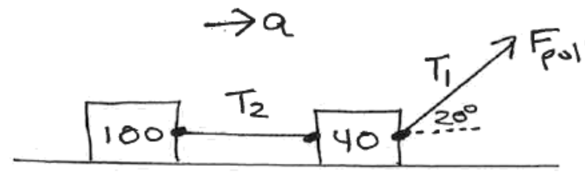
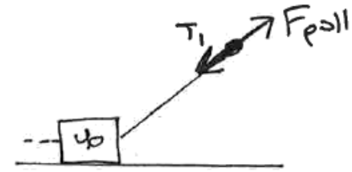
PH2213 Fox : Lecture 09
Chapter 4 : Dynamics : Newton's Laws of Motion

Example : Connected Sleds : Suppose we have two sleds being pulled across flat, horizontal (frictionless) ground. The sleds are connected together with a short horizontal piece of rope, and the front sled is being pulled with a rope that has some amount of pulling force applied to it at the 20^o angle shown in the figure. The sleds are <u>observed</u> to be accelerating to the right at 0.5 m/s^2. Determine the pulling force and the tension in the rope connecting the two sleds.



We have to apply Newton's Laws to each object and point of interest **separately**, so let's start at the front and work backwards. (We'll find this was the wrong direction to work the problem.)

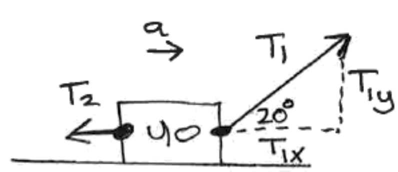
<u>Focusing on the point where the person is pulling on the front rope,</u> we have their pulling force F_{pull} in line with the front rope, which is pulling back with a tension of T_1.
That point actually <u>is</u> accelerating here. $\sum \vec{F} = m\vec{a}$ and $\vec{a}$ is nonzero here. The mass of that point is very small though, and in fact for the remainder of this course we're going to assume that all our connecting elements (ropes, wires, strings, cables, and so on) are all <u>massless</u>.



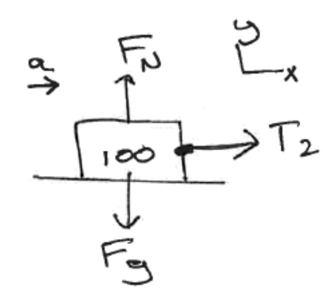
That means the $m\vec{a}$ right-hand side of Newton's Laws will be zero, not because the point isn't accelerating but because we're ignoring the mass of the connecting element. This assumption is fairly common, even for real materials, and really only becomes a problem is the connecting elements have significant mass.

So, assuming our ropes are massless, the pulling force is exactly equal to the tension in the front cable (just in the opposite direction).

<u>Moving on to the sled up front</u>, we have T_1 (at a 20 degree angle, so we'll need it's components), T_2 pulling back to the left (our negative X direction), and also (not shown) gravity downward and the normal force upward. This object is accelerating to the right at 0.5 m/s^2 so applying Newton's Laws in the X direction:
$\sum F_x = ma_x$ yields $-T_2 + T_1 \cos(20^o) = (40\ kg)(0.5\ m/s^2)$ or $-T_2 + 0.93969T_1 = 20$. Two unknowns, so can't go any further (yet).



<u>Looking at the sled at the back now</u>, we have gravity downward, a normal force upward, and the tension T_2 in the connecting rope pulling to the right.
Here then: $\sum F_x = ma_x$ becomes $T_2 = (100\ kg)(0.5\ m/s^2) = 50\ N$.
(Well, clearly we should have started here at the back since applying Newton's Laws immediately provided one of our unknowns.)



Now that we know T_2, we can work backwards. We earlier found that $-T_2 + 0.93969T_1 = 20$ so substituting in the now known value of $T_2 = 50\ N$ yields $T_1 = 74.5\ N$.

Sleds : continued

(a) What force would be needed to cause the sleds to accelerate to the right at 10 m/s^2? What will F_N be in this case? Is this situation possible?

This time we'll start with the sled at the back. $\sum F_x = ma_x$ becomes $T_2 = (100\ kg)(10.0\ m/s^2) = 1000\ N$.

Moving up to the front sled, we have:

$\sum F_x = ma_x$ which yielded $-T_2 + T_1 \cos{(20^o)} = ma = (40\ kg)(10.0\ m/s^2)$ so now: $-1000 + T_1 \cos{(20^o)} = 400$ which yields $T_1 = 1490\ N$.

Fine, but before we go, let's look at the normal force on the front sled. $\sum F_y = 0$ on that object, so collecting the Y components of the forces acting on the sled, we have F_N upward, the vertical component of T_1 upward, and gravity downward: $\sum F_y = F_N + T_1 \sin{(20^o)} - mg = 0$ or $F_N + (1490\ N)\sin{(20^o)} - (40)(9.8) = 0$ but here we end up with $F_N = -118\ N$.

What does a **negative** normal force mean? We drew F_N as acting upward on the front sled, preventing it from moving through the ground. A negative F_N means it's actually acting downward, pulling the sled towards the ground. That's not how the normal force works though.

It looks like if we want to cause these sleds to accelerate at 10 m/s^2 to the right, there isn't enough normal force to keep the front sled in contact with the ground and the picture we drew isn't possible.

(b) What is the maximum force and acceleration allowed here if the front sled is to remain in contact with the ground?)

With $a = 0.5\ m/s^2$ everything worked out; moving up to $a = 10.0\ m/s^2$ we failed to find a valid solution where the front sled remained on the ground.

What's the maximum acceleration we can allow here so that F_N doesn't go negative? Let's look for a solution where $F_N = 0$.

Starting with the front sled's Y equation: $\sum F_y = F_N + T_1 \sin{(20^o)} - mg = 0$ and we're letting the normal force get down to 0 here, so $0 + T_1 \sin{(20^o)} - mg = 0$ or $T_1 = mg/\sin{(20^o)} = (40)(9.8)/\sin{(20^o)} = 1146\ N$.

Looking at the front sled's X equation: $\sum F_x = ma_x$ so $-T_2 + T_1 \cos 20 = ma$ so: $\boxed{-T_2 + 1077 = 40a}$. Well that's as far as we can go for now, so let's look at the sled at the back:

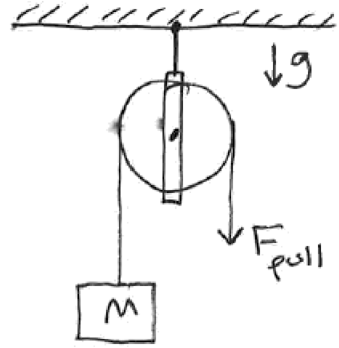
$\sum F_x = ma_x$ so $\boxed{T_2 = 100a}$.

Aha - now we have two equations and two unknowns. We're trying to find the acceleration for this edge case, so let's eliminate T_2 in the first equation using the second equation:

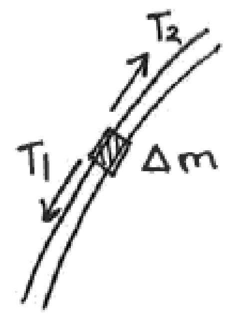
$-(100a) + 1077 = 40a$ or $1077 = 140a$ from which $a = 7.69\ m/s^2$.

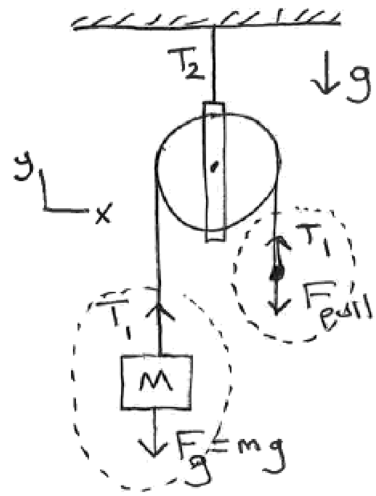
Any higher acceleration and the upward component of T_1 will overcome the force of gravity on the front sled and lift it off the ground (at which point our figure would no longer represent the situation to be analyzed).

Example: Simple Pulley
A 40 kg box is hanging from a cable that runs over a pulley (which is attached to the ceiling).
(a) How much force is the person exerting downward on the right to be able to hold the box in place?
(b) How much tension is in the cable connecting the pulley to the ceiling?



MIDDLE FIGURE: First, we have to consider how many tension variables we have here. If we focus on the cable connecting the box to the person holding it, let's consider a tiny element of the cable as it wraps around the pulley. If the tension is changing in the cable, we'll have (say) T_1 pulling counterclockwise and T_2 pulling clockwise, so using a coordinate system tangent to the pulley at that point, $\sum F = ma$ implies that $-T_1+T_2 = (\Delta m)a$. In this problem, there's no acceleration, so the right side of that equation is zero, which means that $T_1 = T_2$ and we can make that argument millimeter by millimeter along the cable to verify that the tension in that cable is the same everywhere: all the way from the box to the person holding on to the other end of the cable. That wouldn't be true if the system is accelerating though, unless the cable itself has zero mass. In the real world, these connecting wires and cables do have mass, so the tension actually will be different on the two ends of the cable, but for the rest of this course we'll assume that all our 'connecting elements' (ropes, string, cables, etc) are all mass-less. (Actually even in real systems that assumption is often used even when it's not technically true...)
Now we can label the system properly (BOTTOM FIGURE). Let's have T_1 be the tension in the cable running from the box up around the pulley and then down to where the person is holding onto it. T_2 will be the tension in the cable connecting the pulley to the ceiling.

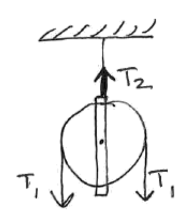




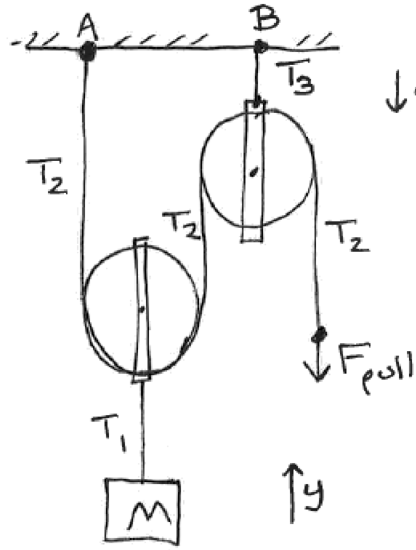
Applying Newton's Laws in the Y direction to the crate: $\sum F_y = ma_y = 0$ so $T_1 - mg = 0$ or $\boxed{T_1 = mg = 392\ N}$.

Applying Newton's Laws in the Y direction to the point where the pulling force is being applied: $T_1 - F_{pull} = ma_y = 0$ so $T_1 = F_{pull}$ which tells that $F_{pull} = T_1 = mg = 392\ N$ as well.

Applying Newton's Laws in the Y direction to the pulley itself, we have T_2 upward and <u>two</u> copies of T_1 pulling downward: one on each side of the pulley, so $\sum F_y = ma_y = 0$ or $T_2 - T_1 - T_1 = 0$ so $T_2 = 2T_1 = (2)(mg) = 784\ N$.



Example: Double Pulley
In the previous example, we didn't really gain any mechnical advantage: we had to pull on the box with a force equal to it's full weight. In addition, the pulley ended up exerting a force on the ceiling equal to <u>twice</u> the weight of the box.
Suppose we add a second pulley as shown in this figure. How much force do we need to exert now, and determine the tensions in all the cables present. (The mass of the box remains 40 kg.)



(Note that the pulley on the left is just resting on the cable, being pulled down by the box, but this pulley isn't connected to anything else like a wall or ceiling.)

Based on the arguments given in the first problem, the cable that starts at point A, wraps around the 'floating' pulley, then back up around the pulley on the right, and then finally down to where the person is pulling on it represents a single tension throughout that cable, which I'm calling T_2.

There's also a tension T_1 in the cable connecting the box to the floating pulley, and a tension T_3 in the cable connecting the fixed pulley to the ceiling.

Let $+Y$ be vertically upward here.

- Applying Newton's laws to the box, we have T_1 upward and the force of gravity downward, so $\sum F_y = ma_y = 0$ becomes $T_1 - mg = 0$ or $\boxed{T_1 = mg}$. (Full weight of the box.)
- Applying Newton's laws to the floating pulley itself, we have T_2 pulling vertically upward on both sides, and T_1 pulling down, so $\sum F_y = ma_y = 0$ becomes $T_2 + T_2 - T_1 = 0$ or $\boxed{T_2 = \frac{1}{2}T_1 = 0.5mg}$. (HALF the weight of the box.)
- Applying Newton's laws to the point where the person is holding onto the cable: $\sum F_y = ma_y = 0$ becomes $T_2 - F_{pull} = 0$ or $\boxed{F_{pull} = T_2 = 0.5mg}$. (HALF the weight of the box.)
- Finally, applying Newton's laws to the pulley attached to the ceiling: $\sum F_y = ma_y = 0$ becomes $T_3 - T_2 - T_2 = 0$ or $T_3 = 2T_2 = 2(0.5mg) = mg$.

In this configuration, the person is only having to exert a force equal to <u>half</u> the weight of the box to hold it in place.

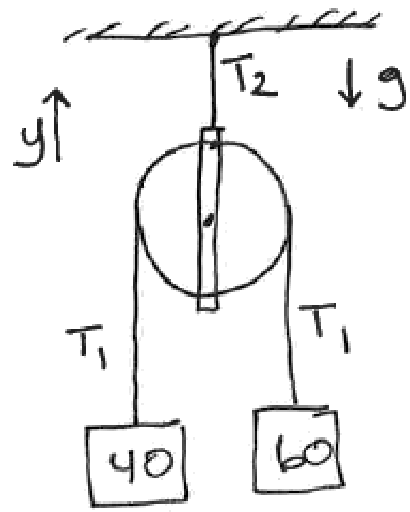
Considering the forces on the ceiling, we see that the left pulley is pulling down on the ceiling with a force of $T_1 = 0.5mg$ and the right pulley is pulling down on the ceiling with a force of $T_3 = mg$, so in total the ceiling has a force of $1.5mg$ being exerted on it. (So less than in the first example, and we've gained a considerable mechanical advantage here when it comes to how much force the person has to exert.)

Example: Atwood Machine

Let's expand on the first pulley problem and have a second box hanging on the right, instead of a person pulling on the cable there. And let's make this box heavier so that the system is no longer in equilibrium and will accelerate.

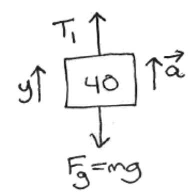
Here, we have our original 40 kg box on the left, and now we also have a 60 kg box on the right.

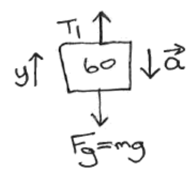
Determine the tensions present and the acceleration of the boxes.



Based on the previous problems, we have two tension variables here: one in the cable connecting the pulley to the ceiling, and one in the cable that runs from the 40 kg box, loops around the pulley and then connects to the 60 kg box.

We need to apply Newton's Laws separately to each object (to each box, for example). They're connected with a cable that isn't changing it's length, so whatever the left box is doing upward, the right box is doing downward. They'll have the same relative displacements, the same speeds, and same accelerations.





If we use a coordinate system with $+Y$ upward throughout this problem, that means that the acceleration vector for the left box is $\vec{a} = a\hat{j}$ (some magnitude a in the $+Y$ direction, which is represented by $\hat{j}$), BUT the acceleration vector for the box on the right will be a in the negative Y direction: $\vec{a} = -a\hat{j}$.

(This detail is often missed, so we'll attack this problem with a smarter choice of coordinate systems once we're done here.)

Applying Newton's Laws to the box on the left: $\sum F_y = ma_y$ becomes: $T_1 - F_g = ma$ so $T_1 - (40)(9.8) = 40a$ or finally $\boxed{T_1 - 392 = 40a}$.

The box on the right has an acceleration of the same magnitude, but it's accelerating in the $-Y$ (negative Y) direction, so we'd have to write it's vector acceleration as $\vec{a} = -a\hat{j}$. That means when we pull out all the $\hat{j}$ components of Newton's Laws, we end up with $-a$ for the a_y component of acceleration now.

Applying Newton's Laws to the box on the right: $\sum F_y = ma_y$ becomes: $T_2 - F_g = ma$ so $T_2 - (60)(9.8) = 60(-a)$ or finally $\boxed{T_1 - 588 = -60a}$.

Rearranging each equation to solve for T_1:

- left box: $T_1 = 392 + 40a$
- right box: $T_1 = 588 - 60a$

Setting the two right-hand sides equal to one another: $392+40a = 588-60a$ which we can rearrange into $40a + 60a = 588 - 392$ or $100a = 196$ from which $\boxed{a = 1.96\ m/s^2}$.

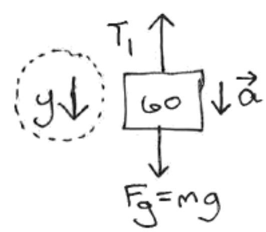
Then using the above equations we can determine the tension present:

- left box: $T_1 = 392 + 40a = 392 + 40(1.96) = 470.4\ N$
- right box: $T_1 = 588 - 60a = 588 - 60(1.96) = 470.4\ N$

NOTE that this tension is NOT equal to the weight of either of the two boxes. This is necessary since each box has to be accelerating now, so there needs to be an imbalance of forces to allow that to happen. Box 1 needs to be being pulled upward with a force larger than it's own weight in order to accelerate upward. Box 2 needs to be being pulled upward with a force less than it's own weight in order to accelerate downward.

A safer approach : Getting the sign of the acceleration right is tricky in connected-object problems, so this is often handled by setting up **separate coordinate systems for each object**, where in each case we choose the positive coordinate to be in the direction we think the object will be accelerating. Here, the left object will be accelerating upward so we'll leave $+Y$ being upward for that object and we already did that, leading to $T_1 - 392 = 40a$.

We think the heavier box will accelerate downward though, so we'll let $+Y$ be vertically downward for that object. Applying Newton's laws to that box now leads to: $-T_1 + mg = ma$ since T_1 is pulling upward (our negative direction) and gravity is pulling downward (our positive direction), leading to an acceleration downward (our positive direction).
For the 60 kg box in this new coordinate system then, we have $-T_1 + 588 = 60a$.



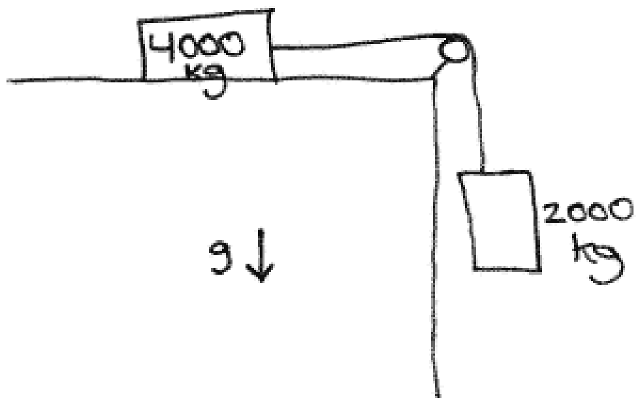
We now have two equations in two unknowns:

- $T_1 - 392 = 40a$ from analyzing the left box
- $-T_1 + 588 = 60a$ from analyzing the right box

Solving these is now trivial since we can just add the two equations together. That immediately eliminates the T_1 variable, leaving us with: $-392 + 588 = 100a$ and the same solution we had before: $a = 1.96\ m/s^2$.

> Good rule of thumb: when dealing with connected objects, we have to apply Newton's Laws separately to each object anyway, so it's a good idea to choose separate coordinate systems for each object, letting the positive direction be the direction we think the objects will be accelerating.

Example : Jurassic Park (version 1)
In one of the early Jurassic Park movies, a dinosaur has pushed half of a connected trailer over the edge of a cliff. Suppose the two objects are initially at rest and we have no friction yet (we'll look at a more 'realistic' version of this problem in the next chapter that does include friction).



Determine the acceleration of the two objects.

Looking at the 2000 kg object, we'll have F_g acting downward on it, and some tension T_1 acting upward.

Following the cable back up to the 4000 kg object, we see that the same T_1 will be pulling that object to the right, and there aren't any other forces in that direction, so it will accelerate to the right.

The objects are connected, so if the 4000 is accelerating to the right at some acceleration 'a', the 2000 kg object will be accelerating downward with that same acceleration value.

$\boxed{\text{4000 } kg \text{ object}}$: Using a coordinate system with +X to the right, $\sum F_x = ma_x$ becomes $\boxed{T_1 = 4000a}$.

$\boxed{\text{2000 } kg \text{ object}}$: Using a coordinate system with +X pointing vertically **downward**, $\sum F_x = ma_x$ becomes $-T_1 + mg = ma$ with $M = 2000$ and $g = 9.8$ so: $\boxed{-T_1 + 19600 = 2000a}$

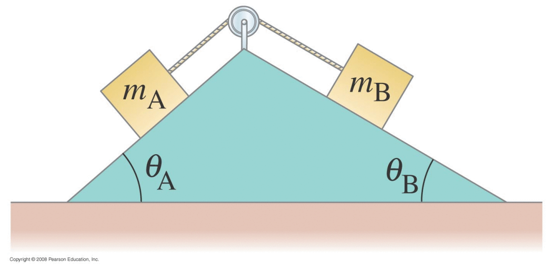
We have two equations and two unknowns to solve for. If we just ADD the two equations together, the T_1 terms will cancel since they're of opposite signs, leaving us with: $19600 = 2000a + 4000a$ or finally $\boxed{a = 3.267\ m/s^2}$.

Now that we have the acceleration, we can go back and find the tension. Using the 4000 kg object's equation: $T_1 = 4000a = 13067\ N$. Using the 2000 kg object's equation: $-T_1 + 19600 = 2000a$ or $T_1 = 19600 - 2000a = 13067\ N$.

NOTE: if the 2000 kg were just hanging there not accelerating, the tension in the rope would just be it's weight of $mg = (2000)(9.8) = 19600\ N$ which is NOT the value we found to be present when the object was accelerating. That's an important thing to take away from examples like this. When objects are accelerating, the tension will be DIFFERENT from what it is when the object isn't accelerating.

Example : Connected Objects on Ramps

Suppose we place our 40 kg and 60 kg boxes on a double-ramp as shown in the figure.
I borrowed this figure from a homework problem but here we have $m_A = 40\ kg$ on the left (with the left side of the ramp at a $\theta_A = 60^o$ angle), and $m_B = 60\ kg$ on the right, where the ramp has an angle of $\theta_B = 30^o$.
Determine the acceleration of the boxes.

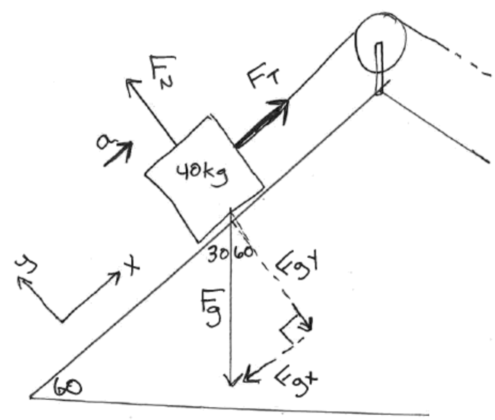


Encountering a problem like this, I have no idea which way the boxes will end up moving (or even if they'll move at all), so we'll just have to guess a direction. Let's assume that the heavier box will accelerate down it's ramp, causing the lighter box to accelerate up it's ramp. We'll apply Newton's Laws to each object, using separate coordinate systems for each.

In the end, if we guessed wrong, we'll end up with a negative acceleration, since an acceleration of (say) $-2\ m/s^2$ in the positive X direction just means we really have an acceleration of 2 m/s^2 in the negative X direction, since $-2\hat{i} = 2(-\hat{i})$.

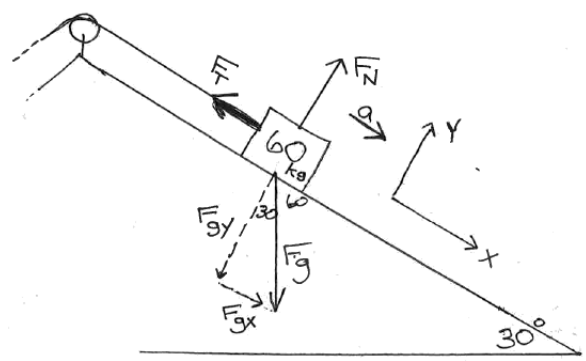
$\boxed{\text{Applying Newton's Laws to the 40 } kg \text{ object}}$

Using the rotated coordinate system shown, in the X direction we have F_T pulling up the slope, and the X component of $F_g = mg = (40)(9.8) = 392\ N$ pulling down the slope so $\sum F_x = ma_x$ becomes $-F_g \sin(60^o) + F_T = 40a$ or $-392 * (0.86603...) + F_T = 40a$ or finally $\boxed{-339.48 + F_T = 40a}$



$\boxed{\text{Applying Newton's Laws to the 60 } kg \text{ object}}$

Using the rotated coordinate system shown, in the X direction we have F_T pulling up the slope (the negative X direction now), and the X component of $F_g = mg = (60)(9.8) = 588\ N$ pulling down the slope (our positive direction now) so $\sum F_x = ma_x$ becomes $F_g \sin(30^o) - F_T = 60a$ or $-588 * (0.500) + F_T = 60a$ or finally $\boxed{294 - F_T = 60a}$



Adding the two boxed equations together eliminates the F_T variable, leaving us with: $-339.48 + 294 = 100a$ or $a = -0.458\ m/s^2$. As argued at the start, that just means that really the boxes are moving the other way: the lighter box is sliding down it's ramp, pulling the heavier box up it's ramp. Even though the left box was lighter, the steeper angle of it's ramp meant that we ended up with a higher along-slope force of gravity in that direction.