PH2213 : Examples from Chapter 3 : Kinematics in 2D, 3D, and Vectors

Key Concepts

A vector is an entity that has both a magnitude and a direction. Common forms:
(a) Cartesian (x,y,z) components: $\vec{A} = (2, 3, 5)$
(b) unit vector notation: $\vec{A} = 2\hat{i} + 3\hat{j} + 5\hat{k}$
(c) 2D polar form: magnitude and direction $\vec{B} = (16\ m, 35^o)$

Summary : 1-D vs Vector Definitions and Equations		
Displacement	Δx	$\Delta \vec{r}$
time interval	Δt	Δt
Average velocity	$\bar{v} = \Delta x / \Delta t$	$\vec{v}_{avg} = \Delta \vec{r} / \Delta t$
Instantaneous velocity	$v = dx/dt$	$\vec{v} = d\vec{r}/dt$
Average acceleration	$\bar{a} = \Delta v / \Delta t$	$\vec{a}_{avg} = \Delta \vec{v} / \Delta t$
Instantaneous acceleration	$a = dv/dt$	$\vec{a} = d\vec{v}/dt$
Equations of Motion : when a or $\vec{a}$ are **constant**		
	$v = v_o + at$	$\vec{v} = \vec{v}_o + \vec{a}t$
	$\bar{v} = v_o + \frac{1}{2}at$	$\vec{v}_{avg} = \vec{v}_o + \frac{1}{2}\vec{a}t$
	$x = x_o + \frac{1}{2}(v_o + v)t$	$\vec{r} = \vec{r}_o + \frac{1}{2}(\vec{v}_o + \vec{v})t$
	$x = x_o + v_o t + \frac{1}{2}at^2$	$\vec{r} = \vec{r}_o + \vec{v}_o t + \frac{1}{2}\vec{a}t^2$
	$v^2 = v_o^2 + 2a\Delta x$	$v^2 = v_o^2 + 2a_x\Delta x + 2a_y\Delta y + 2a_z\Delta z$

Free-fall motion: the object is **only** moving under the influence of gravity, which means it has an acceleration directed downward towards the earth (or moon, or whatever planet). The magnitude of the acceleration due to gravity on the earth is $|a| = g = 9.80\ m/s^2$. The entity g is always a **positive constant**. The acceleration, on the other hand, may be either $+9.8\ m/s^2$ or $-9.8\ m/s^2$ depending on **your** choice of coordinate system.

Specialized projectile motion equations
If we have an object initially moving at some velocity and direction that is only affected by gravity, and using a coordinate system with positive upward and the initial position of the object defining the origin:

$y = (v_{oy}/v_{ox})x - gx^2/(2v_{ox}^2)$ or: $y = (\tan\theta)x - gx^2/(2v_o^2 \cos^2\theta)$
Time to reach maximum altitude: $t_A = (v_o \sin\theta)/g$
Maximum altitude: $h = \frac{v_o^2 \sin^2\theta}{2g}$

If the object **lands** at the **same height** from which it was launched:

Range: $R = \frac{v_o^2 \sin 2\theta}{g}$ Maximum range: $R_{max} = \frac{v_o^2}{g}$ (occurs at $\theta = 45^o$)
Time to land: $t = 2t_A$

Common Errors

Sign of g | Mixing x,y components | trig
Using the specialized projectile equations when $y_{final} \neq y_{initial}$

1. **Basic Vector Math**

   Suppose $\vec{A} = 3\hat{i} - 2\hat{j} + 5\hat{k}$ and $\vec{B} = 2\hat{i} + 4\hat{j} - 7\hat{k}$.

   Compute $\vec{C} = 2\vec{A} - 3\vec{B}$ and find the magnitude of $\vec{C}$.

   We can multiply a vector by a scalar. In Cartesian or unit vector forms, this just means multiplying each term of the vector by that scalar:

   $2\vec{A} = 2 \times (3\hat{i} - 2\hat{j} + 5\hat{k}) = 6\hat{i} - 4\hat{j} + 10\hat{k}$.

   $7\vec{B} = 7 \times (2\hat{i} + 4\hat{j} - 7\hat{k}) = 14\hat{i} + 28\hat{j} - 49\hat{k}$.

   $\vec{C} = 2\vec{A} - 7\vec{B} = (6\hat{i} - 4\hat{j} + 10\hat{k}) - (14\hat{i} + 28\hat{j} - 49\hat{k})$ and collecting terms:

   $\vec{C} = (6 - 14)\hat{i} + (-4 - 28)\hat{j} + (10 + 49)\hat{k}$ or finally: $\boxed{\vec{C} = -8\hat{i} - 32\hat{j} + 59\hat{k}}$.

   The magnitude of a vector is the square root of the sum of the squares of its components, so
   $C = |\vec{C}| = \sqrt{(-8)^2 + (-32)^2 + (59)^2} = \sqrt{64 + 1024 + 3481} = \sqrt{4569} = +67.59$.
   (The **magnitude** is always a positive number (or zero); it can never be negative.)

2. **Vector Simultaneous Equations**

   Occasionally when dealing with forces later, we may encounter situations where we have 2 unknown vectors but we also have 2 equations involving them.

   For example, suppose we have two unknown vectors $\vec{A}$ and $\vec{B}$ but we know that $\vec{A} + 2\vec{B} = 3\hat{i} - 3\hat{j}$ and that $\vec{A} - \vec{B} = 6\hat{i}$.

   We can use the same mathematical machinery we would use for 'normal' simultaneous equations.

   (a) One option is to rearrange one of the equations to solve for one of the variables, then substitute that expression into the second equation. This can get somewhat tedious.

   (b) Another option is to look at the two equations and see if there is some way to combine them to immediately eliminate one of the variables. For example here we see that if we multiply the second equation by 2, we'll have $2\vec{A} - 2\vec{B} = 12\hat{i}$ and we can now **add** this equation to the first equation and eliminate the $\vec{B}$ term altogether:

   $$\begin{array}{rcl} \vec{A} + 2\vec{B} & = & 3\hat{i} - 3\hat{j} \\ 2\vec{A} - 2\vec{B} & = & 12\hat{i} \\ \hline 3\vec{A} + 0 & = & 15\hat{i} - 3\hat{j} \end{array}$$

   Dividing the last equation by 3: $\vec{A} = 5\hat{i} - 1\hat{j}$

   Now that we know $\vec{A}$, we can use either of our original equations to find $\vec{B}$. For example, the second equation was $\vec{A} - \vec{B} = 6\hat{i}$ so $\vec{B} = \vec{A} - 6\hat{i}$ or $\vec{B} = (5\hat{i} - 1\hat{j}) - 6\hat{i} = -1\hat{i} - 1\hat{j}$.

3. Ball Rolling Off Table

Suppose we have a steel ball moving at 0.5 m/s as it rolls off the side of a table that is 1.0 m above the floor. Where will the ball land? How long does it take to reach the floor? How fast is it moving when it hits the floor? What is the complete vector velocity the instant just before it hits the floor?
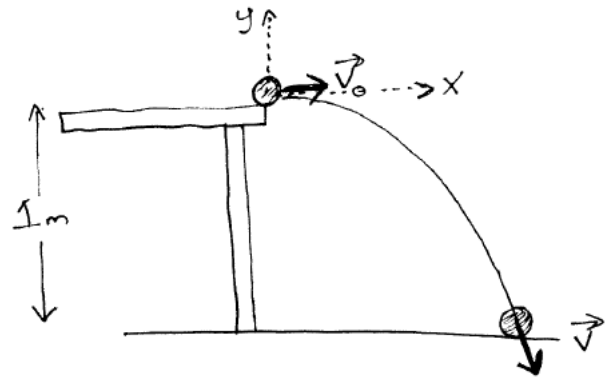
We have 2-dimensional motion here, so let's define an (X,Y) coordinate system with the origin being where the ball is right at the edge of the table, with +Y pointing vertically upward, and +X horizontal in the initial direction the ball is moving. In this coordinate system, the initial position of the ball will be $x_o = 0.0$ and $y_o = 0.0$. The initial velocity will be $v_{ox} = +0.5$ m/s and $v_{oy} = 0.0$ m/s. The acceleration is entirely due to gravity, which is downward, so $a_x = 0$ and $a_y = -9.8$ m/s^2. We'll start our stop watch at the instant the ball leaves the table and starts its free fall towards the ground. When the ball reaches the floor, it will have a y coordinate of $y = -1.0$ m and some unknown x coordinate.

Not seeing any particularly clever shortcut to solve this, we'll just start with the generic equations of motion:

The general equation of motion for the ball in the X direction is: $x = x_o + v_{ox}t + \frac{1}{2}a_xt^2$ so with our choice of coordinates: $x = 0.0 + (0.5)(t) + 0$ or just $\boxed{x = 0.5t}$.

The general equation of motion for the ball in the Y direction is $y = y_o + v_{oy}t + \frac{1}{2}a_yt^2$ so with our choice of coordinates: $y = 0.0 + 0.0 + \frac{1}{2}(-9.8)t^2$ or just $y = -4.9t^2$.

We can use this last equation to find how long it takes the ball to reach the floor, since at that point $y = -1.0$ m so: $-1 = -4.9t^2$ or $t^2 = 1/4.9$ so $\boxed{t = 0.4518 \ s}$.

In the X direction: $x = 0.5t$ so at $t = 0.4518$ s (the point where the ball reaches the floor), we know that $x = (0.5)(0.4518) = 0.2259$ m so we now know that the ball hits the floor about 22.6 cm away from the edge of the table.

Alternate Approach : we could also have done these in reverse order. We found that $x = 0.5t$ which means that $2x = t$. We can thus eliminate the t variable from our y equation of motion and convert $y = -4.9t^2$ into $y = -4.9(2x)^2 = -19.6x^2$. This gives us a direct relationship between x and y at every point along the trajectory. When the ball hits the ground, $y = -1$ so we can find the x value from $-1 = -19.6x^2$ or $x^2 = 1/19.6$ and $x = 1/\sqrt{19.6} = 0.2259$ m. Since we were able to do this in one step, never rounding off an intermediate result, technically this is a more accurate method, although in this case both methods gave the same answer, to 4 significant figures. Now that we know the value of x, we can find the time from $x = 0.5t$ or $t = 2x = (2)(0.2259) = 0.4518$ s.

Speed and Velocity at floor

We can find the speed and velocity when it hits the floor in a couple of ways.

Between the edge of the table and the floor, we have: $v^2 = v_o^2 + 2a_y\Delta y$. Δy is $y_{final} - y_{initial} = (-1.0\ m) - (0.0\ m) = -1.0\ m$ so: $v^2 = (0.5)^2 + (2)(-9.8)(-1) = 0.25 + 19.6 = 19.85$ and finally $v = 4.455\ m/s$. ('Speed' is a magnitude, so is always non-negative.)

The velocity equations for free fall are:

$v_x = v_{ox} + a_x t$ but we have no X component of acceleration here, so $v_x = v_{ox} = 0.5\ m/s$.

$v_y = v_{oy} + a_y t$ so in our case: $v_y = 0.0 - 9.8t$ so $v_y = -(9.8\ m/s^2)(0.4518\ s) = -4.4276\ m/s$. Since we used a time that was only accurate to 4 significant figures, then the answer here will only have, at best, 4 significant figures (and maybe only 3) and should be rounded to $-4.428\ m/s$.

(As a check, the speed would be $v = \sqrt{v_x^2 + v_y^2} = \sqrt{(0.5)^2 + (-4.4276)^2} = \sqrt{0.25 + 19.6} = 4.455\ m/s$ which matches what we computed the other way.)

Alternate Approach : we know that the x component of velocity will remain at $v_x = 0.5\ m/s$ throughout the trajectory. We also found the **speed** at which the ball hits the floor to be $4.455\ m/s$. Well that speed is just the magnitude of the velocity vector when the ball reaches the floor, and that vector has two components: v_x and v_y, so $v^2 = v_x^2 + v_y^2$ or $(4.455)^2 = (0.5)^2 + v_y^2$ or $v_y^2 = 19.6$ which means that $v_y = \pm 4.427\ m/s$. This didn't tell us the **sign** of the y component of the velocity, but we know the ball is moving downward so $v_y = -4.427\ m/s$ would be the correct solution here. Note that the two methods ended up being off in the 4th significant figure, so by rounding off all our intermediate results to 4 significant figures, we will end up with a final result that is slightly less accurate.)

Angle at Floor : However we get the components, we've hit the floor with $v_x = 0.5\ m/s$ and $v_y = -4.43\ m/s$ (we'll go ahead and round that off to 3 significant figures since we have a little uncertainty about the 4th digit...). What angle relative to the horizontal does the ball hit the floor? We've covered this in class several times, but since all the inverse trig functions are multi-valued, I find it safer to find an angle by treating the triangle in the problem as a physical triangle, with all sides having positive values, then relate that to the how they're asking for the angle.
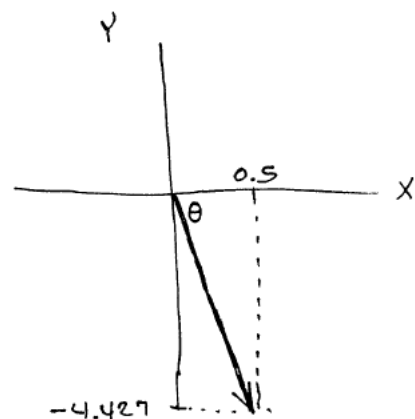
In this case, from the figure we see that $\tan\theta = 4.43/0.5 = 8.86$ and $\theta = 83.6^o$. Given the direction this vector is making in the figure, the correct answer might be something like 83.6^o **below** the horizontal, or in the common polar coordinate convention we might call this angle -83.6^o or $360 - 83.6 = +276.4^o$. (That last one would be the angle measured counterclockwise all the way around starting at the +X axis, which is often (but not always) how the online homework asks for angles.)

4. **Explicit Equation for $\vec{r}$**

Suppose we have a closed-form expression for the position vector as a function of time, say: $\vec{r}(t) = (2\hat{i} - 3\hat{j}) + (4\hat{i} + 6\hat{j})t + (-6\hat{j})t^2$, where the units are such that if the time is provided in seconds, the location vector will be in units of meters.

Determine the velocity as a function of time. $\vec{v} = d\vec{r}/dt$ so differentiating the equation we have for $\vec{r}$: $\vec{v} = 0 + (4\hat{i} + 6\hat{j}) + 2(-6\hat{j})t$ or $\vec{v} = (4\hat{i} + 6\hat{j}) - 12t\hat{j}$.

This gives $\vec{v}$ as a function of time, and we see that it does depend on the time, so the velocity is changing.

The vector acceleration $\vec{a} = d\vec{v}/dt$ so differentiating our equation for $\vec{v}$ as a function of time we get $\vec{a} = -12.0\hat{j}$. That's a constant, so apparently this motion represents some sort of free-fall type motion but the acceleration is 12. (We assumed all our units were in meters and seconds, so this being an acceleration, it must be 12 m/s^2 in the $-Y$ direction, so perhaps we're on a planet with slightly higher gravity than we have on earth.)

(a) Find the average velocity between $t = 0$ and $t = 4$.

The average velocity is defined as $\vec{v}_{avg} = \Delta\vec{r}/\Delta t$. We have an equation giving the position vector $\vec{r}$ as a function of time: $\vec{r}(t) = (2\hat{i} - 3\hat{j}) + (4\hat{i} + 6\hat{j})t + (-6\hat{j})t^2$.

At $t = 0$ this evaluates to: $\vec{r}(0) = 2\hat{i} - 3\hat{j}$.

At $t = 4$ this becomes: $\vec{r}(4) = (2\hat{i} - 3\hat{j}) + (4\hat{i} + 6\hat{j})(4) + (-6\hat{j})(4^2)$ or $\vec{r}(4) = (2\hat{i} - 3\hat{j}) + (16\hat{i} + 24\hat{j}) + (-96\hat{j})$. Combining terms, $\vec{r}(4) = 18\hat{i} - 75\hat{j}$.

Finally, $\vec{v}_{avg} = \frac{\vec{r}(4) - \vec{r}(0)}{4 - 0} = 4\hat{i} - 18\hat{j}$.

(b) Find the instantaneous velocity at $t = 2$.

The instantaneous velocity is the derivative of $\vec{r}$ with respect to time, and we earlier found that to be $\vec{v} = (4\hat{i} + 6\hat{j}) - 12t\hat{j}$ so evaluating that at $t = 2$ we find: $\vec{v}(2) = 4\hat{i} + 6\hat{j} - 24\hat{j} = 4\hat{i} - 18\hat{j}$.

(Why did these turn out to be the same? We found earlier that we have a constant acceleration, that means $\vec{v} = \vec{v}_o + \vec{a}t$ applies: the velocity is increasing linearly with time. That implies that the average velocity over some interval will equal the instantaneous velocity at the midpoint: $\vec{v}_{avg} = \vec{v}_o + \frac{1}{2}\vec{a}t$. This only occurs if the acceleration vector is a constant though, so the times involved here were rigged to produce this result.)

5. **Bullet Fired at a Target (A)**

   This is a classic problem. Suppose we set up a target such that the bullseye is at exactly the same height as the gun and we then fire directly at the bullseye. Where will the bullet strike the target?

   Suppose the bullet is fired directly horizontally at an initial speed of 300 m/s and that the target is 100 m away.
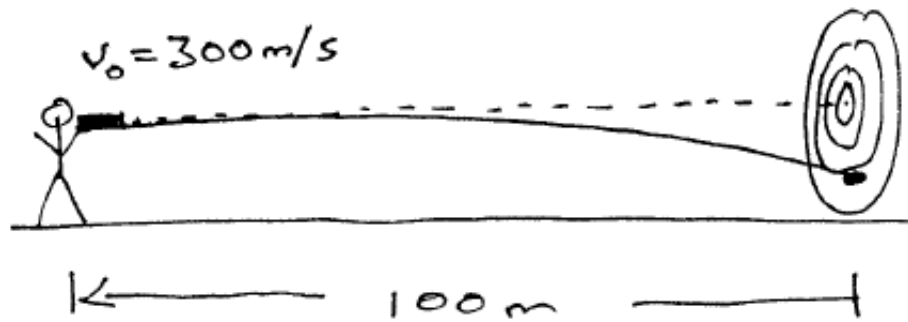
   Starting with our generic equations of motion:

   In the X direction: $x = x_o + v_{ox}t + \frac{1}{2}a_x t^2$
   In the Y direction: $y = y_o + v_{oy}t + \frac{1}{2}a_y t^2$

   We'll define our origin to be the point where the bullet leaves the gun (and we'll start $t = 0$ at that instant also). This makes $x_o = 0$ and $y_o = 0$. Our +Y axis will be pointing vertically upward, and our +X axis will point horizontally from the gun to the target.

   The bullet is flying initially horizontally, so $v_{ox} = 300m/s$ and $v_{oy} = 0m/s$

   Once the bullet leaves the gun, the only force acting is gravity, so our only acceleration will be due to gravity. In our coordinate system: $a_x = 0m/s$ and $a_y = -9.80m/s$.

   Using these values, our generic equations of motion become: $x = (300)(t)$ and $y = -4.9t^2$.

   At the **target** end of the motion: $x = 100$ m, so that lets us find the time it took the bullet to travel that far: $(300) = (100)(t)$ or $t = (1/3)$ sec.

   Now we can find where it is in Y at that point: $y = (-4.9)(1/3)^2 = -4.9/9 = -0.544$ m or about 21 inches low.

   Note : compare this scenario with the earlier problem of the ball rolling off the table onto the floor. Both are the same in the sense that we had to break the motion into separate X and Y directions, but their solutions differ in that in this case we know how far the object travels in the <u>horizontal</u> (X) direction (100 meters to the target) but in the case of the ball we knew how far it travelled in the <u>vertical</u> (Y) direction (1 meter down to the floor)

6. **Bullet Fired at a Target (B)**

In the previous problem we missed the bullseye. What if we aim the gun upward so that the parabolic motion of the bullet causes it to land right at the bullseye. What angle do we need to aim the gun up at to achieve this?
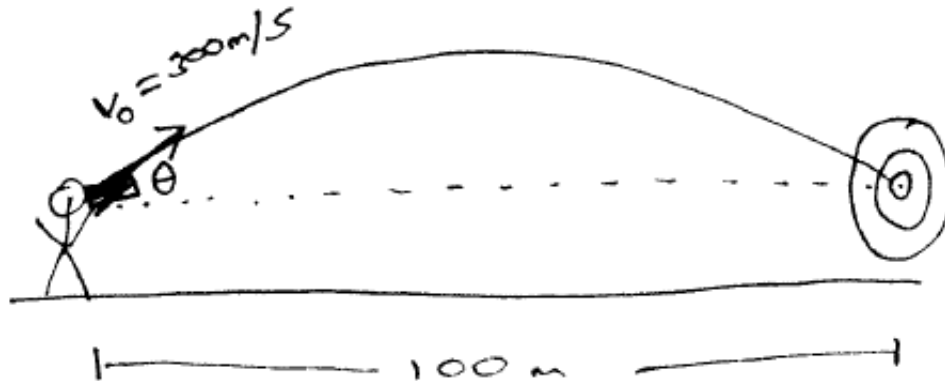

Note that in this case, the bullet leaves the gun and arrives at the target at the **same** y coordinate, so we can use some of our **specialized** projectile motion equations.

In particular: $R = (v_o^2 \sin 2\theta)/g$. Rearranging this a bit: $\sin 2\theta = Rg/v_o^2$ so in our case, $\sin 2\theta = (100 \ m)(9.8 \ m/s^2)/(300 \ m/s)^2 = 0.01089$.

Taking the inverse sine of both sides: $2\theta = 0.624^o$ or $\theta = 0.312^o$ or less than a third of a degree.

Where should we be aiming on the target to achieve this? I.e. how far **above** the bullseye do we need to aim so that the parabolic motion of the bullet causes it to hit the bullseye? (Note in this figure, I've greatly exaggerated the vertical scale.)

Let y be the height above the bullseye where we are aiming. Then $\tan \theta = y/R$ so $y = R \tan \theta = (100 \ m) \tan 0.312^o = 0.544 \ m$.
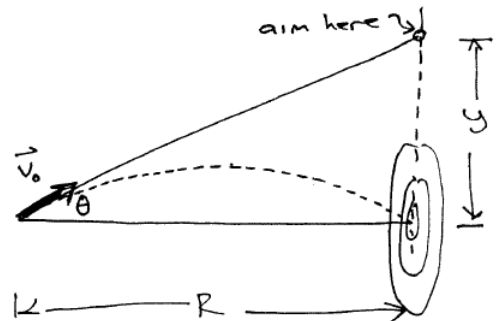
Basically if we aim right for the bullseye, we hit $0.544 \ m$ too low. If we aim $0.544 \ m$ too high, we'll end up hitting the bullseye.

These two numbers are **not** exactly identical if we do the calculations to more significant figures but the angle is so small here that it works out as a good rule of thumb (again, as long as the angle is small).

7. **Baseball Landing on the Roof of a Building**

Suppose we hit a baseball such that it leaves the bat at a speed of 27.0 m/s at an angle of 45°. At that instant, the ball is 1.0 m above the ground. Some time later, it lands on the rooftop of a nearby building at a point that is 13.0 m above the ground level. What horizontal distance did the ball travel? How long was it in the air? What maximum height did it reach? How fast is it moving when it hits the roof?
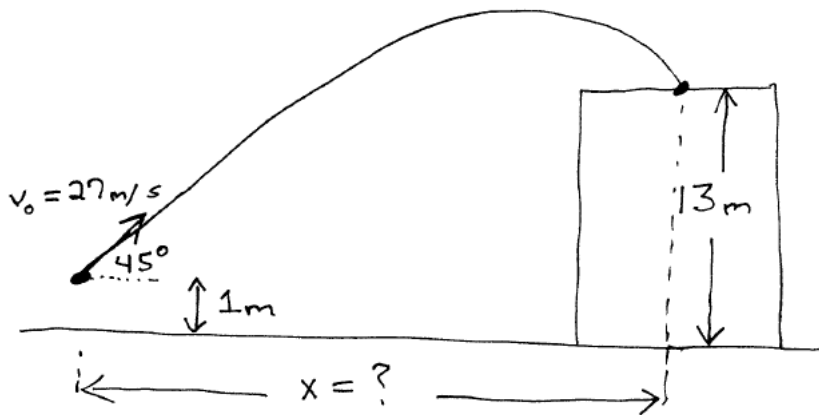
Here the ball is landing at a **different** height from which it was 'launched' so we **can't** use our specialized projectile motion equations. We'll have to resort to the general equations of motion. Let's use coordinates where $x_o = 0$ but we'll set the y axis such that the ground is at $y = 0$. That means the ball starts at $y_o = 1.0$ m and ends at $y = 13.0$ m. (We could also have used coordinates where the origin is the point where the ball was struck, which makes $x_o = y_o = 0.0$ but in these coordinates, $y = 0$ is shifted upward by a meter from the ground, which would mean when the ball lands on the roof, that would represent a y coordinate of $y = 12$ m instead of 13. Either way is fine, as long as you remember to adjust everything in the problem to be in **your** choice of coordinates.)

The ball is 'launched' at a speed of 27.0 m/s at an angle of 45° up from the horizontal, which means that $v_{ox} = v_o \cos \theta = (27.0 \ m/s) \cos 45^o = 19.09 \ m/s$ and $v_{oy} = v_o \sin \theta = (27.0 \ m/s) \sin 45^o = 19.09 \ m/s$.

The general equation of motion for the ball in the X direction is: $x = x_o + v_{ox}t + \frac{1}{2}a_xt^2$ so with our choice of coordinates: $x = 0.0 + (19.09)(t) + 0$ or just $\boxed{x = 19.09t}$.

The general equation of motion for the ball in the Y direction is $y = y_o + v_{oy}t + \frac{1}{2}a_yt^2$ so with our choice of coordinates: $y = 1.0 + 19.09t + \frac{1}{2}(-9.8)t^2$ or just $\boxed{y = 1 + 19.09t - 4.9t^2}$.

We can use this last equation to find how long it takes the ball to reaches the roof, since at that point $y = 13.0$ m so:

$13 = 1 + 19.09t - 4.9t^2$ which we can rearrange into: $4.9t^2 - 19.09t + 12 = 0$. This quadratic equation will have two solutions: $t = \frac{19.09 \pm \sqrt{(19.09)^2 - (4)(4.9)(12)}}{(2)(4.9)}$ or $t = 0.788 \ s$ and $t = 3.108 \ s$.

Both of these are positive, but which is the correct solution? Looking at the figure, we see that the ball passes through $y = 13$ on the way up and then again reaches it on the way down (at which point it hits the roof) so apparently it's the second solution that corresponds to the 'hitting the roof' situation.

Now that we know that the ball lands on the roof at $t = 3.108 \ s$, we can find its X coordinate at that point: $x = (19.09 \ m/s)(t) = (19.09 \ m/s)(3.108 \ s) = 59.33 \ m$.

How high up did the ball fly? If we're careful, we can use a shortcut here. The ball doesn't know it's not going to hit the ground after it passes through the apogee point, so we can still use the specialized projectile motion equations to find h from $h = \frac{v_o^2 \sin^2 \theta}{2g}$. That equation was derived by looking for the point where v_y becomes 0, which is still what's happening here at that point. $h = \frac{(27.0 \ m/s)^2 \sin^2 45}{(2)(9.8)} = \frac{(27)^2(0.5)}{19.6} = 18.6 \ m$ **But**, those specialized equations required

us to define our origin of coordinates to be where the motion starts. So in this case, that means the ball will reach a height of 18.6 m **above** where it **started**, which was 1 m above the ground, so overall the height above the ground will be $18.6 + 1 = 19.6$ m.

How fast was the ball moving when it hit the roof?

$v^2 = v_o^2 + 2a\Delta y$ and here $\Delta y = y_{final} - y_{initial} = 13.0 - 1.0 = 12.0$ m so $v^2 = (27.0)^2 + (2)(-9.8)(12) = 729 - 235.2 = 493.8$ or $v = \sqrt{493.8} = 22.22$ m/s.

Alternate Solution :

Suppose we start the problem by finding the speed at which the ball lands on the root. Using our v^2 equation, we found that it hits the roof at a speed of $|v| = 22.22$ m/s. Let v_x by the X component of the velocity when the ball hits and roof, and v_y by the Y component. Then $v^2 = v_x^2 + v_y^2$. The only acceleration here is due to gravity (downward) which means the X velocity **never changes** along the ball's trajectory. It started at $v_{ox} = 19.09$ m/s and will remain that until it hits the roof. So in $v^2 = v_x^2 + v_y^2$ we have: $(22.22)^2 = (19.09)^2 + v_y^2$ or $v_y^2 = 129.3$ from which $v_y = \pm 11.37$ m/s. We know the ball is falling downward onto the roof, so $v_y = -11.37$ m/s.

But : $v_y = v_{oy} + a_y t$ so $-11.37 = +19.09 + (-9.8)(t)$ from which $t = 3.108$ s (which gave us how long the ball was in flight without having to deal with a quadratic equation, and matches the solution we got doing it that way).

Now that we know how long the ball was in the air, we can find how far it flew horizontally: $x = x_o + v_{ox}t + \frac{1}{2}a_x t^2$ but $x_o = 0$ and $a_x = 0$ so $x = 0 + (19.09)(3.108) + 0 = 59.33$ m (again, same result we got the other way).

8. **Longest Football Throw (A)**

The longest throw in football was apparently about $76\ m$.

(a) Assuming the ball is caught at the same height from which it was thrown, how fast was the ball thrown? (Assume the ball was thrown at a 45° angle.)

Since y is the same at each end of this projectile motion, we can use our ultra-specialized equations. In particular, $R = (v_o^2 \sin 2\theta)/g$.

The maximum range (for a given initial speed) occurs when the object is launched at a 45° angle: $R_{max} = (v_o^2)/g$ so let's assume the quarterback threw the ball at that angle. Then rearranging that equation, we find that $v_o^2 = gR_{max}$ or $v_o = \sqrt{gR_{max}} = \sqrt{(9.8)(76)} = 27.29\ m/s$ which is about 61 $miles/hr$.

(b) How long was the ball in flight before being caught?

The total time of flight for this special type of motion (both ends being at the same y value) is $t = 2t_A = (2) \times (v_o \sin \theta)/g = (2)(27.29) \sin 45/9.8 = 3.94\ s$.

(c) How high up did the ball go during this pass? (We don't know the height above the ground where the ball started, although we can probably estimate it to be about $2\ m$, so here we're really asking how high up, relative to it's initial height, will the ball go?) The maximum height (relative to the starting y coordinate) will be $h = \frac{v_o^2 \sin^2 \theta}{2g}$ so with $v_o = 27.29\ m/s$ and $\theta = 45^o$ we end up with $h = 19.0\ m$ (about 60 feet up).

9. **Longest Football Throw (B)**

In the previous example, we found that the ball was in the air for quite a long time: almost 4 seconds. We can also reach the same target by throwing the ball faster at a smaller angle.

Suppose the ball is thrown at a 30° angle instead of 45°. How fast will the quarterback have to throw it to reach the same target $76\ m$ away? How long is the ball in flight now?

We still have the special situation where the start and end points of the motion of the ball are at the same y level, so we can still use our specialized equations.

Range : $R = (v_o^2 \sin 2\theta)/g$ so $v_o^2 = gR/\sin 2\theta = (9.8)(76)/\sin 60 = 860.0$ or $v = 29.33\ m/s$. This is a little bit faster than it was thrown in the previous problem.

How long is the ball in the air now? $t = 2t_A = (2) \times (v_o \sin \theta)/g = (2)(29.33) \sin 30/9.8 = 2.99\ s$.

Throwing the ball at 30° instead of 45°, the quarterback had to throw the ball about 7 percent faster, but it reached the target about 25 percent more quickly, giving the opposing team less time to react.

To complete the comparison, how high up does the ball go this time? $h = \frac{v_o^2 \sin^2 \theta}{2g}$ so with $v_o = 29.33\ m/s$ and $\theta = 30^o$ we get $h = 10.97\ m$, which is only about half the height it reached before when the ball was thrown at a 45° angle. (And again, this is just the height relative to it's starting point, which was already a couple meters above the ground, so technically all we can say here is that the ball reached a height of $10.97\ m$ above the height from which it was thrown. Any actual exam question would be more specific, giving you the initial height for example, or just asking for the height relative to the initial height...)