PH2213 Fox : Lecture 05
Chapter 3 : Kinematics in 2 or 3 Dimensions; Vectors

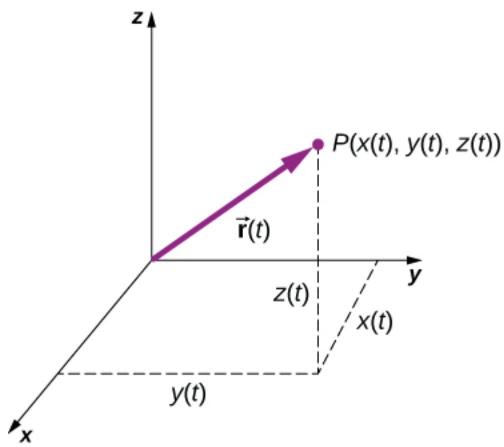
Analyzing 2- and 3-dimensional Motion

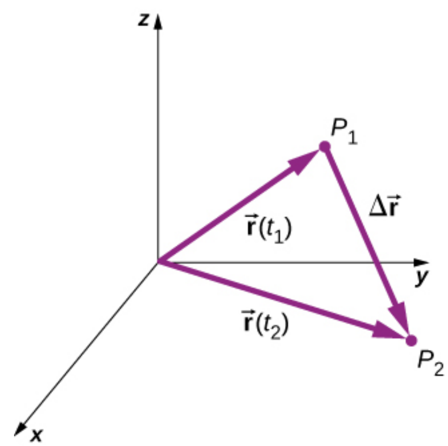
If we roll a ball off the edge of a table, it follows a path (which will turn out to be a parabola) through the air on the way to the ground. The ball is no longer moving in a straight line, so we'll need to use a 2D or 3D coordinate system to fully describe it's motion.

An object moving in a straight line only needs a single coordinate (x perhaps) to specify it's motion, but an object moving arbitrarily through space potentially needs x, y, and z. This triple (x, y, z) can be seen as the components of a **position vector** giving the location of the object.

Position: $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$

Displacement: $\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$



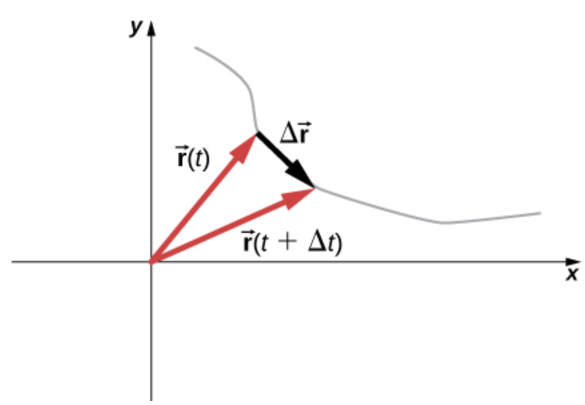


Average Velocity:

- $\vec{v}_{avg} = \Delta\vec{r}/\Delta t$

Velocity:

- $\vec{v}(t) = \frac{d\vec{r}}{dt}$
- $\vec{v}(t) = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$
- $\vec{v}(t) = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$



Average acceleration: $\vec{a}_{avg} = \Delta\vec{v}/\Delta t$

Acceleration: $\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j} + \frac{dv_z}{dt}\hat{k} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$

Constant Acceleration:

- $\vec{a}$ is the SLOPE of the $\vec{v}$ 'graph'
- IF $\vec{a} = constant$, THEN $\vec{v}(t)$ is a 'line' (in 3-space):
- $\vec{v}(t) = \vec{v}_o + \vec{a}t$

Constant Acceleration: $\vec{a}$ is the SLOPE of the $\vec{v}$ 'graph', so if $\vec{a} = constant$, then $\vec{v}(t)$ is a 'line' (in 3-space).

The $\vec{v}(t)$ equation must be such that when we differentiate it, we get a constant (the constant acceleration). That means it must be in the form: $\boxed{\vec{v}(t) = \vec{v}_o + \vec{a}t}$
giving us our first **vector equation of motion**.

We can keep going the same way we did with 1-D motion and the result is a complete set of vector equations of motion that look identical to the 1-D equations we already have:

Vector Equations of Motion

Summary : 1-D vs Vector Definitions and Equations		
Location	x	$\vec{r}$
Displacement	Δx	$\Delta\vec{r}$
time interval	Δt	Δt
Average velocity	$v_{avg} = \Delta x/\Delta t$	$\vec{v}_{avg} = \Delta\vec{r}/\Delta t$
Instantaneous velocity	$v = dx/dt$	$\vec{v} = d\vec{r}/dt$
Average acceleration	$a_{avg} = \Delta v/\Delta t$	$\vec{a}_{avg} = \Delta\vec{v}/\Delta t$
Instantaneous acceleration	$a = dv/dt$	$\vec{a} = d\vec{v}/dt$
Equations of Motion : when a or $\vec{a}$ are CONSTANT		
	$v = v_o + at$	$\vec{v} = \vec{v}_o + \vec{a}t$
	$v_{avg} = v_o + \frac{1}{2}at$	$\vec{v}_{avg} = \vec{v}_o + \frac{1}{2}\vec{a}t$
	$x = x_o + \frac{1}{2}(v_o + v)t$	$\vec{r} = \vec{r}_o + \frac{1}{2}(\vec{v}_o + \vec{v})t$
	$x = x_o + v_ot + \frac{1}{2}at^2$	$\vec{r} = \vec{r}_o + \vec{v}_ot + \frac{1}{2}\vec{a}t^2$
	$v^2 = v_o^2 + 2a\Delta x$	$v^2 = v_o^2 + 2a_x\Delta x + 2a_y\Delta y + 2a_z\Delta z$

Free-fall motion : the object is **only** moving under the influence of gravity, which means it has an acceleration vector directed **downward** towards the earth (or moon, or other object). The **magnitude** of the acceleration due to gravity at the surface of the **earth** is approximately $|\mathbf{a}| = \mathbf{g} = \mathbf{9.80\ m/s^2}$.

- The symbol g is always a **positive constant**.
- IF our vertical coordinate is **Y** with **positive UP**: $a_y = -9.8\ m/s^2$ (and $a_x = a_z = 0$)
- IF our vertical coordinate is **Y** with **positive DOWN**: $a_y = +9.8\ m/s^2$ (and $a_x = a_z = 0$)

Note that a vector equation is really three separate 1-D equations combined: one for each coordinate direction.

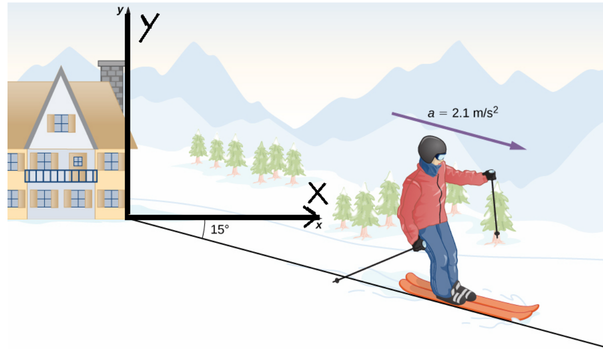
$\vec{v} = \vec{v}_o + \vec{a}t$ really means:

- $v_x = v_{ox} + a_x t$ and
- $v_y = v_{oy} + a_y t$ and
- $v_z = v_{oz} + a_z t$

In practice then, 2-D and 3-D motion starts off with vector equations but calculators (and computers) rarely work directly on vector quantities, so we end up turning the problem into (potentially) three separate problems: what's happening in X, Y and Z separately.

We'll exercise this idea in today's examples.

Example 1 : A skier starts at rest and slides down a perfectly flat slope that is angled 15^o below the horizontal. They are observed to have an acceleration of $2.1\ m/s^2$ along the slope.

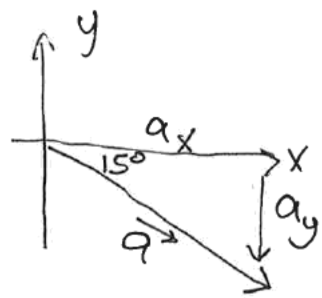


Four seconds later, how far have they travelled and how fast are they moving?

- Using the coordinate system shown...
- Using a coordinate system where the (X,Y) axes have been rotated so that X points down along the slope.

Using the coordinate system shown in the figure.

In this coordinate system, our vector $\vec{a}$ is making an angle 15^o below the X axis, so converting this into X and Y components. Ignoring signs, the **magnitudes** of these components will be:
$a_x = a\cos\theta = (2.1\ m/s^2)(\cos(15^o)) = 2.028\ m/s^2$
$a_y = a\sin\theta = (2.1\ m/s^2)(\sin(15^o)) = 0.5435\ m/s^2$
And looking at the figure, a_x will be positive, but a_y will be negative, so $a_x = +2.028\ m/s^2$ and $a_y = -0.5435\ m/s^2$.



Better: since that angle is **below** the +X axis and we're supposed to measure positive angles around counter-clockwise from that axis, technically this angle is actually $\theta = -15^o$ so we can get the actual signed values of the components directly:
$a_x = a\cos\theta = (2.1\ m/s^2)(\cos(-15^o)) = +2.028\ m/s^2$
$a_y = a\sin\theta = (2.1\ m/s^2)(\sin(-15^o)) = -0.5435\ m/s^2$

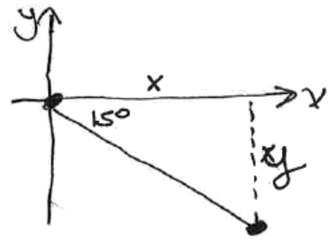
X equations of motion

- $x = x_o + y_ot + \frac{1}{2}a_xt^2$ so here $x = 0 + 0 + (0.5)(2.028)t^2$ or $\boxed{x = 1.014t^2}$ so at $t = 4$ the skier will be located at $x = 16.22\ m$.
- $v_x = v_{ox} + a_xt$ so here $v_x = 0 + 2.028t$ and at $t = 4$ we have $v_x = +8.112\ m/s$.

Y equations of motion

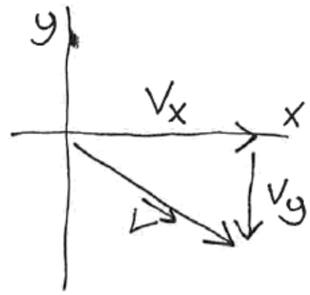
- $y = y_o + y_ot + \frac{1}{2}a_yt^2$ so here $y = 0 + 0 + (0.5)(0.5425)t^2$ or and at $t = 4$ the skier will be located at $y = -4.34\ m$.
- $v_y = v_{oy} + a_yt$ so here $v_y = 0 - 0.5425t$ and at $t = 4$ we have $v_y = -2.17\ m/s$.

Converting these components into the overall distance travelled and the speed at $t = 4$:



Distance Travelled

$d = \sqrt{x^2 + y^2} = \sqrt{(16.22)^2 + (-4.34)^2} = 16.79\ m$



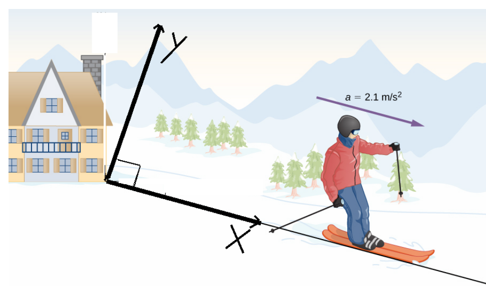
Speed

$v = |\vec{v}| = \sqrt{(8.112)^2 + (-2.17)^2} = 8.397..\ m/s$

Using a rotated coordinate system.

If we rotate our axes so that X actually lines up with the sloping ground, then our vector acceleration $\vec{a}$ is **entirely** in the X direction now, which means that $a_x = 2.1\ m/s^2$ and $a_y = 0$.

This greatly simplifies our equations of motion.



In the Y direction, the skier starts at rest at the origin so $y(t) = y_o + v_{oy}t + a_y t^2 = 0 + 0 + 0 = 0$. The skier's Y coordinate just remains at $y = 0$ throughout the scenario. Also, $v_y = v_{oy} + a_y t$ becomes $v_y = 0 + 0 = 0$: the Y component of the velocity is also zero throughout.

In the X direction, since the acceleration vector is entirely in the X direction now, $a_x = 2.1\ m/s^2$.

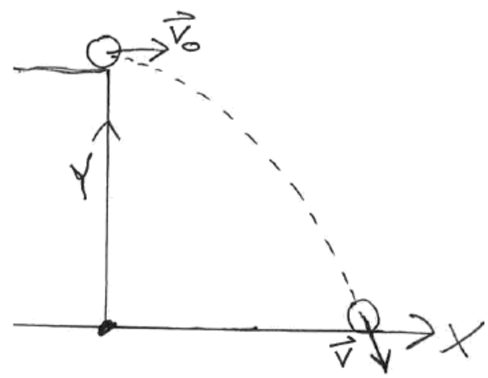
Starting at rest at the origin, we have $x(t) = x_o + v_{ox}t + a_x t^2 = 0 + 0 + (0.5)(2.1)t^2$ or simply $x(t) = 1.05t^2$. At $t = 4\ s$, the skier will be located at $x = 16.8\ m$ (and $y = 0$), so the total **distance travelled** will be $d = \sqrt{(16.8)^2 + (0.0)^2} = 16.8\ m$ **exactly**.

The X velocity will be $v_x(t) = v_{ox} + a_x t = 0 + (2.1)(t)$ so at $t = 4$ we'll have $v_x = (2.1)(4) = 8.4\ m/s$ **exactly**. The skier's **speed** at that point then will be $v = |\vec{v}| = \sqrt{(8.4)^2 + (0.0)^2} = 8.40\ m/s$ **exactly**.

Choosing this rotated coordinate system significantly reduced the amount of work needed to solve the problem (and gave us a more accurate result).

Example 2 : A ball moving (horizontally) at 2 m/s rolls off the side of a table that is 1.0 m above the floor.

1. Where will the ball land?
2. How long does it take to reach the floor?
3. How fast is it moving when it hits the floor?
4. At what angle does it hit the floor?



NOTE: this is essentially example 3 in the examples pdf for chapter 3, but here I'll use a different coordinate system (and the initial velocity of the ball is different here). That pdf has a much wordier description of the process which I won't repeat here, so you may want to review that example first.

Here then, with our choice of coordinate system, the ball's initial velocity is entirely in the X direction: $\vec{v}_o = (2\ m/s)\hat{\imath}$. The acceleration will be $\vec{g}$ which is a vector with a magnitude of $|\vec{g}| = g = 9.8\ m/s^2$ and a direction that is straight down, so we would write that as $\vec{a} = (-9.8\ m/s^2)\hat{\jmath}$.

In terms of components then, we have:

$v_{ox} = 2\ m/s \qquad v_{oy} = 0\ m/s \qquad v_{oz} = 0\ m/s$

$a_x = 0\ m/s^2 \qquad a_y = -9.8\ m/s^2 \qquad a_z = 0\ m/s$

Looking at the x equation of motion : $x(t) = x_o + v_{ox}t + \frac{1}{2}a_x t^2$ becomes $x(t) = 0 + 2t + 0$ or simply $\boxed{x(t) = 2t}$. (Everything was in standard metric units already so I dropped writing the units; here with t in seconds, this equation would give the x coordinate in meters.)

Looking at the y equation of motion : $y(t) = y_o + v_{oy}t + \frac{1}{2}a_y t^2$ becomes $y(t) = 1 + (0)(t) + \frac{1}{2}(-9.8)t^2$ or simply $\boxed{y(t) = 1 - 4.9t^2}$.

Those two boxed equations are what's called **parametric** equations: they basically give the values (here) of x and y as a function of t. We'd use these directly if we wanted to create a computer animation of the ball flying through the air realistically. Each frame of the video represents a particular time, and these equations give us where the ball should be rendered in the scene.

When the ball hits the floor, we don't know where or when that happens, but we do know that $\boxed{y = 0}$ at that point, so we can use our $y(t)$ equation of motion to determine <u>when</u> that happens:

$y(t) = 1 - 4.9t^2 \ \rightarrow\ 0 = 1 - 4.9t^2$

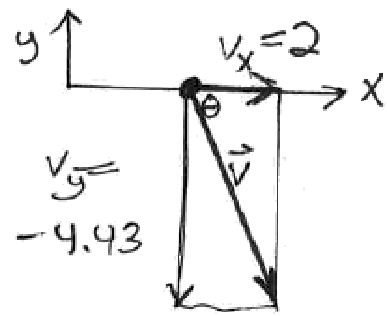
Rearranging: $t = \sqrt{(1/4.9)} = \pm 0.452\ s$.

We know the ball hits the ground <u>after</u> it was released, so the correct solution must be $\boxed{t = +0.452\ s}$.

Now that we know how long the ball was in flight, we can find how far it's travelled horizontally: $x(t) = (2\ m/s)(t) = (2\ m/s)(0.452\ s) = 0.904\ m$.

The ball was apparently in flight for just under a half a second, and it hit the floor about 90 cm from the edge of the table.

Speed and angle at which the ball hits the floor



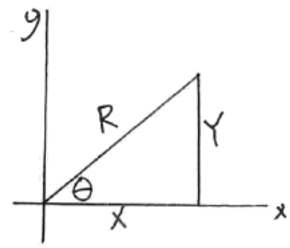
Please look over the similar section in example 3 in the examples03.pdf file.

We can use our velocity equations of motion to determine the components of the velocity vector at the instant the ball hits the floor:

$v_x(t) = v_{ox} + a_x t$ so here $v_x(t) = 2 + (0)(t) = 2\ m/s$ (constant)

$v_y(t) = v_{oy} + a_y t$ so here $v_y(t) = 0 + (-9.8)(t) = -9.8t$ so at the floor $v_y(0.452\ s) = -9.8(0.452) = -4.42\ m/s$.

The speed will be the magnitude of $\vec{v}$, which is essentially just the hypoteneuse of the triangle in the figure: $|\vec{v}| = \sqrt{(v_x^2 + v_y^2 + v_z^2)} = \sqrt{(2)^2 + (-4.43)^2 + (0)^2} = +4.86\ m/s$. (Note there's no ambiguity about the sign here, since the magnitude of a vector is defined as $|\vec{v}|$ and those bars act just like they do when we take the absolute value of a number. The result is always non-negative.)

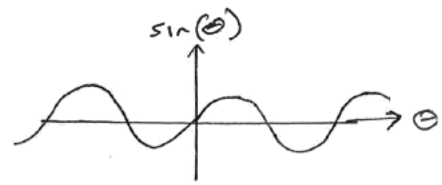


What should the angle be?

From the figure on the right, we can write $\sin\theta = Y/R$, $\cos\theta = X/R$, and $\tan\theta = Y/X$, so what do we get here?

$\sin\theta = -4.43/4.86$ from which $\theta = -65.7^o$
$\cos\theta = +2.00/4.86$ from which $\theta = +65.7^o$
$\tan\theta = -4.43/2.00$ from which $\theta = -65.7^o$

We obviously have a problem with the sign here. Which should it be? Looking at the figure, θ is obviously some angle below the X axis, so the negative angle would be the right one but why are we getting different results?



The basic problem is that the trig functions repeat themselves. If we're looking for the solution of, say, $\sin\theta = 0.5$ there are actually an infinite number of solutions, so which one should a calculator provide? The algorithm for ARCSIN in a calculator has to choose a particular range of angles, and in the case of ARCSIN the answer will always be in the range of -90^o to $+90^o$. Ditto for ARCTAN. In the case of ARCCOS, the answer will always be in the range of 0^o to 180^o.

It's important to treat the output from any ARC-trig function on your calculator with a grain of salt, and we'll see this occurring repeatedly throughout the semester.

The safest way to find the angle is probably to initially ignore the signs of the sides of the triangle. The physical triangle has sides of 2 and 4.43 and $\tan\theta = 4.3/2$ nominally gives us an angle of $\theta = 65.7^o$. Now we can look at the picture and see what that angle **really** is. It's 65.7^o below the X axis, to that would be either $\theta = -65.7^o$ or if we want to measure all the way around positive from the X axis, it would be $\theta = 360 - 65.7 = 294.3^o$.

Example 3 : Hilbun has a flat roof and we'd like to measure it's height. We roll a ball off the roof (so it's moving horizontally initially) at some unknown speed v_o. Exactly 1.60 sec later, the ball hits the flat ground below after travelling 10 m horizontally.

1. What was the initial velocity v_o of the ball?
2. How tall is the building?
3. How fast is the ball moving when it hits the ground?
4. At what angle does it hit the ground?

This looks similar to the previous problem, but we have different information now. We don't know the initial height or velocity, just that it's horizontal. Instead, we have information about the later point: when and where the object hit the ground.

Coordinate System and Knowns : The same solution method applies though: we'll break the vector equation of motion into separate equations in X and Y and see what we know where. Let's use the same coordinate system as in the previous problem. Then $\vec{a} = \vec{g}$ becomes $a_x = 0$ and $a_y = -9.8\ m/s^2$ (since our positive Y axis is upward). We also know that the ball was initially moving horizontally, so $v_{ox} = v_o$ and $v_{oy} = 0$. In this coordinate system, the ball starts at $x_o = 0$ and $y_o = h$ (the height of the building). When it hits the ground, we'll have $y = 0$ and $x = 10\ m$.

Looking at the x equation of motion : $x(t) = x_o + v_{ox}t + \frac{1}{2}a_x t^2$ becomes $x(t) = 0 + v_{ox}t + 0$ or simply $\boxed{x(t) = v_{ox}t}$. (Everything was in standard metric units already so I dropped writing the units; here with t in seconds, this equation would give the x coordinate in meters.)

Now we know that at $t = 1.60\ s$ the object is located at $x = 10\ m$ so we can immediately find the X component of the initial velocity: $10 = (v_{ox})(1.6)$ or $v_{ox} = 6.25\ m/s$ and since the ball was launched horizontally, this is in fact v_o also. $\boxed{v_o = 6.25\ m/s}$ and we're halfway done already.

Looking at the y equation of motion : $y(t) = y_o + v_{oy}t + \frac{1}{2}a_y t^2$ becomes $y(t) = h + (0)(t) + \frac{1}{2}(-9.8)t^2$ or simply $\boxed{y(t) = h - 4.9t^2}$.

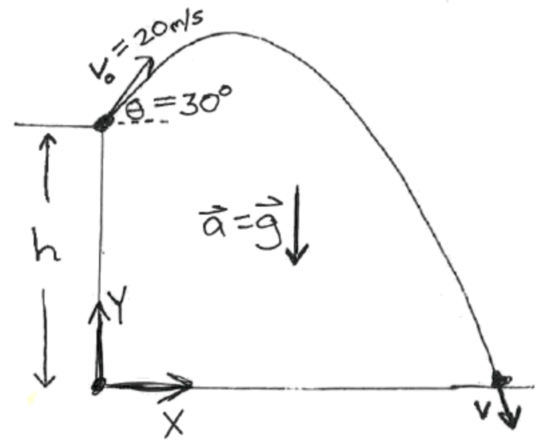
Now we know that at $t = 1.6\ s$, the ball has reached the ground ($y = 0$) so: $0 = h - (4.9)(1.6)^2$ or $\boxed{h = 12.544\ m}$.

Ball Kicked at Angle

A ball is kicked from the top edge of a 15 m tall building at 20 m/s, at an angle of $\theta = 30^o$ above the horizontal.

(a) How much time does the ball spend in the air?

(b) How far from the building will the ball land?



Using the coordinate system shown, what do we know?

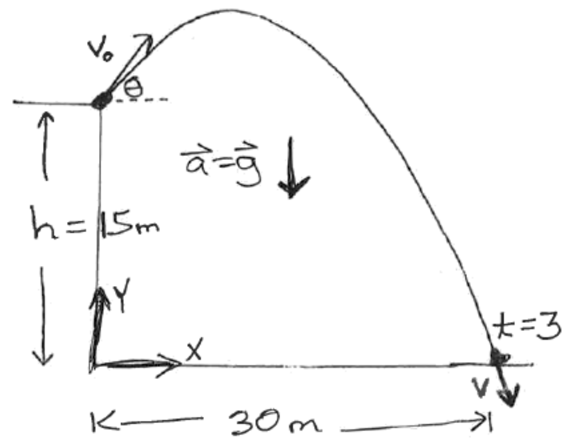
- the ball starts at $x_o = 0$, $y_o = 15\ m$
- the ball has an initial velocity of 20 m/s at 30^o above the horizontal (i.e. above the X direction) so $v_{ox} = v_o \cos(30^o) = 17.32\ m/s$ and $v_{oy} = v_o \sin(30^o) = 10.00\ m/s$
- the acceleration here is $\vec{a} = \vec{g}$ so 9.8 m/s^2 straight down. That makes $a_x = 0$ and $a_y = -9.8\ m/s^2$.
- At the landing point, we know nothing except that $y = 0$ there.

Breaking the problem into the X and Y direction:

X direction	**Y direction**
$x(t) = x_o + v_{ox}t + \frac{1}{2}a_x t^2$	$y(t) = y_o + v_{oy}t + \frac{1}{2}a_y t^2$
Substituting in the variables we know:	
$x(t) = 0 + 17.32t + 0$ Which is as far as we can go for now. Evaluating at $t = 3.046\ s$: $x = (17.32\ m/s)(3.046\ s) = 52.76\ m$	$y(t) = 15 + 10t - 4.9t^2$ At the ground, $y = 0$ so we can find the time when that occurs: $0 = 15 + 10t - 4.9t^2$ Solving that quadratic equation: $t = -1.005\ s$ or $t = +3.046\ s$ The ball hits the ground some time **after** it was launched, so the solution must be $t = +3.046\ s$

The ball was in the air for $\boxed{3.026\ sec}$ and landed $\boxed{52.76\ m}$ away from the building.

Let's look at a 'reverse' version of the previous problem. Suppose we know where the ball landed and how long it spent in the air (both easily measureable) and use that information to determine the initial launch velocity (speed and angle).



A ball is kicked from the top edge of a 15 m tall building at some unknown speed and angle. Exactly 3 seconds later, it lands 30 meters from the building.

(a) Determine the velocity at which the ball was launched (speed and angle).

Using the coordinate system shown, what do we know?

- the ball starts at $x_o = 0$, $y_o = 15\ m$
- the acceleration here is $\vec{a} = \vec{g}$ so $9.8\ m/s^2$ straight down. That makes $a_x = 0$ and $a_y = -9.8\ m/s^2$.
- At the landing point, we know that $x = 30\ m$ and $y = 0$ at $t = 3\ s$.

Breaking the problem into the X and Y direction:

X direction	**Y direction**
$x(t) = x_o + v_{ox}t + \frac{1}{2}a_x t^2$	$y(t) = y_o + v_{oy}t + \frac{1}{2}a_y t^2$
Substituting in the variables we know:	
$x(t) = 0 + v_{ox}t + 0$	$y(t) = 15 + v_{oy}t - 4.9t^2$
We know the X and Y values at $t = 3$:	
$30 = 0 + (v_{ox})(3)$	$0 = 15 + (v_{oy})(3) - 4.9(3)^2$
And we can immediately find the initial velocity:	
$v_{ox} = 30/3 = 10\ m/s$	Rearranging and solving for v_{oy} : $v_{oy} = 9.70\ m/s$

We know the components of $\vec{v}_o$ now so can find the speed and angle of the ball when it hits the ground:

$v_o = \sqrt{(v_{ox})^2 + (v_{oy})^2} = \sqrt{(10)^2 + (9.7)^2} = 13.93\ m/s$

$tan(\theta) = v_{oy}/v_{ox} = 9.7/10$ from which $\theta = 44.1^o$

(From the original picture, we see that the launch angle is somewhere between 0^o and 90^o, putting it in the first quadrant. **All** the inverse-trig functions on your calculator 'work' and return the correct angle when we're in this quadrant.)

Addendum for Example 2

(Not part of this class, but interesting for Math nerds...)

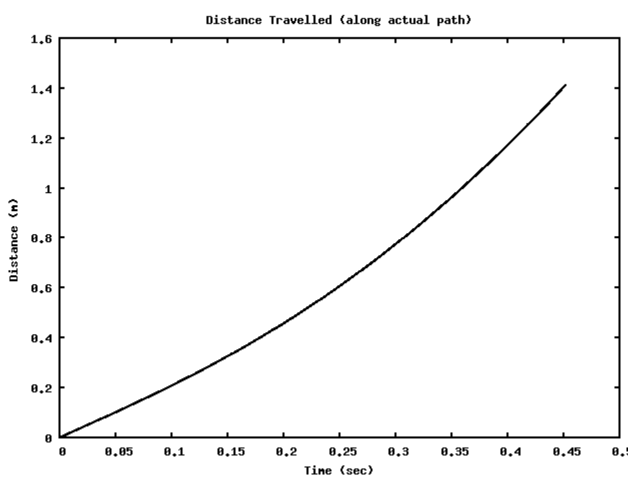
Let's look at Example 2 again, where we had a ball rolling off the side of a table.

Originally we used a fixed (x, y) coordinate system to define the position of the ball. This time, let's try to use a **1-d** coordinate where the position is just how far the ball has travelling **along it's path through the air.**

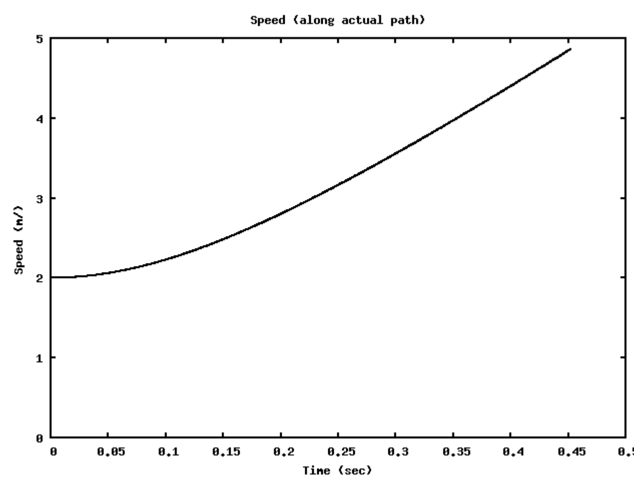
We do know the exact $x(t)$ and $y(t)$ positions of the ball in our original coordinate system, so we can calculate that millisecond by millisecond and determine how far the ball travelled in each millisecond, then just keep accumulating those results to find how far it's travelled along its path. (This can also be done exactly using a **parametric** integral which I think you see in Cal 2 or Cal 3 but doing it numerically like this is good enough to illustrate a problem.)

Now that we have the distance travelled (millisecond by millisecond) we can determine the ball's speed along the path (millisecond by millisecond) and from that we can find it's acceleration.

Position (along path)



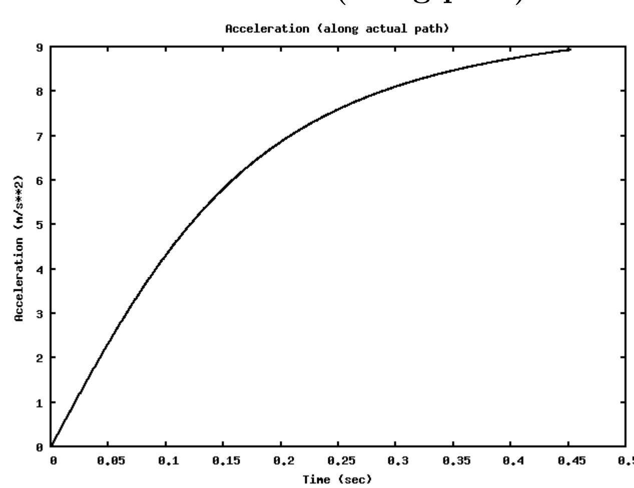
Speed (along path)



Now, acceleration is the (instantaneous) slope of the 'velocity' graph, which isn't a perfectly straight line, so it's derivative won't be a constant value. This graph shows the acceleration vs time for the ball. It's not constant, starting at 0 the instant the ball leaves the table, and gradually increasing. If the ball could fall long enough, it would eventually reach 9.8 m/s^2.

The problem here is that acceleration isn't constant, which means **we can't use our equations of motion** to analyze its motion.

Acceleration (along path)



The package of equation of motion we have **only** applies when acceleration is constant. Using the fixed coordinate system makes that happen here since $\vec{a} = \vec{g}$ which is a constant as long as our scenario allows us to pretend the Earth is flat.