**PH2213 Fox : Lecture 13**
**Chapter 6 : Gravitation**

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| NOTE: this is just a summary of the highlights of the chapter, mostly from sections 1, 3 and 4. We don't go into enough detail on this material to include it on any of the tests, though. |

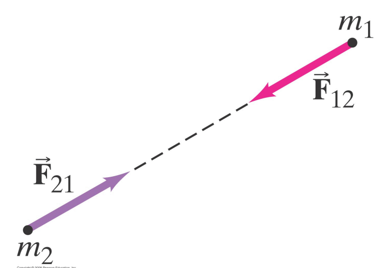
This chapter introduces the 'real' force of gravity that exists between any masses. Near the surface of the Earth, we've been approximating this as $\vec{F}_g = m\vec{g}$ with $\vec{g}$ varying slightly but for the most part being a vector with a magnitude of about 9.8 $m/s^2$ and a direction 'downward' towards the center of the planet.

The actual force of gravity is somewhat more complicated than this though, and if we are <u>not</u> near the surface, we'll need to use a more accurate version.

**Newton (1687)** : (point) masses attract with a force proportional to each of their masses, and inversely proportional to the distance between them.

Magnitude: $F_G = G\frac{m_1 m_2}{r^2}$

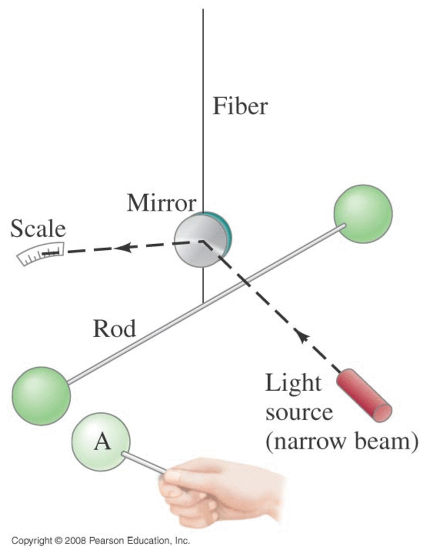
Direction: (always attractive)




**Determining G** : The value of the universal gravitational constant G is tiny, but measurable even here on Earth.

**Cavendish experiment (1798)** : two masses in the form of a dumbell hanging from a thin wire. When another mass is brought near, they will attract one another and cause the wire to twist a tiny amount. The twist isn't visible, but using a mirror and a light-source, that tiny change can be amplified and measured, leading to an estimate for $G$.

**Latest result** : $\boxed{G = 6.6743 \times 10^{-11} N\ m^2/kg^2}$




(See example 2 in the examples06.pdf file for example.)

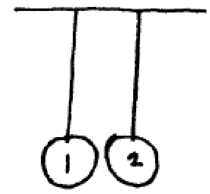
**What about non-point masses?** Calculus to the rescue. Break up objects into an infinite number of infinitesimal point masses, and 'do the integral.'

The result is surprisingly simple: as long as an object is <u>spherical</u> and <u>uniform</u>, mathematically it acts (mathematically) exactly like a point mass with all the mass located at it's center. This even holds if the density is NOT uniform, but at least only depends on r (so 'spherically symmetric').

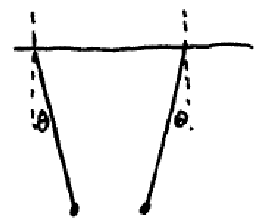
For planet-sized objects and larger (stars), this is almost always 'good enough'.

## Example: Bowling Balls on Wires

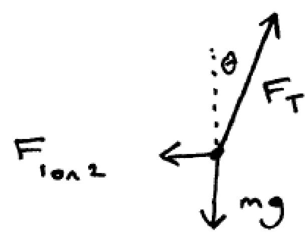
Two 5 *kg* bowling balls are hanging from cables connected to the ceiling. The balls each have a diameter of 16 *cm*, so let's hang them from 1 *m* long wires such that the balls are 1 *cm* apart (meaning their centers are exactly 17 *cm* apart). What is the gravitational force between them? The cables appear vertical but actually are not. What angle does each cable make with the vertical?




The upper figure shows the nominal situation. We see the two balls hanging 'vertically' on the cables. The two bowling balls are attracting each other gravitationally though, there is a force pulling them towards each other. An exaggerated diagram of this is shown in the middle figure, where we've also replaced each bowling ball with its equivalent point mass.




Focusing on the bowling ball on the right, what are the forces acting on it? We have the force the earth is exerting on it (we'll use the usual $F_g = mg$ we've been using before). We also have the gravitational force that the ball on the left is exerting on this one, and we have some tension in the cable.




Everything is at rest here, so $\sum \vec{F} = 0$. Breaking this into components:

$\sum F_x = 0$ so $-F_{1\ on\ 2} + F_T \sin\theta = 0$ which we can write as $F_T \sin\theta = F_{1\ on\ 2}$.

$\sum F_y = 0$ so $-mg + F_T \cos\theta = 0$ which we can write as $F_T \cos\theta = mg$

Dividing the first equation by the second will let us cancel out the tension term:

$\frac{F_T \sin\theta}{F_T \cos\theta} = \frac{F_{1\ on\ 2}}{mg}$ or just $\tan\theta = F_{1\ on\ 2}/mg$.

How much force is ball 1 exerting on ball 2? $F = Gm_1m_2/r^2$ in general so here: $F = (6.67 \times 10^{-11})(5)(5)/(0.17)^2 = 5.77 \times 10^{-8}\ N$.

The weight of the ball is $F_g = mg = (5\ kg)(9.8\ m/s^2) = 49\ N$ so:

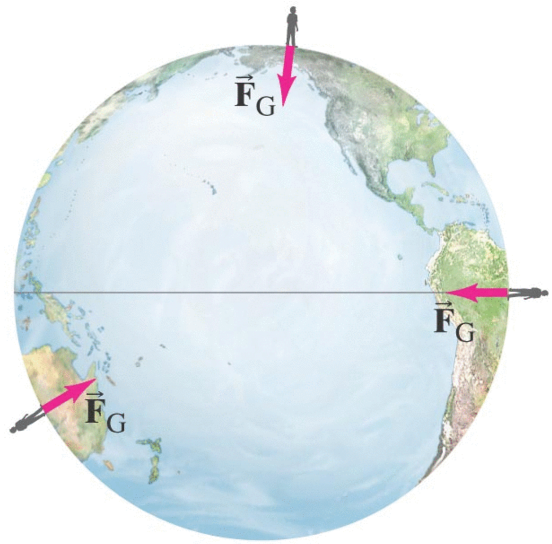
$\tan\theta = (5.77 \times 10^{-8})/(49) = 1.178 \times 10^{-9}$ from which $\theta = 6.75 \times 10^{-9}\ deg$. (Which is close enough to vertical that the cables certainly look like they're hanging straight down, even if they're not quite...)

Physically how far did the ball move? $\sin\theta = (x)/L = x/(1\ m)$ which leads to $x = 1.12 \times 10^{-9}\ m$. That's barely over one nanometer, or only about 10 atoms wide.

Measuring $G$ using an experiment like this would be hopeless, hence the clever approach done by Cavendish.

$\boxed{\text{Relating } F_g \text{ and } F_G}$

On surface of planet (like Earth), we can use the calculus result just mentioned and treat the Earth as a sphere of radius R, and pretend all it's mass M is located at the very center. In that case, the 'real' force of gravity acting on an object of mass $m$ (on the surface) would be: $F_G = G\frac{Mm}{R^2}$
The Earth is spherical-ish, so R is (sort of) constant, meaning we can relate this to the $F_g = mg$ we've been using:
$F_G = G\frac{Mm}{R^2} = mg$, which implies:





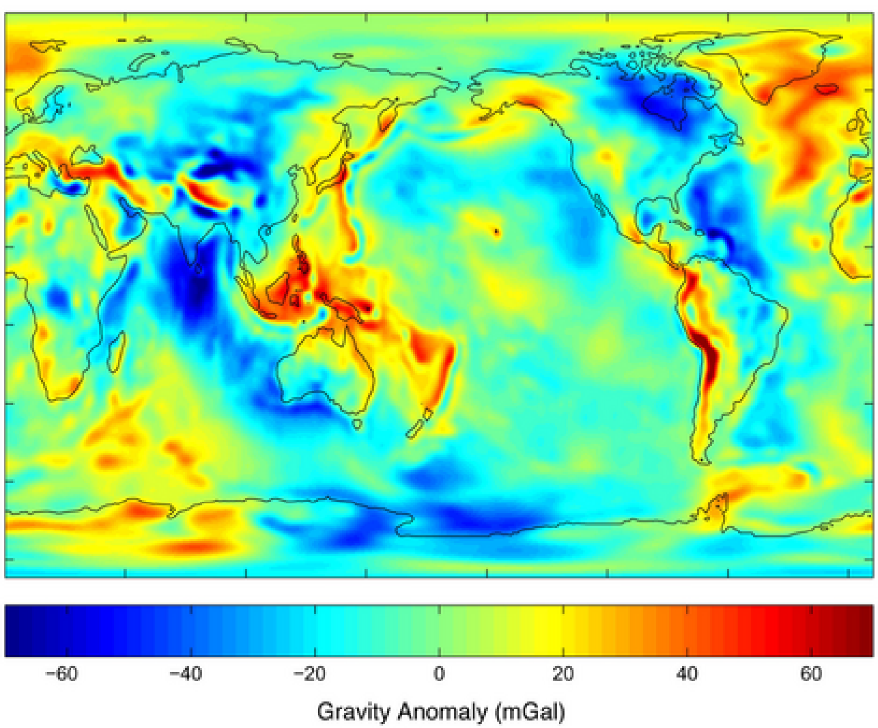
$$\boxed{g = GM/R^2}$$

We can measure $g$ and $R$ directly, so can use this to calculate the mass of the Earth:

$M = gR^2/G = (9.8)(6.38 \times 10^6)^2/(6.67 \times 10^{-11}) = 5.98 \times 10^{24}\ kg$

We can determine the mass of any planet/moon we land on the same way. Knowing it's size and measuring the local value of $g$ (by just dropping something and timing how long it takes to hit the 'ground'), this directly provides the mass of the planet/moon.

This figure shows how $g$ varies around the Earth, due mostly to the variations in the density or thickness of the crust.

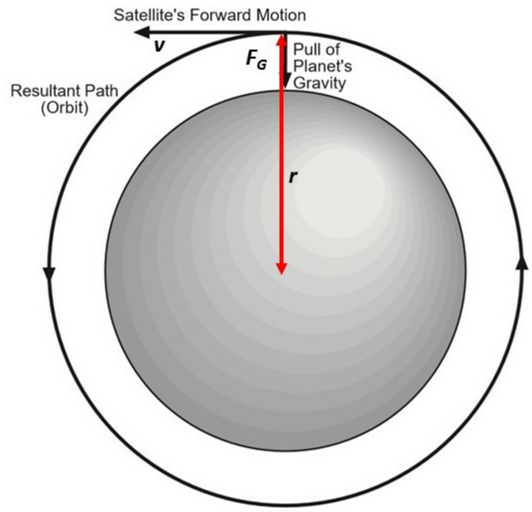


Gravity is a 'radially inward' force, which is just what we need for circular motion, so handles situations like planets around stars, and moons and other satellites around planets.

Consider a (relatively) light object of mass $m$ orbiting a much heavier object of mass $M$ in a circular path (the radius of that 'orbit circle' being $r$).

$\sum F_r = ma_r$ so $GMm/r^2 = mv^2/r$ or $\boxed{GM/r = v^2}$

Note that the mass of the satellite (or moon) cancels out here. (REMINDER: we're assuming $m << M$ here. If that's not the case, a more complicated analysis is needed.)

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The International Space Station orbits about 254 *miles* above the surface of the Earth. Determine it's orbit speed and period.

**ORBIT RADIUS** : Referring to the figure above, the ISS is orbiting 254 *miles* above the surface of the Earth, so the actual radius of it's orbit (the circle centered at the center of the Earth) would be the radius of the Earth plus that additional distance:

- radius of earth : $R_E = 6.38 \times 10^6\ m$
- height above surface: $h = 254\ miles \times \frac{1609\ m}{1\ mile} = 4.26 \times 10^5\ m$
- radius of orbit circle (sum of above): $\boxed{r = R_E + h = 6.81 \times 10^6\ m}$.

We found for circular orbits: $v^2 = GM/r$ so $\boxed{v = \sqrt{(GM_E/r)}}$

$G = 6.67 \times 10^{-11}\ N\ m^2/kg^2$
$M_E = 5.98 \times 10^{24}\ kg$
$r = 6.81 \times 10^6\ m$ (radius of orbit circle)

Result: $v = 7653\ m/s$ (about 17,120 miles/hour).

Period: $v = 2\pi r/T$ so $T = 2\pi r/v = 5588\ sec = 93.1\ min$

Note: satellites that orbit within a few hundred miles of the Earth's surface are referred to as Low-earth orbit (LEO) satellites, with orbit periods of slightly over 90 minutes.

This web page shows the orbits of a couple of the current LEO satellite constellations used to provide internet: `https://satellitemap.space`

# Geostationary Orbits

For any light object orbiting a much heavier object, we have:

- $v^2 = GM/r$ so the farther away the object is orbiting, the slower its speed
- $T = 2\pi r/v$ so if we're farther away (larger $r$) and moving at a slower speed $v$, the period will be much longer.

One very convenient orbit is to put the satellite in an equatorial plane with a period is exactly **one day**, which means that the satellite takes the same amount of time to go once around in it's orbit as the Earth takes to rotate once on it's axis. The effect is that the satellite always appears in the same spot in the sky, which means that the dish mounted on your house or in your yard can always point in the same direction. Objects at that distance are called geostationary satellites.

Satellites in the DirecTV and Dish networks are located out there, as are some weather satellites and others (older satellite internet providers).

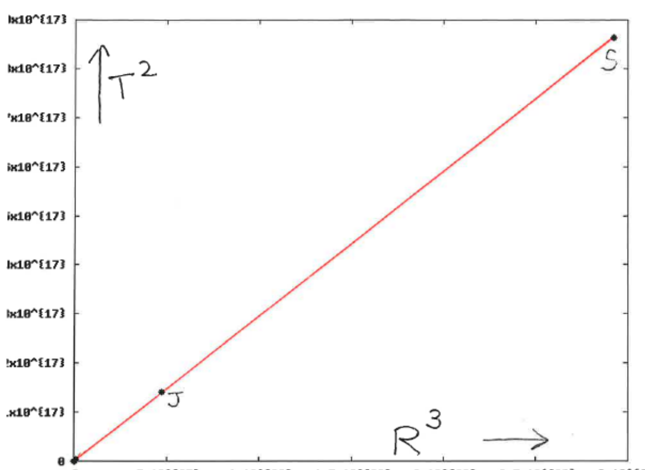
Let's determine how far away from the Earth they need to be located.

$v^2 = GM/r$ and $v = 2\pi r/T$ so replacing $v$ in the first equation yields: $\boxed{4\pi^2 r^3 = GMT^2}$

giving us a direct relationship between the period and orbit radius for objects orbiting a given heavy central object of mass M. For example, if the central body is the Sun we could use this to compute the periods for the various planets orbiting the Sun. If the central body is the Earth, this would connect the periods of satellites (including the Moon) to their orbit radii.

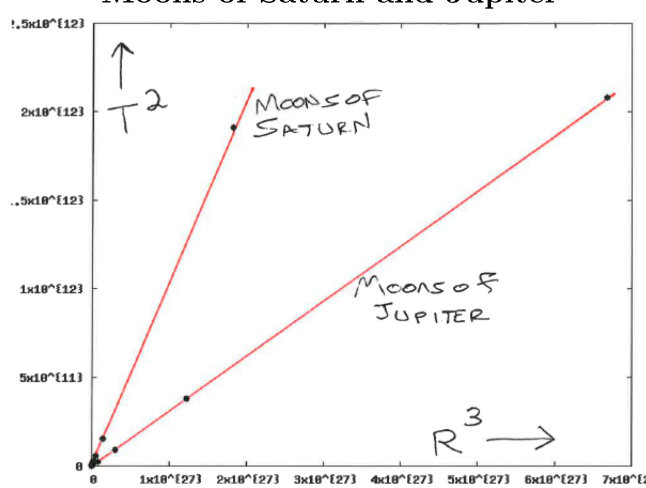
These figures plot $T^2$ vs $r^3$ for the planets (out to Saturn) around the Sun, and also for the moons orbiting Saturn and Jupiter.

**Planets orbiting the Sun**




Slope of line ultimately yields $M_{sun}$.

**Moons of Saturn and Jupiter**




Slopes yield masses of these planets. Note Jupiter is only 10% larger than Saturn. If they were made of the same 'stuff' that would make Jupiter about 33% heavier but the slope here shows that Jupiter has about **three times** the mass of Saturn.

Returning to our **geostationary orbit** situation, we're trying to find where to put a satellite so that it has a period matching the Earth's rotation, so the satellite stays at the same point in the sky at all times.

One complete (360 deg) rotation of the Earth takes 23 hours, 56 minutes, and 4.09 *sec*, for a total period of $T = (23\ hours) \times \frac{3600\ sec}{1\ hour} + (56\ min) \times \frac{60\ s}{1\ min} + 4.09\ s = 86164.1\ sec$.

Using $4\pi^2 r^3 = GMT^2$, we find we need an orbit radius of $r = 4.217 \times 10^7\ m$ or $42,174\ km$ or $26,211\ miles$ (that would be from the center of the Earth, so that's about $22,250\ miles$ above the surface of the Earth).

**Satellite-based Internet**

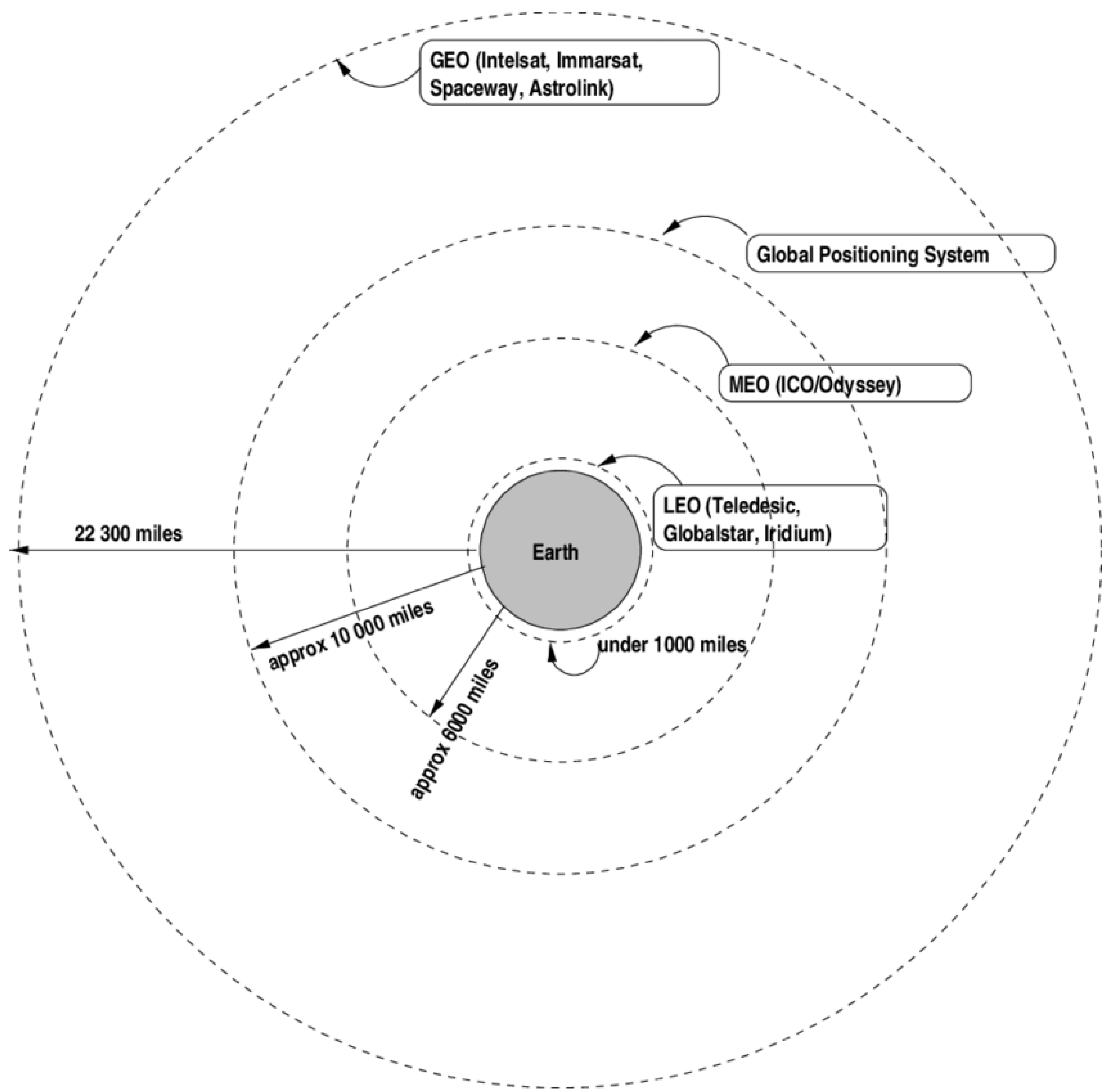
For geostationary satellite-based internet, this distance introduces a considerable amount of gaming **lag**. The signals have to go from your computer up to the satellite, then down to the server to calculate the result, which then has to go back up to the satellite and back down to your computer. The net result is that the signal has to travel an additional path of $4 \times 22,250\ miles = 89,000\ miles$.

These (radio) signals travel at the speed of light (about $186,282\ miles/sec$) so this introduces a total lag of about **478 ms** (on top of the usual lags that terrestrial internet providers have, resulting in about a **half-second delay** overall.

In recent years, several LEO internet satellite constellations have been launched, cutting the travel distance down by factor of 40 or so, so this 'speed of light lag' effect only adds about 12 *ms* to the other lags present (instead of 478 *ms*). (Unfortunately the 'cost' is that nearly all ground-based telescope images are cluttered with streaks representing light reflected from the thousands of such satellites now in orbit.)

See `https://satellitemap.space` for a 'live' graphic showing some of these constellations.

The figure below illustrates some of the common regimes for satellites with some examples. The International Space Station and Hubble space telescope are both about 340 miles above the surface (along with thousands of other satellites and space-debris). GPS satellites are in what is called a semi-synchronous orbit where they take exactly 12 hours to orbit the Earth. Over the past year, several companies have started to launch large fleets of satellites into low-Earth orbit to provide satellite-based internet with very low lag (since they're only a few hundred miles above the surface). Since their orbit periods are on the order of 90 minutes though, each individual satellite will zip by overhead quickly: thousands (or tens of thousands) of such satellites are needed to provide continuous internet access (which is driving astronomers nuts).



## Determining Mass of Planets and Moons

We previously found that for a (light) object orbiting a (much heavier) object, the orbit radius and period were related:

$$\boxed{4\pi^2 r^3 = GMT^2}$$

That means that if we can actually measure both $r$ and $T$, we can determine the mass of the body it's orbiting:

$$\boxed{M = \frac{4\pi^2 r^3}{GT^2}}$$

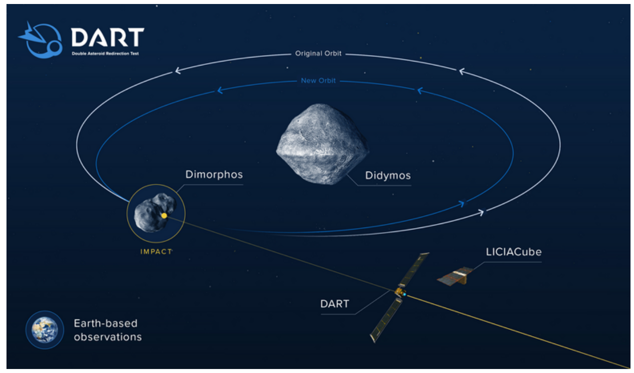
- Observing the (multiple) moons orbiting Jupiter, Saturn, Mars, etc, the masses of those planets can be found very accurately. (See example 5 in the examples06.pdf file where we estimate the mass of the Sun using information about the Earth's orbit.)
- Putting satellites in orbit around the Earth's moon allowed it's mass (and therefore it's surface value of $g = GM_{moon}/r^2_{moon}$) to be determined accurately prior to the moon landings.
- Any object that is observed to have other objects orbiting it can have it's mass determined, including other stars now. As of now, 5587 planets have been found orbiting 4150 stars (with 945 of those stars having multiple planets), allowing estimates of the masses of those stars to be found. Even some large asteroids in our own solar system have tiny moons (really just rocks) orbiting them, so their masses have been determined as well.

An interesting application of this was to 'measure' the mass of the black hole at the center of the Milky Way (called Sagittarius A*).

About 20 years of images taken of that location shows stars orbiting that object. From the orbit radii and periods, the mass of the black hole was estimated to be about 4.1 million times the mass of our Sun. A short movie, condensing these observations can be found here:

`https://www.eso.org/public/videos/eso1825e/`

**ADDENDUM** : Example: just recently (late 2022), NASA purposely crashed a spacecraft into a small asteroid to test the idea of deflecting asteroids on collision courses with Earth. The 'impactor' spacecraft had a mass of about 600 $kg$ and struck the asteroid travelling at 6600 $m/s$ (about 15,000 $miles/hr$).
The target asteroid was called Dimorphos, which is orbiting a larger asteroid called Didymos.




- Dimorphos is observed to be orbiting Didymos with a period of $T = 11\ hr\ 55\ min$ at a distance of 1.18 $km$ from Didymos. Use that information to estimate the mass of the central body (Didymos).

  Converting the given information, we have a period of $T = 42900\ sec$ and Dimorphos' orbit radius would be $r = 1180\ m$.

  For a light object orbiting a much heavier object, the mass of the central body is:
  $M = \frac{4\pi^2 r^3}{GT^2} = \frac{(4)(\pi^2)(590\ m)^3}{(6.67\times 10^{-11}\ N\ m^2/kg^2)(42900\ s)^2} = 5.28 \times 10^{11}\ kg$

  The value quoted in the literature is $M = 5.23 \times 10^{11}\ kg$, about 1 percent off, which is explained by the fact that the lighter object isn't really moving in an exact circle, but an ellipse.

- Didymos is roughly 780 meters across, while Dimorphos is only about 163 meters across. If they're made of the same material (i.e. have the same density), estimate the mass of Dimorphos.

  We can short-cut this calculation a bit. Density is mass/volume so mass is volume times density: $M = \rho V$. If we assume the objects are spheres (they're not quite), then $V = \frac{4}{3}\pi r^3$. Thus the mass is proportional to the volume, which itself is proportional to the cube of the radius. Ultimately then, $M \propto r^3$. That means we can set this up as a ratio problem:

  $\frac{M_{Dimorphos}}{M_{Didymos}} = \frac{r^3_{Dimorphos}}{r^3_{Didymos}}$ or

  $M_{Dimorphos} = M_{Didymos} \times \frac{r^3_{Dimorphos}}{r^3_{Didymos}} = (5.28 \times 10^{11}\ kg) \times \frac{160^3}{780^3} = 4.56 \times 10^9\ kg$. (The estimate quoted in the literature is $4.8 \times 10^9\ kg$; here the difference is probably due to the fact that the bodies are only approximately spherical.)

- What density did we find? $\rho_{water} = 1000\ kg/m^3$ and the density of the rocks in the Earth's crust are around 2600 $kg/m^3$. Metals like iron can have densities of around 8000 $kg/m^3$. What do these things appear to be made of?

  Density is mass/volume so $\rho = \frac{M}{V}$. We assumed the same density for each, so could use data from either object to estimate the density. Using Didymos, $M = 5.28 \times 10^{11}\ kg$ and $V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(390)^3 = 2.485 \times 10^8 m^3$ (note we needed the radius here, not the diameter, so $r = (780\ m)/2 = 390\ m$), we find $\rho = 2126\ kg/m^3$. (The value quoted in the literature is $2170 \pm 350\ kg/m^3$, so looks like we're pretty close on this one too.) They're both probably a mixture of rock and water (well, ice).

- What is **g** on the surface of Didymos?

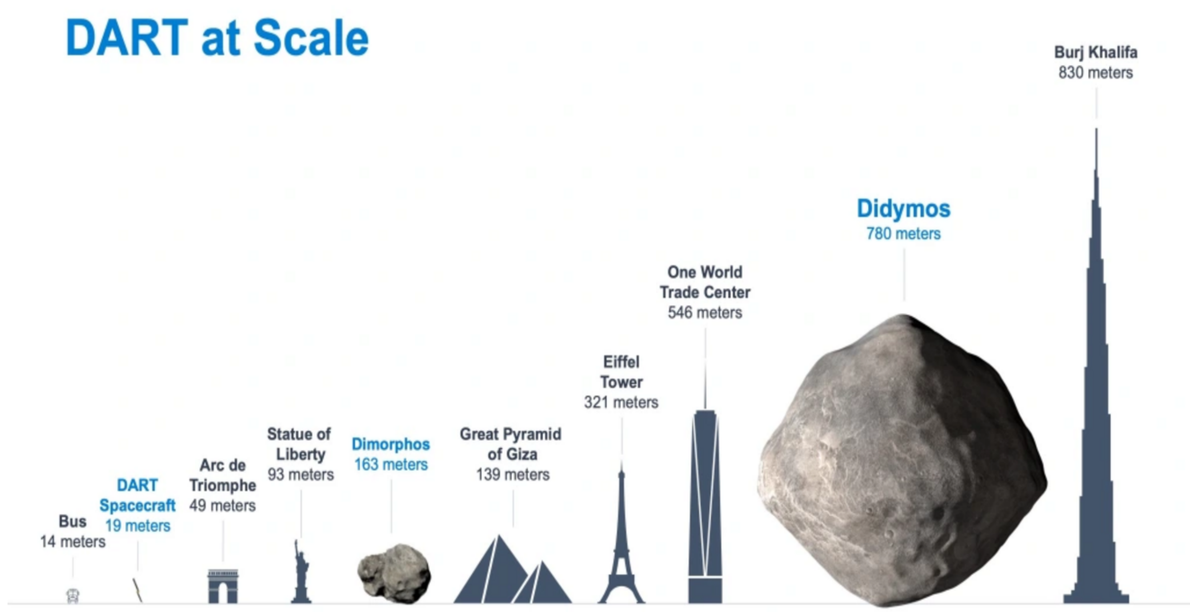
  We found earlier that $g = GM/r^2$ with $G = 6.6743 \times 10^{-11} N\ m^2/kg^2$ so $g = (6.6743 \times 10^{-11})(4.8 \times 10^9)/(390)^2 = 2.1 \times 10^{-6}\ m/s^2$

- If a person can jump 50 $cm$ vertically on Earth, how high would they jump on Didymos?

  Using our $v^2$ equation in the vertical direction, $v^2 = v_o^2 + 2a_y \Delta y$ so from the ground to the 'apogee' of this jump: $(0)^2 = v_o^2 + (2)(-g)(h)$ resulting in $v_o^2 = 2gh = (2)(9.8)(0.5)$ (on Earth) resulting in an initial vertical speed of $v_o = 3.13\ m/s$.

  On Didymos, with it's tiny value for $g$ : $h = v_o^2/2g = 2.33 \times 10^6\ m$ or 2330 $km$ (!).

  (That's far enough away from the asteroid that we can't use a constant value for $g$ and we'll have to tackle this another way.)




This figure just shows Didymos (the larger, central object) and Dimorphos (the smaller asteroid orbiting it) in comparison to some of the buildings and structures here on Earth.

If the smaller asteroid hit the Earth, it would destroy a city block. The larger one would take out a major city.

(The acronym DART stands for 'Double Asteroid Redirection Test.')