PH2213 Fox : Lecture 10
Chapter 5 : Using Newton's Laws

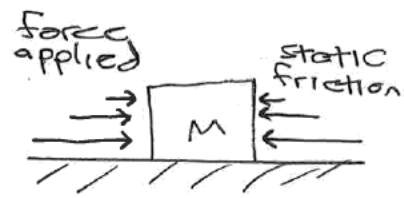
This chapter introduces another force type (friction), another type of motion (circular), and yet another force type (drag forces).

In one sense, friction is just another force to add to the left hand side of $\sum \vec{F} = m\vec{a}$, but unfortunately there are two types of friction, and the direction depends on how the object is moving (or if it is moving at all).

Our textbook uses a single name to represent both types of friction $\vec{F}_{fr}$ but I prefer the notation our previous book uses since it clearly differentiates between the two different types of friction.

STATIC FRICTION

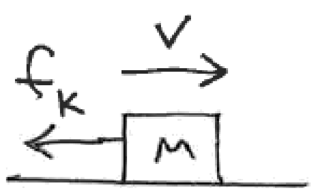
- objects 'stuck together' in a sense
- static friction provides force needed to oppose motion that **other forces would have created** without friction present
- maximum static friction: $f_{s,max} = \mu_s F_N$, so:
- magnitude: $f_s \leq f_{s,max}$ where $f_{s,max} = \mu_s F_N$
- direction: opposed to the motion that all the other forces would have produced if there were no friction

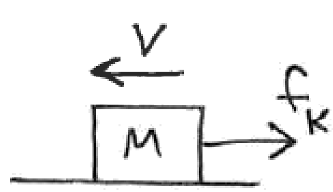


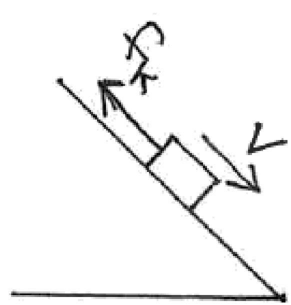
As the applied force increases, the amount of static friction increases as well, up to a maximum value at which point the applied force 'wins' and the object breaks free and starts sliding, at which point f_s is gone and is replaced with kinetic friction (see below).

KINETIC FRICTION

- object sliding along another object (floor, etc)
- magnitude: $f_k = \mu_k F_N$
- direction opposite to object's motion

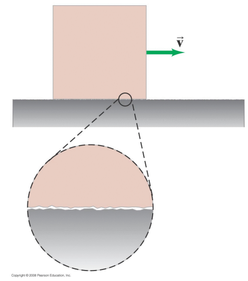






Actual mechanism is complicated

- microscopically rough surfaces
- stiction (molecular bonds)



SUMMARY

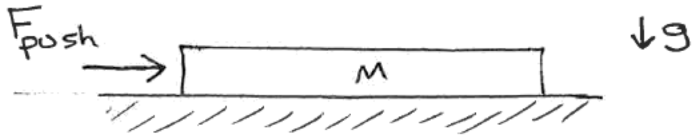
$$\vec{F}_{fr} = \begin{cases} \text{Kinetic:} & f_k = \mu_k F_N \\ \text{Static:} & f_s \leq f_{s,max} \text{ where } f_{s,max} = \mu_s F_N \\ \text{Direction:} & \text{Opposes motion} \end{cases}$$

TABLE 5–1 Coefficients of Friction[†]

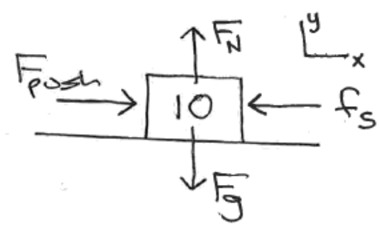
Surfaces	Coefficient of Static Friction, μ_s	Coefficient of Kinetic Friction, μ_k
Wood on wood	0.4	0.2
Ice on ice	0.1	0.03
Metal on metal (lubricated)	0.15	0.07
Steel on steel (unlubricated)	0.7	0.6
Rubber on dry concrete	1.0	0.8
Rubber on wet concrete	0.7	0.5
Rubber on other solid surfaces	1–4	1
Teflon® on Teflon in air	0.04	0.04
Teflon on steel in air	0.04	0.04
Lubricated ball bearings	<0.01	<0.01
Synovial joints (in human limbs)	0.01	0.01

[†] Values are approximate and intended only as a guide.

Example : Two sheets of smooth glass are in contact as shown in the figure. How much force (applied horizontally) is required to cause the top sheet (which has a mass of 10 kg to start moving?
Once it starts moving, what will it's acceleration be?
Glass on glass: $\mu_\mathbf{k} = \mathbf{0.4}$, $\mu_\mathbf{s} = \mathbf{0.94}$



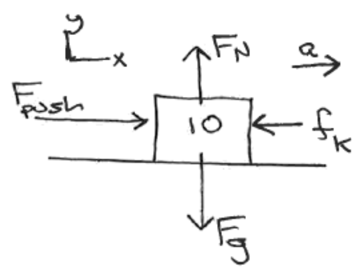
Apply Newton's Laws to the problem. We start off by choosing our object (here, the top sheet of glass) and add all the force vectors acting on it:



- Object has mass, so $\vec{F_g} = m\vec{g}$ acting vertically downward
- Object is not moving through the material it's sitting on, so there's some normal force $\vec{F_N}$ perpendicular to the interface (in this problem, straight up)
- External pushing force
- Object isn't moving, so static friction will be present. If there were no friction the object would slide to the right, so f_s must be pointing to the left

(a) In the X direction we only have the pushing force and static friction, so $\sum F_x = ma_x = 0$ implies $F_{push} - f_s = 0$. The more we push, the larger f_s will be to prevent the object from moving. Static friction can only reach a maximum value of $f_{s,max} = \mu_s F_N$ though so once the pushing force reaches that value, the object will move. Here then, that occurs when $F_{push} \geq f_{s,max} = \mu_s F_N$. Looking in the vertical direction, $\sum F_y = ma_y = 0$ so $F_N - mg = 0$ or $F_n = mg = (10\ kg)(9.8\ m/s^2) = 98\ N$. $f_{s,max} = \mu_s F_N = (0.94)(98\ N) = 92.12\ N$. As long as we push with a force less than that, the object will just sit there. Once we exceed that pushing force, the object will move.

(b) For the second part, we're assuming that we continue to push with $F_{push} = 92.12\ N$ but now the object has broken free and has started moving, so we have <u>kinetic friction</u> present now. $f_k = \mu_k F_N$ and we already determined the normal force above, so $f_k = (0.4)(98\ N) = 39.2\ N$.
$\sum F_x = ma_x$ so $F_{push} - f_k = ma$ or $92.12 - 39.2 = 10a_x$ from which $a_x = 5.292\ m/s^2$.



The upper sheet of glass will have a large acceleration to the left, so even if we stop pushing quickly, it will be able to pick up a considerable speed. (Often, sheets of glass are separated with paper or some other material to make it safer to deal with.)

Addendum : The average reaction time for humans is 0.25 *sec* to a visual stimulus, 0.17 *sec* for an audio stimulus, and 0.15 *sec* for a touch stimulus.

Here then, for the first 0.15 *sec* after we feel the glass moving, we're still pushing on it. How fast is the sheet of glass moving at that point? How far has it moved?

We found on the previous page that the top sheet of glass will be accelerating at $a_x = 5.292\ m/s^2$ as long as we keep pushing on it, and the 'reaction time' value means that we will be pushing on it for 0.15 *seconds*.

How fast will the sheet of glass be moving at the end of that time:

$$v_x = v_{ox} + a_x t = 0 + (5.292\ m/s^2)(0.15\ s) = 0.7938\ m/s.$$

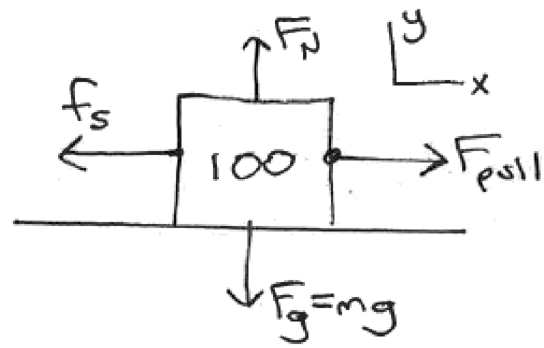
How far will the glass sheet move in this time?

$$x = x_o + v_{ox}t + \frac{1}{2}a_x t^2 = 0 + 0 + (0.5)(5.292)(0.15)^2 = 0.0595\ m.$$

Apparently this sheet, with very sharp edges, will slide about 6 cm before we can take our hands off it. (My cousin's husband was a talented stained glass artist and his hands were covered with scars as a result of this effect.)

Example: Forces at Angles : A 100 kg crate is sitting on the floor. The coefficients of friction between the crate and the floor are $\mu_s = 0.8$ and $\mu_k = 0.6$.
(a) If we pull horizontally, how much force would be needed to cause the crate to start moving?
(b) What happens if we pull on the crate with a 650 N force 30^o above the horizontal?
(c) What happens if we push on the crate with a 1200 N force 30^o below the horizontal?

(a) PULL HORIZONTALLY : We essentially did this problem already (see the glass sheet example). The pulling force and static friction are both entirely in the X direction and are the only forces in that direction, so the object won't move until F_{pull} exceeds the maximum amount of static friction, which would be $f_{s,max} = \mu_s F_N$. In the vertical direction, we only have the weight of the object and the normal force, so $F_N = mg$. Finally $f_{s,max} = \mu_s mg = (0.8)(100\ kg)(9.8\ m/s^2) = 784\ N$. We have to pull with at least that much force to cause the crate to start moving.

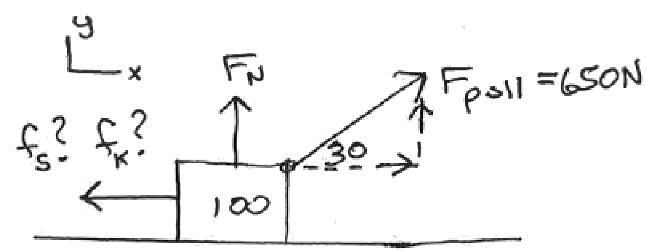


If we keep pulling on the crate with that 784 N of force, what will the crate's acceleration be?

The crate has broken free from the static friction holding it in place, so now that it's moving we'll replace f_s with $f_k = \mu_k F_N = (0.6)(980\ N) = 588\ N$.

$\sum F_x = ma_x$ so $+784 - 588 = 100a$ from which $a = 1.96\ m/s^2$.

(b) PULLING UP AT ANGLE : Here, we pull with a smaller force than we found in part (a), but we're doing so at an angle (above the horizontal).
Friction is present, so we'll need the normal force: let's find that first.
$\sum F_y = ma_y = 0$ so $F_N + F_{pull} \sin(30^o) - mg = 0$ or $F_N = mg - F_{pull} \sin(30^o) = (100)(9.8) - (650)(0.5) = 655\ N$.

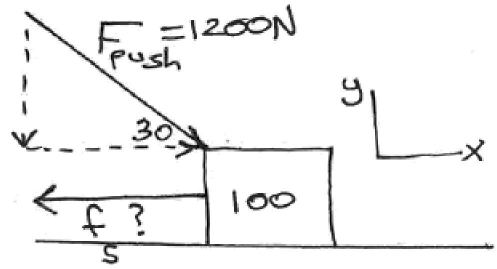


Friction: $f_{s,max} = \mu_s F_N = (0.8)(655\ N) = 524\ N$.

Looking in the X direction now, we have $F_{pull} \cos(30^o)$ to the right, which is $(650\ N)(0.866...) = 562.9\ N$. Looking at all the forces in the X direction, that pulling force component EXCEEDS the maximum amount of static friction, which means the crate will NOT just sit there: it will accelerate to the right meaning we have KINETIC friction present, with $f_k = \mu_k F_N = (0.6)(655\ N) = 393\ N$.

Switching to kinetic friction : $\sum F_x = ma_x$ becomes $F_{pull} \cos(30^o) - f_k = ma$ and substituting in those values: $562.9 - 393 = 100a$ or $a = 1.70\ m/s^2$.

$\boxed{\text{(c) PUSHING DOWN AT ANGLE}}$: Here, we push on the crate with a considerably higher force that was needed before, but do so at an angle where we're partly pushing down on the crate. Repeating the steps we did in part (b): $\sum F_y = ma_y = 0$ so $F_N - F_{push}\sin(30^o) - mg = 0$ or $F_N = mg + F_{push}\sin(30^o) = (100)(9.8) + (1200)(0.5) = 1580\ N$. Static friction: $f_{s,max} = \mu_s F_N = (0.8)(1580\ N) = 1264\ N$.



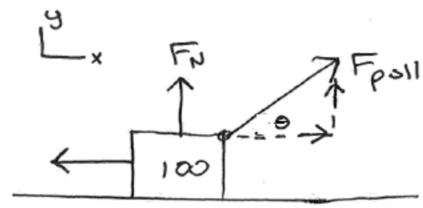
Looking in the X direction now, we have $F_{push}\cos(30^o)$ to the right, which is $(1200\ N)(0.866...) = 1039\ N$. Looking at all the forces in the X direction, that pushing force component is SMALLER than the maximum amount of static friction that can be present, so this time **the crate won't move**.

The pulling force's X component of $1039\ N$ to the right is cancelled out with a static frictional force of $1039\ N$ to the left.

Remember, the actual amount of static friction present is just what's needed to stop the object from moving. Here, we only needed $1039\ N$ of static friction to hold the object in place, so that's all the static friction that's actually present.

In this example, static friction can be UP TO $1264\ N$ but we didn't need that much to hold the crate in place.

ADDENDUM: Let's look at the (b) scenario, with the crate being pulled with a force directed at some angle above the horizontal.
Once we get the crate moving, suppose we want to continue pulling it across the floor, but we want to maintain some **constant speed** (i.e. $a = 0$ now). What pulling angle minimizes the amount of force needed?



'Constant speed' so $a = 0$ here.

$\sum F_y = ma_x = 0$ so: $F_N + F\sin\theta - mg = 0$ which means that $\boxed{F_N = mg - F\sin\theta}$.

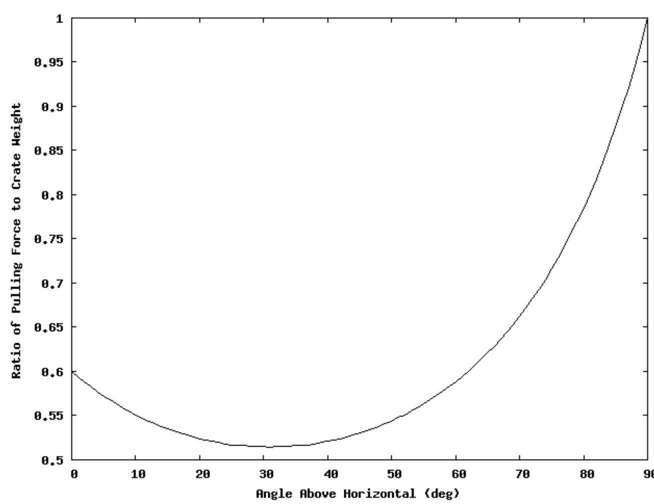
$\sum F_x = ma_x = 0$ so: $-f_k + F\cos\theta = 0$ where $f_k = \mu_k F_N$ which we can write as $F\cos\theta = f_k = \mu_k F_N = \mu_k(mg - F\sin\theta)$

Expanding everything out here: $F\cos\theta = \mu_k mg - \mu_k F\sin\theta$

Collecting the F terms together: $F\cos\theta + \mu_k F\sin\theta = \mu_k mg$ or $F(\cos\theta + \mu_k\sin\theta) = \mu_k mg$ and finally:

$$\boxed{F = (\tfrac{\mu_k}{\cos\theta + \mu_k\sin\theta})mg}$$

With $\mu_k = 0.6$ as we had in this problem, we can plot the force needed as a function of angle:
Note that there's a magic angle that minimizes the force needed. If we pull horizontally, our force has to overcome kinetic friction. If we pull at some angle above the horizontal, some of that force is reducing the normal force and thus reducing the amount of friction present, making it easier to move. At the same time though, less of our force is in the horizontal direction and available to overcome that friction, so it looks like there's a sweet spot: a particular angle that minimizes the amount of force we need to provide to keep the crate moving (at a constant speed).



You've done min/max problems via calculus, so let's look at using that approach here. The factor we're trying to find the min/max for is $\frac{1}{\cos\theta + \mu_k\sin\theta}$.

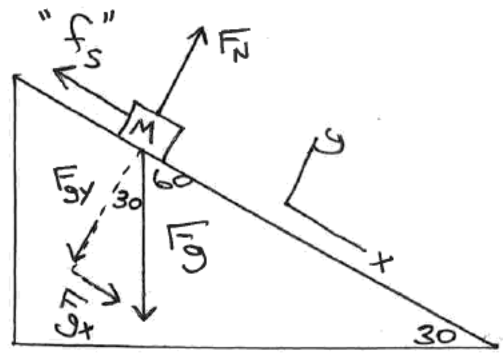
The θ only appears in the denominator here, so where the overall function reaches it's minimum will be where that denominator alone reaches its maximum. So we're looking for where:

$\frac{d}{d\theta}(\cos\theta + \mu_k\sin\theta) = 0$

Doing the derivative, that turns into: $-sin\theta + \mu_k\cos\theta = 0$ or rearranging: $\boxed{\tan\theta = \mu_k}$. For our $\mu_k = 0.6$ situation, this optimum angle is $\theta = 30.96^o$ which looks about where the curve reaches its minimum.

Example: Object on Incline : Static Case

An object of some mass M is placed on an incline that makes an angle of θ relative to the horizontal. As we increase the angle, eventually the object will start sliding down the incline. At what angle does that happen?



What condition will allow the object to just sit there, even when on an incline? Looking at the force vectors, we have a component of $\vec{F}_g$ acting down along the ramp, so there must be a force acting up the ramp to counteract that. We have friction, so let's assume the situation is static (nothing's moving) and see what that implies:

X direction

- $\sum F_x = 0$ means: $-f_s + mg\sin\theta = 0$ or: $mg\sin\theta = f_s$
- Unfortunately, f_s isn't unlimited: $f_s \leq f_{s,max} = \mu_s F_N$
- Our X equation really becomes: $\boxed{mg\sin\theta \leq \mu_s F_N}$.

Y direction (Object never changes it's Y coordinate, so:)

- $\sum F_y = 0$ means: $F_N - mg\cos\theta = 0$ or:
- $F_N = mg\cos\theta$.

Substituting that expression for F_N into our X equation:

$$mg\sin\theta \leq \mu_s(mg\cos\theta)$$

Cancelling the common mg factor from both sides: $\sin\theta \leq \mu_s\cos\theta$ or $\boxed{\tan\theta \leq \mu_s}$.

For example, if $\mu_s = 1$ (a pretty high value), then $\tan\theta \leq 1$ means $\theta \leq 45^o$. Any higher angle and the object will accelerate down the incline.

If we have a particular μ_s present between the two materials, we can gradually raise the angle of the ramp and the object will still just sit there until we reach the point where $\boxed{\tan\theta = \mu_s}$. **Any higher angle and we won't have enough (static) friction to hold the object in place.**

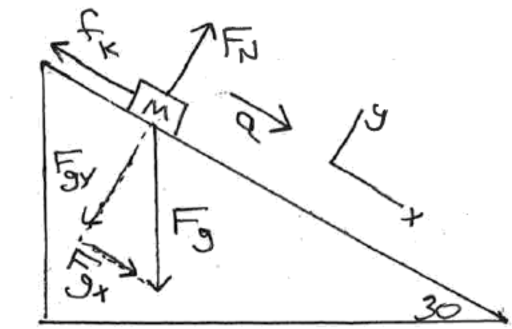
That actually gives us a quick way to estimate the μ_s between two objects. Put one on the other and gradually increase the angle until the one on top starts slipping down the ramp.

Example: Object on Incline : Kinetic Case

An object of some mass M is **sliding** on an incline that makes an angle of θ relative to the horizontal.

(a) What does the acceleration ultimately depend on?

(b) At what angle will the object slide down the incline at a constant speed? (I.e. where $a = 0$?)



We've raised the angle high enough to start the object sliding. What will its acceleration be?

X direction

- $\sum F_x = ma_x$ so $-f_k + mg\sin\theta = ma_x$
- $f_k = \mu_k F_N$ so we'll need the normal force to go any further.

Y direction (Object never changes it's Y coordinate, so:)

- $\sum F_y = 0$ means: $F_N - mg\cos\theta = 0$ or:
- $F_N = mg\cos\theta$.

Now we can substitute that into our X equation:

$-\mu_k F_N + mg\sin\theta = ma_x$ becomes: $-\mu_k mg\cos\theta + mg\sin\theta = ma$.

Every term in the equation (on both sides) has the same m we can cancel out, leaving us with: $g\sin theta - \mu_k\cos\theta = a_x$ or finally:

$$\boxed{a_x = g(\sin\theta - \mu_k\cos\theta)}$$

Note that $a_x = 0$ (object sliding down the ramp at a constant speed) happens when $\sin\theta - \mu_k\cos\theta = 0$ and rearranging that, when $\boxed{\tan\theta = \mu_k}$.

The coefficient of kinetic friction μ_k is never higher than the coefficient of static fruction μ_s, so this angle is always a bit smaller than the angle we found in the previous example.

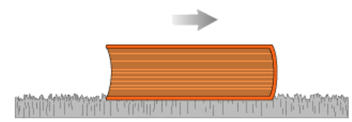
If we place an object on a ramp and increase the angle until it starts slipping, it'll always keep sliding down, picking up speed as it does.

Technically, if we then quickly adjust the angle to the point where it's just sliding at a constant speed, that angle would tell us what μ_k is for these materials, but that's not very practical.

Let's look at a better way to estimate μ_k next.

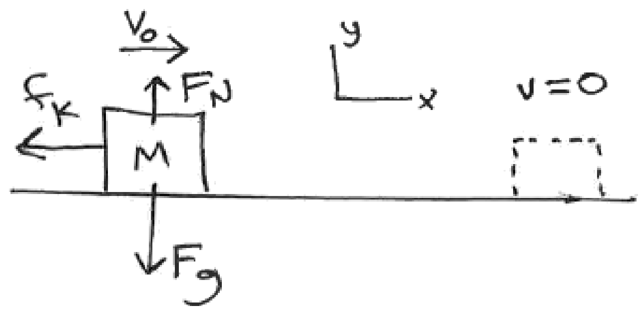
Example: Stopping Distance

An object of mass M is initially moving at v_o to the right on a flat, horizontal surface. The coefficient of kinetic friction between the object and the surface is μ_k.



(a) How far will the object move before coming to a stop?

- Object: the book, crate, car, etc
- Coordinates: The object is sliding to the right, so we'll let +X be in that direction, with +Y vertically upward.
- Forces acting on the object:
  (*) F_g downward
  (*) F_N upward (keeping the object from passing through the table)
  (*) friction (here f_k since the object is moving, and it will be pointing to the left since the object is moving to the right)



Applying Newton's Laws in the Y direction: $\sum F_y = ma_y = 0$ so $F_N - mg = 0$ or $F_N = mg$.

Kinetic friction: $f_k = \mu_k F_N$ so $f_k = \mu_k mg$.

Applying Newton's Laws in the X direction: $\sum F_x = ma_x$ so $-f_k = ma_x$

Substituting in the expression we found for f_k: $-(\mu_k mg) = ma_x$ or $\boxed{a_x = -\mu_k g}$.

How far will the object slide before coming to a stop?

We have 1-D motion here. The object is initially moving at v_o but has an acceleration of $a_x = -\mu_k g$ until it comes to a stop after travelling a distance of d:

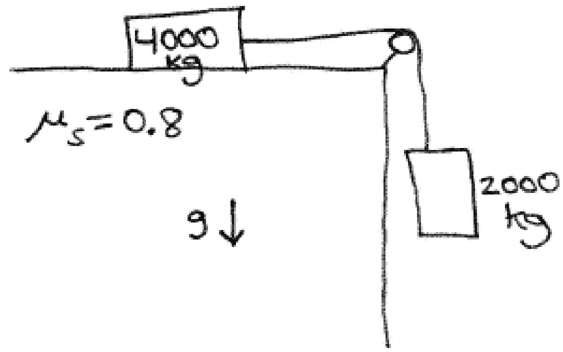
$v_x^2 = v_{ox}^2 + 2a_x\Delta x$ so here $(0)^2 = v_o^2 + (2)(-\mu_k g)(d)$ which we can arrange into: $\boxed{d = \frac{v_o^2}{2\mu_k g}}$

In the original version of this example, the object was a book initially sliding across a table at $v_o = 2\ m/s$, with a coefficient of kinetic friction between the book and the table of $\mu_k = 0.4$, which yields a stopping distance of $d = 0.51\ m$.

An equation like this would apply to **any** object sliding to a stop on a **flat, horizontal surface** where no other forces are present (just gravity, F_N, and kinetic friction). A car skidding to a stop on pavement would be an example. Note the stopping distance goes up as the square of the initial speed, so a car going twice as fast would need $2^2 = 4$ times longer to come to a stop. Also, $d \propto 1/\mu_k$, so the stopping distance would go up if the road were wet.

(And technically, $d \propto 1/g$, so roads on the Moon or Mars (where g is much lower than on the earth) would have considerably longer stopping distances, so should probably have much lower speed limits!)

Example : Jurassic Park 2 : at one point a dinosaur has pushed a trailer of mass **2000 kg** over the edge of a cliff. It hangs there, connected by a cable to another trailer of mass **4000 kg** which remains on the ground as shown in the figure. Let the coefficients of friction here be $\mu_s = \mathbf{0.8}$ and $\mu_k = \mathbf{0.6}$.



(a) Verify that the present situation is **stable** (i.e. static friction is providing enough force to keep the trailers from sliding over the edge).

(b) The dinosaur is not happy with that and decides to **push horizontally** on the trailer. How much force does the dinosaur need to provide to overcome static friction and cause the trailers to start to move?

(c) If the dinosaur **continues** to push with the same amount of force, determine the **acceleration** present, and the **tension** in the connecting cable.

(a) Verify the present situation is stable.

For this part, we are assuming that everything is static - nothing is moving. Applying Newton's Laws to the hanging object, the sum of all the forces in the vertical direction must be zero, since the object is not accelerating. We have some tension in the cable, so looking at this object, we have T upward, and the weight of the object downward. $\sum F_y = ma_y = 0$ so $+T - mg = 0$ or $T = mg = (2000\ kg)(9.8\ m/s^2) = 19,600\ N$

Let's look at the part of the trailer that is sitting on the ground now. In the X direction, we have the tension pulling it to the right, and static friction acting to the left. That's all we have in the X direction, so will this static friction be enough to resist the tension trying to pull the object to the right?

The **maximum** amount of static friction will be $f_{s,max} = \mu_s F_N$ so we'll need to find the normal force here. Looking in the vertical direction and all the forces acting on **this** object, we have it's weight downward, and the normal force upward, so $+F_N - mg = 0$ or here $F_N = mg = (4000\ kg)(9.8\ m/s^2) = 39,200\ N$. The **frictional** force then can be **up to** $f_{s,max} = \mu_s F_N = (0.8)(39,200\ N) = 31,360\ N$.

Thus it looks like we have more than enough 'friction budget' available to keep the trailer in place.

(b) Additional Horizontal Force Needed to Start Moving

The dinosaur needs to add some addition force in the +X direction: just enough force that it plus the tension that is already there will be enough to reach the maximum static friction, so $F_{push} + 19600 = 31360$ or $F_{push} = 11,760\ N$. Anything less and there's still enough left in the static friction budget to resist the motion. Once we add this much force though, static friction has reached its limit and the object can start to move.

$\boxed{\text{(c) Acceleration and Tension in Moving Scenario}}$

If we continue to push the (now moving) 4000 kg object to the right with $F_{push} = 11,760\ N$, what will be the acceleration of the objects?

- The 4000 kg object is moving to the right, so we'll let +X be to the right for that object.
- The 2000 kg object is moving downward, so we'll let +X be downward on that object.

Applying Newton's Laws to the hanging object

$\sum F_y = ma_y$ so $-F_T + mg = ma$ or here $-F_T + (2000)(9.8) = (2000)(a)$ or $\boxed{-F_T + 19600 = 2000a}$.

Applying Newton's Laws to the object on the ground

$\sum F_x = ma_x$ so $F_{push} + F_T - f_k = ma$

How much kinetic friction is present? (Remember: the objects are MOVING now, so we no longer have static friction - we have kinetic friction instead.) $f_k = \mu_k F_N$ and we already found F_N earlier, so $f_k = (0.6)(39200\ N) = 23520\ N$.

$\sum F_x = ma_x$ so $F_{push} + F_T - f_k = ma$ or $11760 + F_T - 23520 = 4000a$ or $\boxed{-11760 + F_T = 4000a}$

We now have two equations and two unknowns. F_T appears in each with opposite sign, so let's just add these two equations together, leaving us with:

$19600 - 11760 = 6000a$ or $\boxed{a = 1.307\ m/s^2}$.

Plugging that value of a back into either of the boxed equations leads to $\boxed{F_T = 16987\ N}$. (Less than the 19600 N of tension we had back before the objects started moving, but remember we need that to be the case so that the unbalanced forces can create this acceleration.)