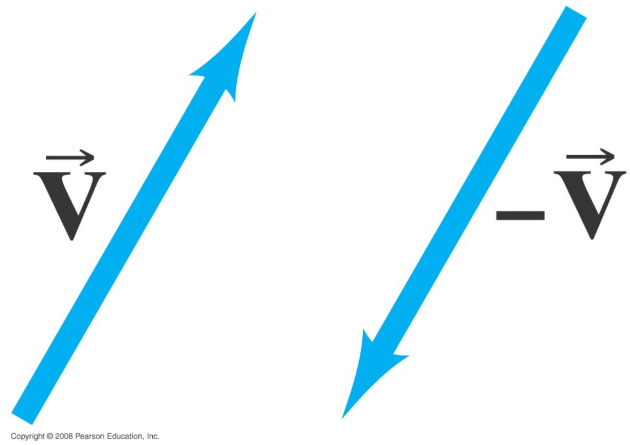
Extending the picture from the previous page, adding two (or more) vectors basically means we're putting each vector's tail where the previous vectors head was and accumulating the results as we move along. The subfigure labelled $\boxed{\text{b}}$ is worth highlighting because in the next chapter we'll be dealing with force vectors that are acting on some object in perhaps different directions and we'll need to add those vectors up to determine how the object reacts (moves). Even though the forces are acting on the object, the act of summing the vectors is more generic than that. Technically, a vector just has information about magnitude and direction and doesn't refer to any particular starting or ending point.



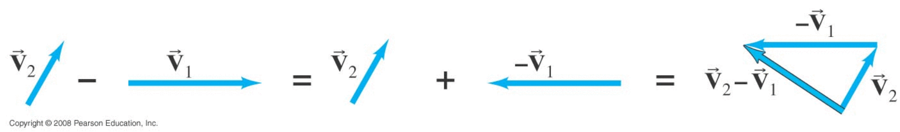
Multiplying a Vector by a Scalar

There are three types of multiplication involving vectors that we'll end up using during the course, but for now we'll just focus on one of them: the result of multiplying a **vector** by a **scalar**.

The **negative** of a vector is a vector with the same magnitude, but the opposite direction.

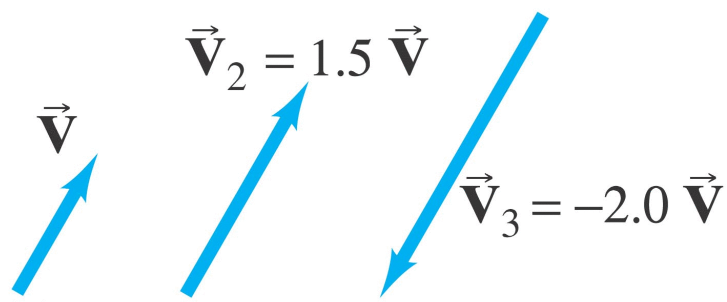


Subtraction can then be seen as adding the negative of a vector:



(Interestingly, computer math chips often do the same thing: rather than have special parts of the chip to handle subtraction, they just have a much quicker operation that 'flips the sign' of the second number and then adds that to the first number.)

Multiplying a vector by a **scalar** yields a vector in the same **direction** but whose **magnitude** has been multiplied by that scalar.



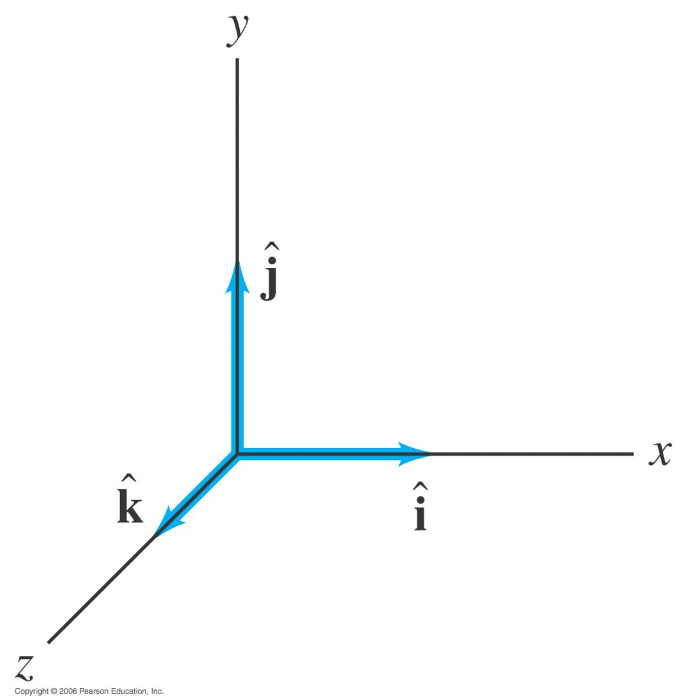
Unit Vector (Cartesian) Representation

Since we can break down vectors into components (so much in the X direction, then so much in the Y direction and so on, to construct the original vector), we can create another compact form to represent them. First, we have one last definition:

The **unit vectors** are basically short-hand for:

$\hat{i} =$ *in the X direction*
$\hat{j} =$ *in the Y direction*
$\hat{k} =$ *in the Z direction*

(Note: some textbooks may use a different notation for these 'unit vectors': $\hat{x}$ for 'in the X direction', $\hat{y}$ for 'in the Y direction' and so on. Our textbook uses the $\hat{i}, \hat{j}, \hat{k}$ convention, so please try and stick with what our text uses.)

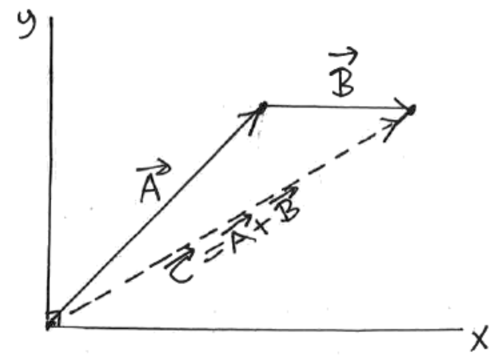


Looking back at page 3, vector $\vec{D}_1$ represented a displacement of '10 km to the east', and we defined east to be our X direction so this would be 10 km in the +X direction, or $\vec{D}_1 = 10\hat{i}$ (in units of km).

Similarly, $\vec{D}_2$ was '5 km to the north', which would become $\vec{D}_2 = 5\hat{j}$ (in units of km).

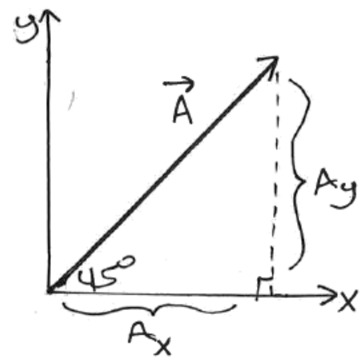
Example : A hiker leaves camp and travels 200 m to the northeast, then 100 m directly to the east. If we want to fly a drone directly from the camp to where they are now located, how far and in what direction does the drone need to fly?

Let's sketch this out first, using a coordinate system where East is in the $+X$ direction, and North is the $+Y$ direction (a fairly typical choice).
$\vec{A}$ is 200 m long and is to the 'northeast' which means it's midway between north and east, representing a 45^o angle counterclockwise from the $+X$ axis.
$\vec{B}$ is 100 m long and is entirely towards the 'east' which is our $+X$ direction.
The dotted line then represents the vector sum of these two segments of the trip.

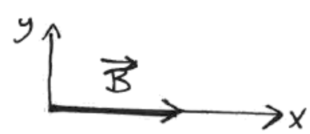


In class, we first 'solved' this problem just using geometry and trig, but let's skip straight to the method of converting each of these segments separately into our unit-vector notation scheme.

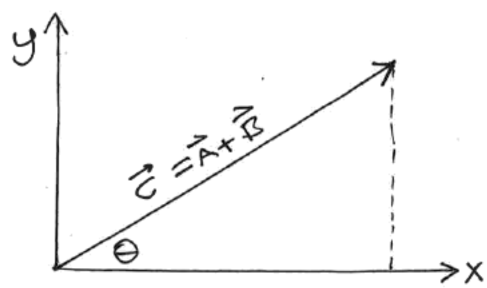
$\vec{A}$ is 200 m long and makes a 45^o angle with the X axis, so we can create the same vector by walking A_x along the X axis (in the $\hat{\imath}$ direction), then turning north and walking A_y along the Y axis (i.e., in the $\hat{\jmath}$ direction).
$A_x = A\cos\theta = (200\ m)\cos(45^o) = 141.4\ m$
$A_y = A\sin\theta = (200\ m)\cos(45^o) = 141.4\ m$
So: $\boxed{\vec{A} = 141.4\hat{\imath} + 141.4\hat{\jmath}}$ (in units of meters).



$\vec{B}$ is 100 m long and is entirely in the X direction, so we can convert this vector easily: $\vec{B}$ represents walking 100 m in the X (that is, $\hat{\imath}$) direction, then walking not at all (zero meters) in the Y direction. $\boxed{\vec{B} = 100\hat{\imath} + 0\hat{\jmath}}$



We can do the sum now:
$\vec{C} = \vec{A} + \vec{B} = (141.4\hat{\imath} + 141.4\hat{\jmath}) + (100\hat{\imath} + 0\hat{\jmath})$
Now, the X and Y directions are independent, so we need to combine only like terms. Collecting the $\hat{\imath}$ and $\hat{\jmath}$ terms separately:
$\vec{C} = (141.4 + 100)\hat{\imath} + (141.4 + 0)\hat{\jmath}$ or finally:
$\vec{C} = 241.4\hat{\imath} + 141.4\hat{\jmath}$

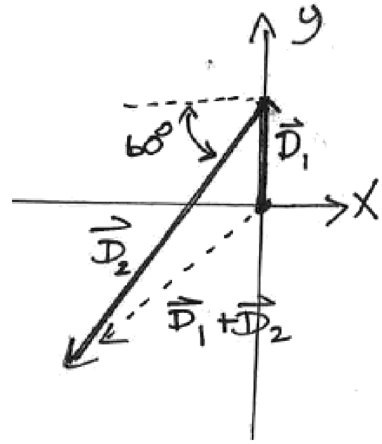


The magnitude of $\vec{C}$ will be $C = |\vec{C}| = \sqrt{(241.4)^2 + (141.4)^2} = 279.8\ m$.

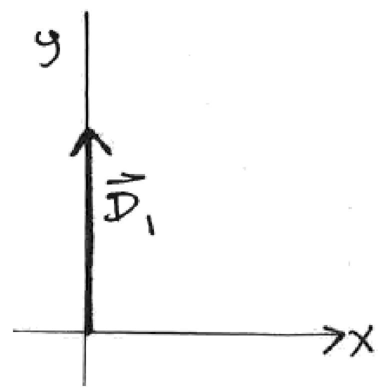
We have all three sides of the triangle now so could use any trig function to determine the angle θ but since C was calculated from the lengths of the two sides, it's less accurate. The preferred way of getting θ here would be $\tan\theta = 141.4/241.4$ from which $\theta = 30.4^o$.

Example 3-2 (modified) : **Mail carrier's displacement.** A rural mail carrier leaves the post office and drives 22.0 km in a northerly direction. She then drives in a direction 60.0^o south of **west** for 47.0 km (see figure at right). What is her displacement from the post office?

NOTE : see the book version of this example first. In that one, the mail carrier's second segment is in a different direction. The modification here creates a resultant vector in the third quadrant, which causes difficulties properly determining the angle, which is the point I'm focusing on in this modified version of the problem, since it's something you'll often encounter moving forward.



APPROACH We choose the positive X axis to be east and the positive Y axis to be north, since those are the compass directions used on most maps. The origin of this xy coordinate system is at the post office. We resolve each vector into its x and y components. We add the x components together, and then the y components together, giving us the x and y components of the resultant.



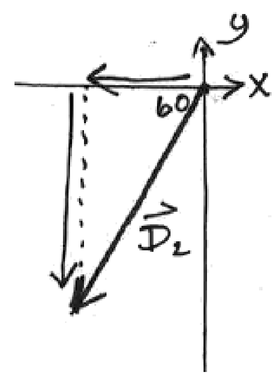
SOLUTION Resolve each displacement vector into its components, as shown in the middle figure on the right. Since $\vec{D}_1$ has magnitude 22 km and points north, it only has a y component:

$D_{1x} = 0$, $D_{1y} = 22.0\ km$.

$\vec{D}_2$ has both x and y components:

$D_{2x} = -(47\ km)\cos(60^o) = -23.5\ km$

$D_{2y} = -(47\ km)\sin(60^o) = -40.7\ km$



Notice that D_{2y} and D_{2x} are negative because those components points in the negative y and negative x directions respectively. We could also get the signs to come out correctly in the first place if we use the ACTUAL angle for this vector, which means using an angle that starts with $\theta = 0$ in the $+X$ axis direction. In that case, our ACTUAL angle here would be $180 + 60 = 240^o$ so for example: $D_{2x} = (47\ km)(\cos(240^o) = -23.5\ km$ with the sign coming out correct automatically.

The resultant vector $\vec{D} = \vec{D}_1 + \vec{D}_2$ has components:

$D_x = D_{1x} + D_{2x} = 0.0 - 23.5 = -23.5\ km$

$D_y = D_{1y} + D_{2y} = 22.00 + (-40.7) = -18.7\ km$

CONVENTIONS : Let's express that answer in the three conventions we talked about.

<u>Cartesian</u> : $\vec{D} = (-23.5\ km\ ,\ -18.7\ km)$

<u>Unit vectors</u> : $\vec{D} = -23.5\hat{\imath} - 18.7\hat{\jmath}$ (in units of km)

(see next page for the conversion to polar form)

Polar : Here we'll need to get the magnitude and angle for $\vec{D}$. The magnitude we can find via the pythagorean theorem: $D = |\vec{D}| = \sqrt{(D_x)^2 + (D_y)^2} = \sqrt{(-23.5)^2 + (-18.7)^2} = \sqrt{552.25 + 349.69} = 30.03\ km$. (NOTE: since our 'inputs' here only had three significant figures, we should probably write this result using no more than three significant figures also, so $D = 30.0\ km$ would be more appropriate.)

Finding the correct angle turns out to be surprisingly difficult though.

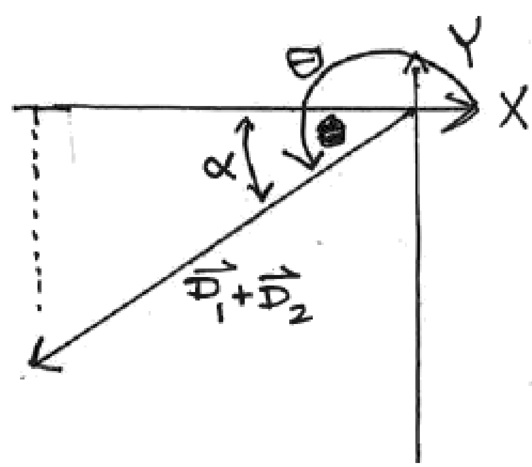
Remember: we define the angle in polar notation to be measured around counter-clockwise, starting at the +X axis. For example:

- the +X axis represents $\theta = 0^o$
- the +Y axis would be $\theta = +90^o$
- the negative X axis would be $\theta = 180^o$
- the negative Y axis would be $\theta = 270^o$

If we use the 'standard' unit-circle definition of $\tan\theta = y/x$ we have $\tan\theta = (-18.7)/(-23.5) = +0.7957...$ from which $\theta = +38.5^o$ but looking at the previous figures that can't be right. The angle we need, measured starting at the X axis and going around counter-clockwise obviously is something larger than 180^o.

The fundamental problem is that all the trig functions are **periodic** which means that all the **inverse trig functions** (arc-sine, arc-tangent, etc) have an **infinite number of solutions**. The math chip in your calculator or computer has to pick just one and it might not be the right one for the problem you're solving.

One safe approach I usually take is to ignore all the signs of the numbers in the figure and treat the triangle as something physical, cut out of cardboard maybe, with the numbers representing the (positive) actual lengths of the sides of that triangle.
In that case, we can find $\tan\alpha = 18.7/23.5$ from which $\alpha = 38.5^o$.
Now we can look back at the actual scenario and see that the angle θ we need can be found by adding 180^o to the angle α we just found. Here then, the angle θ we're looking for is $\theta = 38.5 + 180 = 218.5^o$.



Finally then, in polar notation: $\vec{D} = (30.0\ km\ ,\ 218.5^o)$