**PH2213 Fox : Lecture 06**
**Chapter 3 : Kinematics in 2 or 3 Dimensions; Vectors**

Today we'll wrap up the 2-D motion topic by adding a few more useful equations and work through some additional examples.

**Vector Equations of Motion**

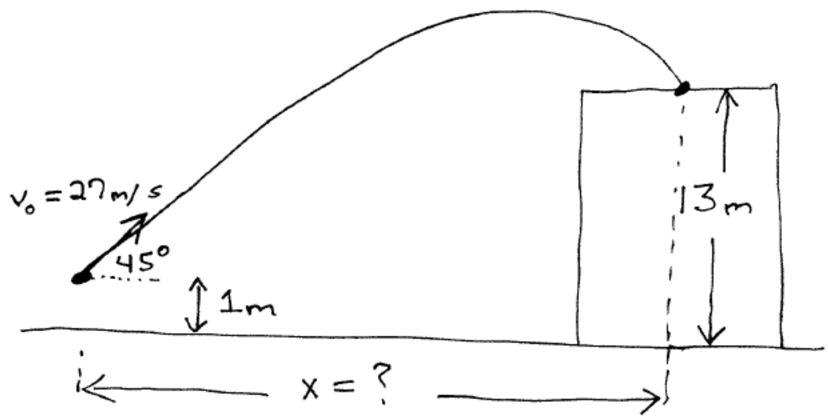
| Summary : 1-D vs Vector Definitions and Equations | | |
|---|---|---|
| Location | $x$ | $\vec{r}$ |
| Displacement | $\Delta x$ | $\Delta\vec{r}$ |
| time interval | $\Delta t$ | $\Delta t$ |
| Average velocity | $v_{avg} = \Delta x/\Delta t$ | $\vec{v}_{avg} = \Delta\vec{r}/\Delta t$ |
| Instantaneous velocity | $v = dx/dt$ | $\vec{v} = d\vec{r}/dt$ |
| Average acceleration | $a_{avg} = \Delta v/\Delta t$ | $\vec{a}_{avg} = \Delta\vec{v}/\Delta t$ |
| Instantaneous acceleration | $a = dv/dt$ | $\vec{a} = d\vec{v}/dt$ |
| Equations of Motion : when $a$ or $\vec{a}$ are CONSTANT | | |
| | $v = v_o + at$ | $\vec{v} = \vec{v}_o + \vec{a}t$ |
| | $v_{avg} = v_o + \frac{1}{2}at$ | $\vec{v}_{avg} = \vec{v}_o + \frac{1}{2}\vec{a}t$ |
| | $x = x_o + \frac{1}{2}(v_o + v)t$ | $\vec{r} = \vec{r}_o + \frac{1}{2}(\vec{v}_o + \vec{v})t$ |
| | $x = x_o + v_o t + \frac{1}{2}at^2$ | $\vec{r} = \vec{r}_o + \vec{v}_o t + \frac{1}{2}\vec{a}t^2$ |
| | $v^2 = v_o^2 + 2a\Delta x$ | $v^2 = v_o^2 + 2a_x\Delta x + 2a_y\Delta y + 2a_z\Delta z$ |

**Free-fall motion** : the object is **only** moving under the influence of gravity, which means it has an acceleration vector directed **downward** towards the earth (or moon, or other object). The **magnitude** of the acceleration due to gravity at the surface of the **earth** is approximately $|\mathbf{a}| = \mathbf{g} = \mathbf{9.80\ m/s^2}$.

- The symbol g is always a **positive constant**.
- IF our vertical coordinate is Y with positive upward: $a_y = -9.8\ m/s^2$
- IF our vertical coordinate is Y with positive downward: $a_y = +9.8\ m/s^2$

**Example : Baseball Landing on the Roof of a Building**

Suppose we hit a baseball such that it leaves the bat at a speed of 27.0 $m/s$ at an angle of $45^o$. At that instant, the ball is 1.0 $m$ above the ground. Some time later, it lands on the rooftop of a nearby building at a point that is 13.0 $m$ above the ground level. What horizontal distance did the ball travel? How long was it in the air? What maximum height did it reach? How fast is it moving when it hits the roof?




**Coordinates** : Let's put the origin on the ground right under where the ball was struck, with +X to the right and +Y vertically upward. That means the ball starts at $x_o = 0, y_o = 1.0\ m$ and ends at $y = 13.0\ m$ (and $x$ unknown).

The ball is 'launched' at a speed of 27.0 $m/s$ at an angle of $\theta = 45^o$ up from the horizontal, which means that $v_{ox} = v_o \cos\theta = (27.0\ m/s)\cos 45^o = 19.09\ m/s$ and $v_{oy} = v_o \sin\theta = (27.0\ m/s)\sin 45^o = 19.09\ m/s$.

The general equation of motion for the ball in the X direction is: $x = x_o + v_{ox}t + \frac{1}{2}a_x t^2$ so with our choice of coordinates: $x = 0.0 + (19.09)(t) + 0$ or just $\boxed{x(t) = 19.09t}$.

The general equation of motion for the ball in the Y direction is $y = y_o + v_{oy}t + \frac{1}{2}a_y t^2$ so with our choice of coordinates: $y = 1.0 + 19.09t + \frac{1}{2}(-9.8)t^2$ or just $\boxed{y(t) = 1 + 19.09t - 4.9t^2}$.

We can use this last equation to find how long it takes the ball to reaches the roof, since at that point $y = 13.0\ m$ so:

$13 = 1 + 19.09t - 4.9t^2$ which we can rearrange into: $4.9t^2 - 19.09t + 12 = 0$. This quadratic equation will have two solutions: $t = \frac{19.09 \pm \sqrt{(19.09)^2 - (4)(4.9)(12)}}{(2)(4.9)}$ or $\boxed{t = 0.788\ s}$ **and** $\boxed{t = 3.108\ s}$.

**Both** of these are positive, but which is the correct solution? Looking at the figure, we see that the ball passes through $y = 13$ on the way up and then again reaches it on the way down (at which point it hits the roof) so apparently it's the second solution that corresponds to the 'hitting the roof' situation.

**Horizontal distance (x)** : Now that we know that the ball lands on the roof at $t = 3.108\ s$, we can find its X coordinate at that point: $x = (19.09\ m/s)(t) = (19.09\ m/s)(3.108\ s) = 59.33\ m$.

**How high up did the ball fly?** At the apogee point, the ball is still moving horizontally but has (momentarily) stopped moving vertically. At that point $v_y = 0$. $v_y = v_{oy} + a_y t$ but we know $v_{oy} = 19.09\ m/s$ and $a_y = -9.8\ m/s^2$ so this becomes: $0 = 19.09 - 9.8t$ or $t = 1.948\ s$. Plugging this into the $y(t)$ equation of motion we found above, we find $y_{max} = 19.6\ m$.

Another approach: Apply $v^2 = v_o^2 + 2a_x\Delta x + 2a_y\Delta y$ between the launch point and the apogee point. $a_x = 0$ so this reduces to: $v^2 = v_o^2 + 2a_y\Delta y$. At launch, $v = 27\ m/s$. At the apogee point $v_x = v_{ox} = 19.09\ m/s$ and $v_y = 0$ so at that point $v = 19.09\ m/s$ so: $(27)^2 = (19.09)^2 + (2)(-9.8)\Delta y$ from which $\Delta y = 18.60\ m$. REMEMBER THOUGH: that's just the CHANGE in y between those two points, and since we started at $y_o = 1$, the y coordinate at the apogee point would be $1 + 18.60 = 19.60\ m$ (same as we found the other way).

**How fast was the ball moving when it hit the roof?**

$v^2 = v_o^2 + 2a\Delta y$ and here $\Delta y = y_{final} - y_{initial} = 13.0 - 1.0 = 12.0\ m$ so $v^2 = (27.0)^2 + (2)(-9.8)(12) = 729 - 235.2 = 493.8$ or $v = \sqrt{493.8} = 22.22\ m/s$.

Another approach: $v_x(t) = v_{ox} + a_x t$ but $a_x = 0$ so $v_x(t) = v_{ox} = 19.09\ m/s$ and never changes. $v_y(t) = v_{oy} + a_y t = 19.09 - 9.8t$ so when the ball hits the roof at $t = 3.108\ s$ we find $v_y = -11.368\ m/s$. Combining those: $v^2 = v_x^2 + v_y^2$ which yields $|v| = 22.22\ m/s$ (same result).

**Anvil Toss : Version 1** : (1-D motion ) : There is a group of people who launch anvils into the air using high explosives. The claim in one video is that the anvil reached a height of 200 $ft$. Given all the dust involved, actually measuring the height would be difficult, so they actually infer the height by measuring the time from when the anvil is launched until it lands on the ground. Suppose this anvil was in the air for **8 seconds** and landed at the same spot it was launched from (i.e. it's only moving vertically up and down).
**Determine the anvil's launch speed and maximum height.**

For this problem, we're assuming the anvil is launched perfectly vertically so it will just go straight up and down. Let's use a Y axis with $+Y$ vertically upward, starting with an origin down on the ground. At $t = 0$ the anvil is launched upward with some vertical velocity $\vec{v}_o$, and exactly 8 seconds later it lands at the same point.

Our Y equation of motion here is: $y(t) = y_o + v_{oy}t + \frac{1}{2}a_y t^2$ and here we're starting at $y_o = 0$ and have an acceleration of $a_y = -9.8\ m/s^2$ ($\vec{a} = \vec{g}$ here, which is a vector of magnitude $9.8\ m/s^2$ and a direction vertically downward, but our +Y is vertically upward).

That lets us write: $y(t) = 0 + v_{oy}t + \frac{1}{2}(-9.8)t^2$ or $\boxed{y(t) = v_{oy}t - 4.9t^2}$.

We do know that at $t = 8$ the object has returned to $y = 0$ so evaluating that equation at $t = 8$ yields:

$0 = (v_{oy})(8) - (4.9)(8)^2$ which we can rearrange to find $\boxed{v_{oy} = 39.2\ m/s}$. (The anvil was apparently launched vertically upward at about 88 $miles/hr$.)

<u>How high up will the anvil fly?</u>

The anvil will fly upwards for a while, reach a maximum height (apogee) where the velocity becomes zero, then fall back down to the ground. We can use our velocity equation of motion to determine the time:

$v_y(t) = v_{oy} + a_y t$ so $v_y(t) = 39.2 - 9.8t$. At the apogee, $v_y = 0$ so $0 = 39.2 - 9.8t$ implies that this apogee is reached at $t = 4\ sec$.

Since we know the launch velocity, we can use the $y$ equation of motion to determine the maximum height, since that will be the anvil's $y$ coordinate at exactly $t = 4\ s$ into the flight.

$y(t) = 39.2t - 4.9t^2$ and evaluating that at $t = 4$ yields $y_{max} = 78.4\ m$, or about 257 feet (a bit higher than the 200 feet the video claimed).

<u>Another Approach</u>

We can also use our $v^2$ equation to determine the maximum height. If we focus on the interval between launch and the time it reaches the apogee (where $v$ drops to zero momentarily), we have:

$v_y^2 = v_{oy}^2 + 2a_y\Delta y$ so here:

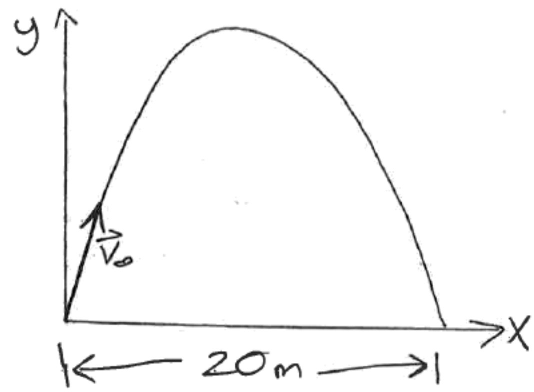
$(0)^2 = (39.2)^2 + (2)(-9.8)\Delta y$ which yields $\Delta y = +78.4\ m$ also.

$\Delta y = y_{apogee} - y_{initial}$ so $78.4 = y_{max} - 0$ or $y_{max} = 78.4\ m$ again.

**Anvil Toss : Version 2** : (2-D motion ) : **What if** another anvil is in the air for 8 seconds **but** lands 20 meters away from where it was launched?
**Determine the anvil's launch speed (and angle) and maximum height.**

**Coordinates**

- origin at launch point
- +X to the right
- +Y up
- $t = 0$ at instant of launch




**Knowns**

- $\vec{a} = \vec{g}$ so $a_x = 0$, $a_y = -9.8\ m/s^2$
- $x_o = 0$, $y_o = 0$
- at $t = 8$ : $x(8) = 20\ m$ and $y(8) = 0$

Let's apply our generic equations of motion and see what we find:

$\boxed{\text{X direction}}$ : $x(t) = x_o + v_{ox}t + \frac{1}{2}a_x t^2$
substituting in the values we know on the RHS: $x(t) = 0 + v_{ox}t + 0$. *At the landing point*, we know $x = 20\ m$ at $t = 8$ so: $20 = (v_{ox})(8)$ from which $\boxed{v_{ox} = 2.5\ m/s}$.

$\boxed{\text{Y direction}}$ : $y(t) = y_o + v_{oy}t + \frac{1}{2}a_y t^2$
substituting in the values we know on the RHS: $y(t) = 0 + v_{oy}t + \frac{1}{2}(-9.8)t^2$ or $y(t) = v_{oy}t - 4.9t^2$. *At the landing point*, we know $y = 0\ m$ at $t = 8$ so: $0 = (v_{oy})(8) - 4.9(8)^2$ from which $\boxed{v_{oy} = 39.2\ m/s}$.

Note that *this* anvil actually had the same vertical velocity as our first anvil.

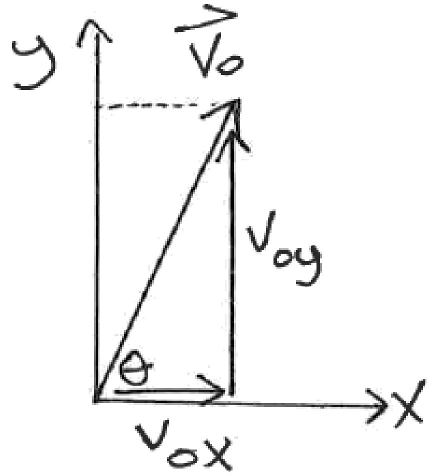
$\boxed{\text{Max height}}$ : How high in the air did this anvil go? Let's apply one of our $v^2$ equations: $v_y^2 = v_{oy}^2 + 2a_y\Delta y$ to the segment between the launch point and the point where the anvil has reached it's maximum height. At that point it's *vertical* velocity component has dropped to zero. It's still moving horizontally at a constant 2.5 $m/s$, but it's stopped moving upward for an instant before starting to drop back down towards the ground. At that point then, $v_y = 0$ so: $(0)^2 = (39.2)^2 + (2)(-9.8)(\Delta y)$ from which $\Delta y = 78.4\ m$. At this point, it's exactly 78.4 $m$ above where it was launched from. That is the *same* height the previous (perfectly vertically launched) anvil reached, so we have a tie here.

$\boxed{\text{Launch speed and angle}}$

We have the X and Y components of the initial velocity, so:
$v_o = +\sqrt{(v_{ox})^2 + (v_{oy})^2}$ or:
$v_o = +\sqrt{(2.5)^2 + (39.2)^2} = 39.3\ m/s$ (about).

The launch angle we can find from: $\tan\theta = v_{oy}/v_{ox} = 39.2/2.5$ from which $\theta = 86.35^o$.

This anvil was apparently launched with a **slightly** higher velocity than the first one, but a few degrees off from vertical.



**Specialized Projectile Motion Equations**

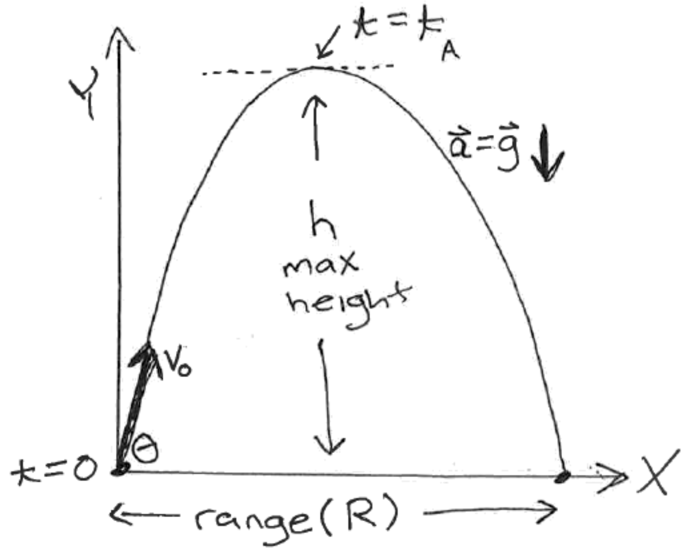
Object launched at origin with $\vec{v}_o = (v_o, \theta)$

First, let's derive some **special-purpose equations** that we can apply for certain types of motion, specifically for trajectories where the starting and ending points are at the same elevation.

We'll use a coordinate system with the origin at the launch point, and with +X to the right (pointing over towards where the object will land) and with +Y vertically upward.

For as long as it's in flight, the object will undergo an acceleration of $\vec{a} = \vec{g}$, so in our chosen coordinate system the components will be $\boxed{a_x = 0}$ and $\boxed{a_y = -g}$. (Note that $g = |\vec{g}|$ is always a positive number. The acceleration may be positive or negative depending on our coordinate system choice, but the symbol $g$ itself **always** represents a **positive** value.)

If the object is launched with an initial speed $v_o$ at an angle of $\theta$ (relative to the horizontal) then $\boxed{v_{ox} = v_o \cos\theta}$ and $\boxed{v_{oy} = v_o \sin\theta}$.




$\boxed{\text{Apogee time: } t_A}$ : at this point $v_y$ has become zero so using our equation of motion for the vertical velocity: $v_y = v_{oy} + a_y t$ or in our coordinate system: $v_y = v_{oy} - gt$. At the apogee point, $t = t_A$ and $v_y = 0$ so $0 = v_{oy} - gt_A$ from which $t_A = v_{oy}/g$ or in its usual form: $\boxed{t_A = v_o \sin(\theta)/g}$

$\boxed{\text{Landing time}}$ : at this point, $y$ has returned to 0, so starting with the Y equation of motion: $y(t) = y_o + v_{oy}t + \frac{1}{2}a_y t^2$ becomes: $0 = 0 + v_{oy}t - \frac{1}{2}gt^2$. Rearranging, we find $t = 2v_{oy}/g$. That is *exactly twice* the apogee time we just found, so the landing time is: $\boxed{t_{landing} = 2t_A = 2v_o \sin(\theta)/g}$ (BE CAREFUL with this one though: this time is ONLY valid if the object returns back to the same elevation (same Y coordinate) it was launched from.)

$\boxed{\text{Max height: } y_{max} \text{ or } h}$ : this would be the $y(t)$ value at $t = t_A$, so we can just evaluate the Y equation of motion at that time. $y(t) = y_o + v_{oy}t + \frac{1}{2}a_y t^2$ or here $y(t) = 0 + v_{oy}t - \frac{1}{2}gt^2$.

Using $t = t_A = v_{oy}/g$ we find that $h = y_{max} = v_{oy}^2/(2g)$ or in the usual form: $\boxed{h = v_o^2 \sin^2\theta/(2g)}$.

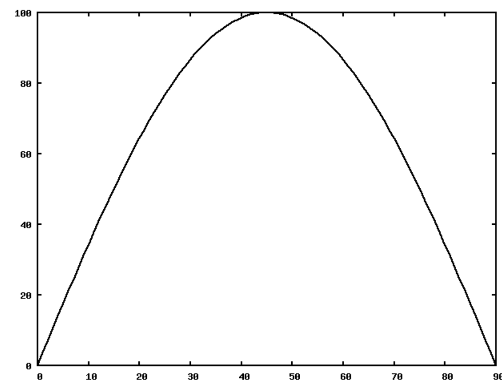
$\boxed{\text{Range: } R}$ : Finally, the range will be the X coordinate at $t = 2t_A$. In general: $x(t) = x_o + v_{ox}t + \frac{1}{2}a_x t^2$ or here $x(t) = 0 + v_{ox}t + 0$ or just $x(t) = v_{ox}t$. Evaluating this at $t = 2t_A$ using the result from above: $R = 2v_{ox}v_{oy}/g$, but $v_{ox} = v_o \cos\theta$ and $v_{oy} = v_o \sin\theta$ so we can write that as $R = 2v_o \cos(\theta) v_o \sin(\theta)/g$ or collecting terms: $R = v_o^2(2cos(\theta)sin(\theta))/g$. That term in parenthesis is just $\sin(2\theta)$ though, which gives the usual form: $\boxed{R = (v_o^2 \sin(2\theta))/g}$.

$\boxed{y(x)}$ : the X equation of motion simplified to $x(t) = v_{ox}t$ which we can rearrange into $t = x(t)/v_{ox}$. If we substitute that expression for $t$ into the Y equation of motion, a bit of trig and algebra later we can produce an equation that gives the Y coordinate directly in terms of X: $y = (\tan\theta)x - (\frac{g}{2v_o^2\cos^2\theta})x^2$. This is sometimes (but rarely) useful, but it does finally show that $y(x)$ is in the form of a parabola.

Let's look more closely at that RANGE equation.

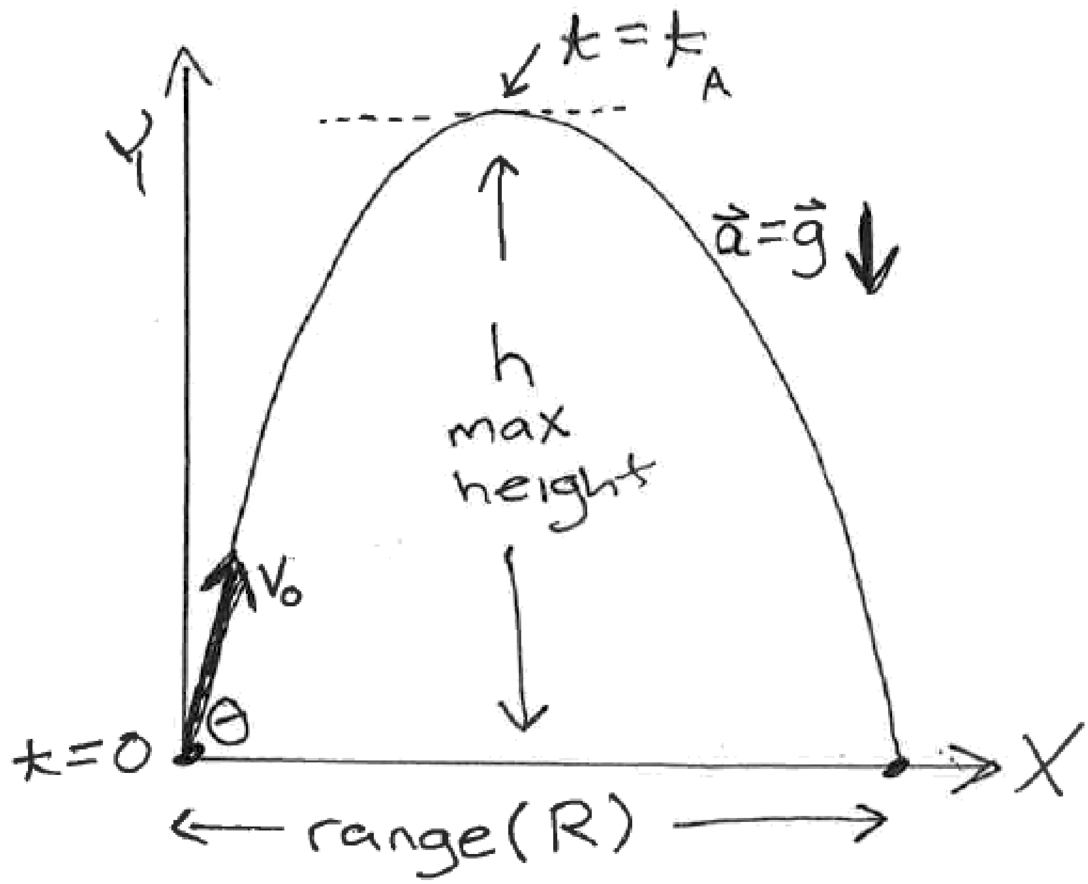
**Range** : $R = \frac{v_o^2 \sin(2\theta)}{g}$

For a given initial speed of the ball, the range peaks when the sine reaches it's maximum value, which is 1, giving us a maximum range of $\boxed{R_{max} = \frac{v_o^2}{g}}$. That max range will occur when the argument of the sine function is $90^o$ so here: $2\theta = 90^o$ or $\boxed{\theta = 45^o.}$.



**Specialized Projectile Motion Equations**

Object launched at origin with $\vec{v}_o = (v_o, \theta)$




$a_x = 0 \qquad a_y = -g$

$v_{ox} = v_o \cos\theta \qquad v_{oy} = v_o \sin\theta$

$x = (v_o \cos\theta)t \qquad y = (v_o \sin\theta)t - \frac{1}{2}gt^2 \qquad y = (\tan\theta)x - (\frac{g}{2v_o^2 \cos^2\theta})x^2$

Apogee:

$t_A = (v_o \sin\theta)/(g)$

$h = (v_o^2 \sin^2\theta)/(2g)$ above the launch point.

If $y_{final} = y_{initial}$ :

Total flight time: $t = 2t_A$

Range $R = (v_o^2 \sin 2\theta)/(g)$

$R_{max} = v_o^2/g$ at $\theta = 45^o$

**Example : World War 2 Battleship**

The US 'Colorado-class' battleships used during World War II fired shells at 2600 $ft/s$ at a maximum angle above the horizon of $30^o$.

- What would the projectile range be?
- How long is the projectile in the air?
- What maximum height did the projectile reach?

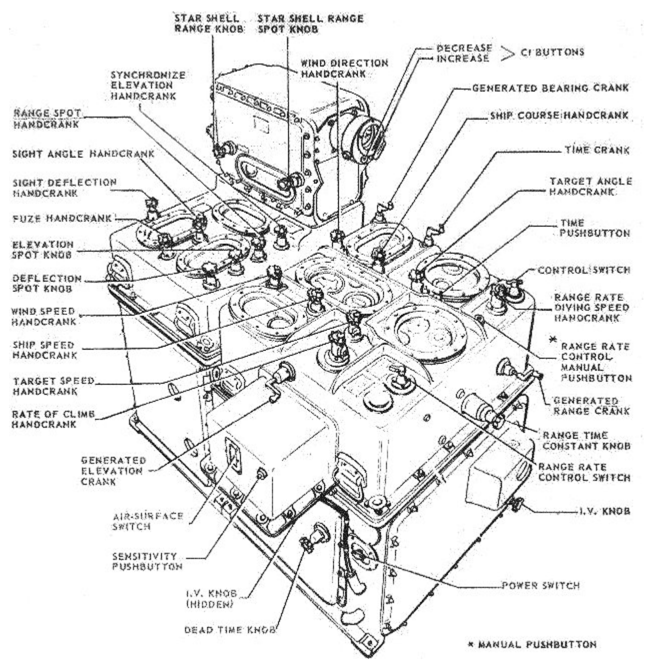
Converting units first: $\frac{2600\ ft}{1\ sec} \times \frac{1\ m}{3.281\ ft} = 792.44\ m/s$ (a little over 1700 $miles/hour$).

The barrel couldn't be aimed up more than $30^o$ above the horizontal, so:

- Range $R = \frac{v_o^2 \sin(2\theta)}{g} = \frac{(792.44)^2 \sin(60^o)}{9.8} = 55492\ m$
  (so 55.5 $km$ or about 34.5 $miles$)
- Flight time: $t = 2t_A = 2(v_o \sin\theta)/(g) = (2)(792.44)\sin(30^o)/9.8 = 80.86\ sec$ (almost a minute and a half)
- Apogee height: $h = (v_o^2 \sin^2\theta)/(2g) = \frac{(792.44)^2 \cdot (\sin(30^o))^2}{(2)(9.8)} = 8010\ m$
  (That's slightly over 8 km or nearly 5 miles above the surface.)

Useful tidbit: assuming the Earth is a sphere and you're standing at sea level with your eye located $h$ meters above sea level, how far away is the horizon (in kilometers)? $d \approx \sqrt{13 * h}$. For an average adult, that puts the horizon about 5 km (3 miles) away. The deck of the battleship is about 12 $m$ above the sea surface yielding a horizon distance of about 12.5 $km$ (just short of 8 miles), so the 'landing point' of these shells was well beyond the horizon.

Problems like this lead to some of the earliest **mechanical** 'computers' back in the 1940's. This gadget, full of wheels and levers, weighted about 3000 pounds!



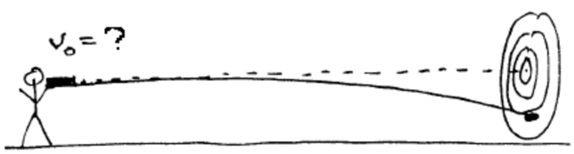
**Additional Example: Archery**

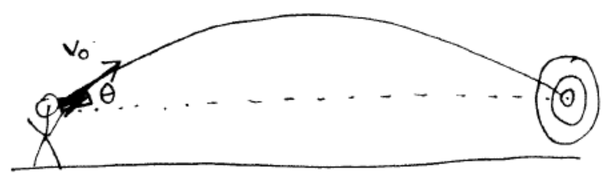
Suppose we fire an arrow at a backyard archery target. Here, the arrow is fired at exactly the same height at the bullseye on the target, and is fired horizontally. If the target is 16 $m$ away, we see that the arrow hits 78.4 $cm$ below the bullseye.

(a) How fast was the arrow moving when initially fired?

(b) In order to hit the bullseye, we'll need to aim the arrow up at some angle relative to the horizontal. Assuming it's fired at the same speed as before, determine that angle.

(c) Approximately what is the maximum range of the arrow? How long would it be in flight?




(a) If we put our coordinate origin at the launch point, we know the ending X and Y coordinates and the launch angle ($0^o$) so let's exercise our new $y(x)$ equation to find the unknown launch speed $v_o$: $\boxed{y = (\tan\theta)x - (\frac{g}{2v_o^2\cos^2\theta})x^2.}$

Here, $\theta = 0$ so the tangent term goes away and the cosine term becomes just 1, leaving us with: $\boxed{y = -\frac{gx^2}{2v_o^2}}$. At the target, we have $x = 16\ m$ and $y = -0.784\ m$, which yields $v_o = 40\ m/s$. (That's a reasonable speed for an arrow fired by a normal person. A good archer can reach 70 $m/s$ with a normal bow, and about 90 $m/s$ with a compound bow.)

Alternate path: use the $y(t)$ equation of motion to find the **time** it takes for the arrow to reach the target; then evaluate the $x(t)$ equation at that time which will yields the launch speed.

(b) Now that we know the launch speed of the arrow, at what angle do we need to launch it so that it exactly hits the bullseye? Since now the launch and landing points are at the exact same elevation, we can use the specialized RANGE equation: Range $R = (v_o^2\sin 2\theta)/(g)$

Here $R = 16\ m$ and $v_o = 40\ m/s$ so: $16 = \frac{(40)^2\sin(2\theta)}{9.8}$ and after rearranging we find that: $\sin(2\theta) = 0.098$.

Taking the inverse sine of both sides of this equation yields: $2\theta = \sin^{-1}(0.098) = 5.624^o$ from which finally $\boxed{\theta = 2.812^o}$.

**So where would you need to aim on the target to hit the bullseye?** Initially the arrow ended up 78.4 $cm$ low, so it's tempting to say you'd just aim that much above the bullseye, but that's not quite correct. If you draw a line from the firing point to the target at the angle we found, the Y coordinate where the line hits the target would be found by $\tan\theta = y/x$ or $y =$

$x \tan\theta = (16\ m)\tan(2.812^o) = +0.786\ m$ or $78.6\ cm$ above the bullseye. Notice that's slightly **higher** than the $78.4\ cm$ that we missed the bullseye by initially, and this difference gets more and more significant the larger the 'miss' distance was.

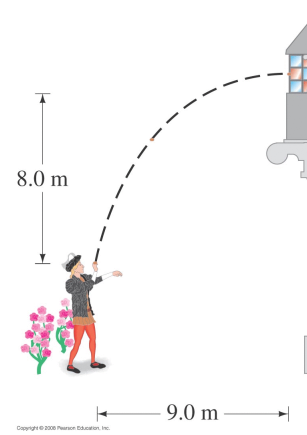
(c) Maximum Range: if we launch the arrow at a $45^o$ angle, the range will be $R_{max} = v_o^2/g = (40)^2/9.8 = 163.3\ m$, so we could theoretically move the target that far away and still hit the bullseye.

Maximum flight time: $t = 2t_A = 2(v_o \sin\theta)/(g) = (2)(40)\sin(45^o)/9.8 = 5.77\ s$

(Technically, the equations for R and $t_A$ require the object to launch and land at the same elevation and they don't *quite* do that here. Try solving this using the full equations of motion for $x(t)$ and $y(t)$ and assume the arrow is launched at $y_o = 2\ m$ above the ground, and still at a $45^o$ angle. You'll find the arrow lands at $x = 165.2\ m$ now, or about 2 meters farther. A fancier solution that allows the angle to vary yields a maximum range of $165.25\ m$ if the arrow is fired at about $44.65^o$.)

**Additional Example: HW Problem 3.52** : Romeo is throwing pebbles gently up to Juliet's window and he wants the pebbles to hit the window with only a horizontal component of velocity. He is standing at the edge of a rose garden 8.0 $m$ below her window and 9.0 $m$ from the base of the wall (see figure). How fast are the pebbles going when they hit her window?

**Coordinates** : we'll put the origin where the rock was thrown, with $+X$ to the right, and $+Y$ vertically upward.




Since the pebble is reaching it's apogee point right at the window (the point where it's travelling only horizontally, so $v_y = 0$ at that point). The figure here looks like the first half of our specialized projectile motion equations, so it's tempting to use those equations, where we know the apogee height is $h = 8\ m$ and the range would be $R = 18\ m$. I'll do it that way first, but this approach turns out to be tricky.

$h = 8 = (v_o^2 \sin^2\theta)/(2g)$ or multiplying both sides by $2g$ we have: $\boxed{156.8 = v_o^2 \sin^2\theta}$

$R = 18 = (v_o^2 \sin(2\theta))/(g)$ and multiplying both sides by $g$ yields: $\boxed{176.4 = v_o^2 \sin(2\theta)}$.

Technically we now have two equations and two unknowns that we could use to find the initial launch speed $v_o$ and launch angle $\theta$, and then the X component of that velocity would be $v_{ox} = v_o \cos\theta$ and that's the velocity the pebble would still have when it hits the window.

How can we combine these equations though? They both contain $v_o^2$ so let's get rid of that term first by just dividing the first equation by the second:

$\frac{156.8}{176.4} = \frac{v_o^2 \sin^2\theta}{v_o^2 \sin(2\theta)} = \frac{\sin^2\theta}{\sin(2\theta)}$

Now we have to use a trig identity that $\sin(2\theta) = 2\sin\theta\cos\theta$ which means the right hand side of that equation is actually $\frac{\sin\theta\sin\theta}{2\sin\theta\cos\theta}$ and we can cancel one of the sines leaving us with $\frac{\sin\theta}{2\cos\theta}$ or just $0.5\tan\theta$.

Finally then: $0.\overline{8888} = 0.5\tan\theta$ or $\tan\theta = 1.\overline{7777}$ from which $\boxed{\theta = 60.64^o}$ and using the height or range equations each yields $v_o = 14.367\ m/s$, so $v_{ox} = v_o\cos\theta = 7.04\ m/s$ as the final answer.

**OK, let's try a somewhat more straightforward solution**, breaking the problem up into X and Y components. In the Y direction, we do know $v_y = 0$ when the pebble hits the window, we know the Y acceleration, and we know the $\Delta y$ between the launch point and the window, so: $v_y^2 = v_{oy}^2 + 2a_y\Delta y$ becomes: $(0)^2 = v_{oy}^2 + (2)(-9.8)(8)$ from which $v_{oy} = 12.52\ m/s$.

We can use that to find the TIME it takes the pebble to reach the window: $v_y = v_{oy} + a_y t$ so here: $0 = 12.52 - 9.8t$ from which $t = 1.2776\ sec$.

Finally, in the X direction, $x(t) = x_o + v_{ox}t + \frac{1}{2}a_x t^2$ becomes: $9 = 0 + (v_{ox})(1.2776) + 0$ from which $v_{ox} = 7.04\ m/s$ (and the pebble maintains that X velocity throughout it's flight, and when the pebble hits the window at the apogee point, that's the only velocity component still present).

**Additional Example: Football Throw** : A quarterback throws the ball to a location where the receiver will be located some (brief) time later. For a particular throw, suppose the ball needs to arrive at a location 30 $m$ from where it was thrown, and needs to arrive at that location 1.5 $sec$ after being thrown.
At what speed and angle must the ball be thrown? (Assume the ball is thrown and caught at the same elevation.)

It's tempting to use the specialized equations for range, height, time, etc here. Looking at them again:

Range $R = (v_o^2 \sin 2\theta)/(g)$ and flight time $t = 2t_A = (2v_o \sin\theta)/(g)$ which means we'd have to use the trig identity to expand $\sin(2\theta)$ again and combine the equations. You should try that (it's not as bad this time), but let's attack this with our previous approach, breaking the motion into separate X and Y motions.

With an origin located at the point where the ball was thrown from, with $+Y$ vertically upward and $+X$ pointing towards where the ball was caught, we have $x_o = 0$, $y_o = 0$, $a_x = 0$ and $a_y = 0$ we have:

X direction : $x(t) = x_o + v_{ox}t + \frac{1}{2}a_x t^2$ or $x(t) = v_{ox}t$ and plugging in what we know: $30 = (v_{ox})(1.5)$ or $\boxed{v_{ox} = 20\ m/s}$.

Y direction : $y(t) = y_o + v_{oy}t + \frac{1}{2}a_y t^2$ and plugging in what we know: $0 = 0 + (v_{oy})(1.5) - (4.9)(1.5)^2$ or $\boxed{v_{oy} = 7.35\ m/s}$.

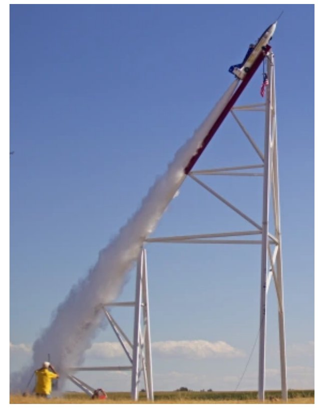
Well - that was a much quicker solution. Now that we know the components of the initial velocity vector, we can find the magnitude and angle:

$v = \sqrt{v_{ox}^2 + v_{oy}^2} = \sqrt{(20)^2 + (7.35)^2} = 21.31\ m/s$ (about 48 $mph$, which is not unusual for a football speed).

$\tan\theta = v_{oy}/v_{ox} = 7.35/20$ from which $\theta = 20.2^o$.

**Additional Example : Snake River Canyon Jump**

: On 16 September 2016, stuntman Eddie Braun successfully jumped a motorcycle (more like a rocket bike) across the 1400-foot-wide chasm of the Snake River Canyon (a feat initially but unsuccessfully attempted by Evel Knievel on 8 September 1974).
The news photograph shows a launch angle of about $54^o$ (instead of the optimal $45^o$), so what must the initial launch speed have been?



(FYI the rocket bike didn't actually land directly, but deployed a parachute on the other side of the chasm. For purposes of this problem though, assume it launched and landed at the same elevation.)

The description of the flight here exactly matches the situation our specialized projectile equations were designed for. We know the range and angle and are looking for the launch speed.

Units: converting the distance first: $(1400\ ft) \times \frac{1\ m}{3.281\ ft} = 426.7\ m$.

Range $R = \frac{v_o^2 \sin{(2\theta)}}{g}$ so here:
$426.7 = \frac{v_o^2 \sin 108^o}{9.8}$ which yields $v_o = 66.3\ m/s$ (about **148** $mph$).

**NOTE** : I noticed the photograph is taken from a point of view that is at an angle. The actual launch angle will be smaller than the apparent angle extracted from the figure. Assuming the bike **was** launched at $\theta = 45^o$: $426.7 = \frac{v_o^2 \sin 90^o}{9.8}$ which yields $v_o = 64.67\ m/s$ (about **145** $mph$).

Another article claimed the rocket bike achieved a maximum height of 2000 $ft$ or about 610 $m$. Let's see what that implies about the launch speed:

Height : $h = (v_o^2 \sin^2\theta)/(2g)$ so $610 = \frac{v_o^2 (\sin 54^o)^2}{(2)(9.8)}$ from which $v_o \approx 135\ m/s$ (about **300** $mph$).

A third article claimed the rocket hit a maximum speed of **400** $mph$.

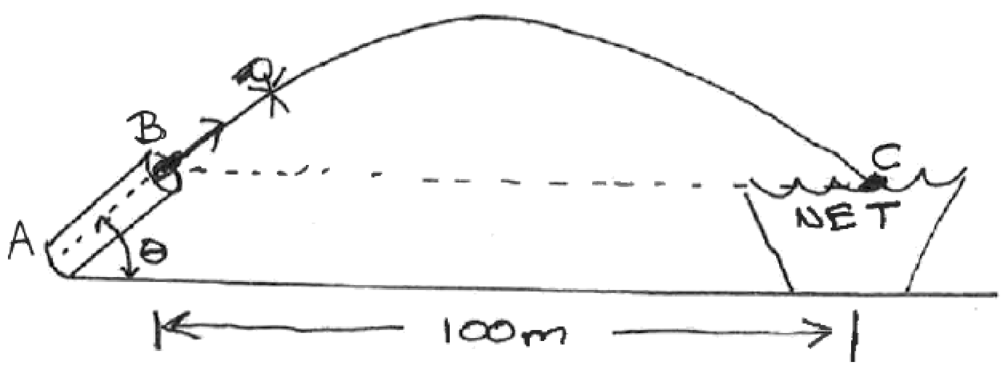
**Why all these inconsistent results?**

Braun's attempt involved a powered rocket bike, where the rocket thrust continued to accelerate the 'bike' even after launch, so really this flight **doesn't** fit the assumptions of our projectile motion equations. Those assume an initial launch speed $v_o$ and then afterwards it's just gravity pulling the object downward.

**Additional Example : Clown Gun Half-time Stunt**
We have been asked to evaluate the feasibility of doing a spectacular stunt during half-time at the Egg Bowl game. Based on the 'clown gun' stunt sometimes seen at carnivals or circuses, a person will be launched from a large cannon and fly through the air across the entire length of the field, landing 'safely' in a large net.

Additional information:

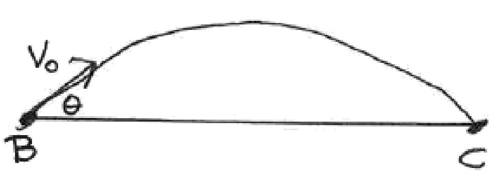
- The launch and landing points are exactly 100 $m$ apart, and are at the same elevation.
- The ~~victim~~ volunteer will accelerate from rest to launch speed over a length of 4 $m$ within the barrel of the cannon.
- A fit human can reasonably withstand an acceleration of 5 to 10 g's for brief periods.

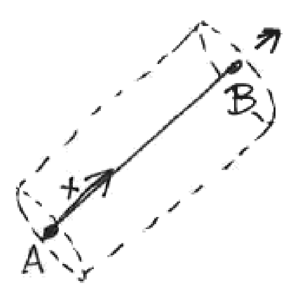



We will find that, as designed, this is not feasible.

- What length barrel is needed to make it (minimally) 'safe'?
- If we can't change the barrel length, what is the maximum safe distance for the stunt (i.e. we'll have to move the net closer in)?

Note what's going on here. The object (the clown) starts at rest at the bottom of the cannon, and leaves the top moving fast enough to fly through the air and land on the net. That means we really have **two connected problems** here: $\boxed{A}$ to $\boxed{B}$ is a 1-D acceleration problem while the object is in the cannon, while $\boxed{B}$ to $\boxed{C}$ is a free-fall problem ( $\vec{a} = \vec{g}$ ) while they're flying through the air.



$\boxed{\text{From B to C in the figure}}$, we have a classic projectile motion situation where the starting and ending points are at the same elevation, so we can use the specialized equations of motion such as: $R = \frac{v_o^2 \sin(2\theta)}{g}$.

For a given launch speed $v_o$, the maximum range will occur when the launch angle $\theta = 45^o$. We need to reach a range of $R = 100\ m$ here, so if we launch at any other angle, the $\sin(2\theta)$ term will be less than 1, which means we'll have to 'make up' the difference by increasing $v_o$. A higher launch speed though means that the object will have to undergo a higher acceleration in the cannon, and we want that value to be as low as possible. If $a$ exceeds $10\ g$ in the cannon, the person being launched could be injured.

So: to put the least stress on the individual, we want to launch with the smallest speed possible, which implies we want to get as much range as possible out of that launch speed. We'll want to aim the cannon so that the launch angle is $45^o$.

At this $45^o$ angle then, we have: $R = \frac{v_o^2 \sin(2\times45)}{g} = \frac{v_o^2 \sin(90)}{g} = \frac{v_o^2}{g}$. Rearranging: $v_o = \sqrt{Rg} = \sqrt{(100\ m)(9.8\ m/s^2)} = 31.305\ m/s$.

$\boxed{\text{From A to B in the figure}}$, we have 1-D motion where the object starts at rest, then accelerates to a launch speed of $31.305\ m/s$ over a distance of $4\ m$, so we can use: $v^2 = v_o^2 + 2a\Delta x$ to find the acceleration required. **NOTE** that this is a new problem, and we've changed what our variables mean. In the cannon, $v_o$ is their initial speed (zero), and $v$ is their final speed of $31.305\ m/s$. (And we've created a 1-D coordinate system, with the X axis being along the line they're travelling in the cannon.

Applying that equation then: $(31.305)^2 = (0)^2 + (2)(a)(4)$ from which $a = 122.5\ m/s^2$.

$g$ represents an acceleration of $9.8\ m/s^2$ so that can be seen as a units conversion factor. Basically $1\ g = 9.80\ m/s^2$. Multiplying our acceleration by that factor:

$a = (122.5\ m/s^2) \times \frac{1\ g}{9.8\ m/s^2} = 12.5\ g$.

Sadly, that exceeds the $10\ g$ limit that we were allowed. What can we do?

$\boxed{\textbf{Option 1}}$ : retain the $100\ m$ range (which requires a launch speed of $31.305\ m/s$ but extend the barrel. Setting $a = 10\ g = 98\ m/s^2$,:

$v^2 = v_o^2 + 2a\Delta x$ implies that
$(31.305)^2 = (0)^2 + (2)(98)(\Delta x)$ or $\Delta x = 5\ m$.

If we lengthen the barrel from 4 to 5 meters, we can adjust the springs or hydraulics or whatever mechanism is causing the acceleration to produce a slightly lower acceleration over a slightly longer distance and still achieve the desired launch speed.

$\boxed{\textbf{Option 2}}$ : retain the existing cannon length, but reduce the range. If $a = 10g = 98\ m/s^2$ in the barrel, the launch speed will be: $v^2 = v_o^2 + 2a\Delta x = (0)^2 + (2)(98)(4)$ from which $v = 28\ m/s$. That implies a max range of $R_{max} = \frac{v_o^2}{g} = \frac{(28)^2}{9.8} = 80\ m$ (which would still be pretty impressive).