PH2213 : Examples from Chapter 6 : Gravitation

$\boxed{\textbf{Key Concepts}}$

Two point-mass objects of masses m_1 and m_2 separated by a distance of r will attract each other with a gravitational force of magnitude $\mathbf{F} = \mathbf{G}\frac{\mathbf{m_1 m_2}}{\mathbf{r^2}}$. The direction of this force is always attractive (i.e. along a line between the two points, and towards each other).

$$\mathbf{G} = \mathbf{6.67 \times 10^{-11} \ N \ m^2/kg^2}$$

A sphere with a uniform density or a density that only depends on the distance to the center of the sphere (i.e. $\rho(r)$ only) can be 'replaced' (as far as the gravitational force calculations go) by a point of the same mass located at the center of that sphere. (Provable via calculus, once you've gotten to volume integrals.)

If a spherical planet has such a density distribution, then the local acceleration due to gravity on the surface of the planet is $\mathbf{g} = \mathbf{GM/R^2}$ where M is the mass of the planet and R is its radius.

For a light mass in orbit around a (very) much heavier mass M (enough heavier that we can treat the larger mass as stationary), we can relate the period and orbital radius of the smaller mass (tiny moon or satelite perhaps):

$$\mathbf{T^2} = (\tfrac{4\pi^2}{\mathbf{GM}})\mathbf{r^3}$$

$\boxed{\textbf{Common Errors}}$

- $\vec{F}_G$ is a vector force that is always attractive

- Newton's third law applies here: this is an interaction force; each object exerts the same force on the other

- math : we're usually dealing with entities with large positive or negative exponents here so be careful in your calculations

- units : when dealing with planets and astronomical bodies, distances and times are often given in units other than simple meters and seconds

Example 1 : Three masses are shown in the figure. Determine the (vector) acceleration of mass 3 due to the gravitational attraction from masses 1 and 2. Assume the masses are uniform spheres (i.e. treat them as point-masses).

Mass 1 is a 1000 kg ball located at the origin. Mass 2 is a 500 kg mass located at $x = 2$ m, and mass 3 is a 10 kg mass located at $y = 4$.

First, we'll draw in vectors representing the force that each mass is exerting on ball 3. Gravity is an attractive force: each pair of objects is exerting a force on the other. From Newton's laws, $\sum \vec{F} = m\vec{a}$ but remember what that means: the forces acting **on an object** are what produce the acceleration of **that** object. So here, we have to consider just the forces that the other two balls are exerting on mass 3. Balls 1 and 3 are exerting equal and opposite forces on one another, ditto with 2 and 3 and between 1 and 2. Most of that we don't care about: it's only the forces being exerted **on** ball 3 that will affect it's motion.

So first, let's draw a figure showing the forces that are being exerted **on** ball 3. It's being attracted towards ball 1 so that force is exactly in the negative Y direction. It's being attracted towards ball 2 at some angle θ which we can determine from the coordinates: $\tan \theta = (2\ m)/(4\ m) = 0.5$ so $\theta = 26.565°$. We'll use that later to determine the components of that force.
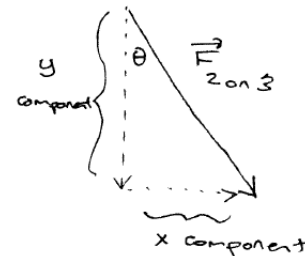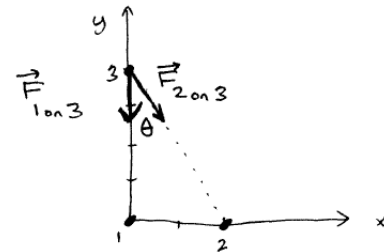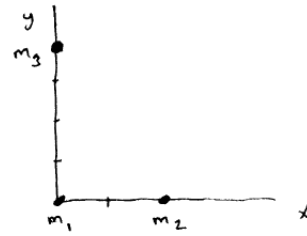
Force of 1 on 3 : the magnitude will be $F = Gm_1 m_3/r_{13}^2$ or $F = (6.67 \times 10^{-11})(1000)(10)/(2)^2 = 1.668 \times 10^{-7}$ N As a vector, it is pointing in the $-Y$ direction, so $\vec{F}_{1\ on\ 3} = -1.668 \times 10^{-7}\hat{j}$ (in newtons).

Force of 2 on 3 : the magnitude will be $F = Gm_2 m_3/r_{23}^2$. The distance (squared) between 2 and 3 will be $r_{23}^2 = (2)^2 + (4)^2 = 20$ m^2 so: $F = (6.67 \times 10^{-11})(500)(10)/(20) = 1.668 \times 10^{-8}$ N

We need to convert this into a vector (see bottom figure). The X component of this force will be $F \sin \theta$ and will be in the $+X$ direction. The Y component of this force will be $F \cos \theta$ and will be in the $-Y$ direction. We've already determine the angle though, so this becomes:

$\vec{F}_{2\ on\ 3} = (7.46 \times 10^{-9})\hat{i} - (1.49 \times 10^{-8})\hat{j}$.

From Newton : $\sum \vec{F} = m\vec{a}$ so we can add the two force vectors and divide by the mass of ball 3 to determine the acceleration of that ball: $\sum \vec{F} = (7.46 \times 10^{-9})\hat{i} - (1.82 \times 10^{-7}) = (10\ kg)\vec{a}$ so finally: $\vec{a} = (7.46 \times 10^{-10})\hat{i} - (1.82 \times 10^{-8})\hat{j}$ (in m/s^2).

Example 2 : Two 1000 kg wrecking balls are hanging from cables connected to the ceiling. Their centers are 20 cm apart. What is the force between them? The cables appear vertical but actually are not. What angle does each cable make with the vertical?

The upper figure shows the nominal situation. We see the two balls hanging 'vertically' on the cables. The two wrecking balls are attracting each other gravitationally though, there is a force pulling them towards each other. An exaggerated diagram of this is shown in the middle figure, where we've also replaced each wrecking ball with its equivalent point mass.

Focusing on the wrecking ball on the right, what are the forces acting on it? We have the force the earth is exerting on it (i.e. the usual $F_g = mg$ we've been using before). We also have the gravitational force that the ball on the left is exerting on this one, and we have some tension in the cable.

Everything is at rest here, so $\sum \vec{F} = 0$. Breaking this into components:
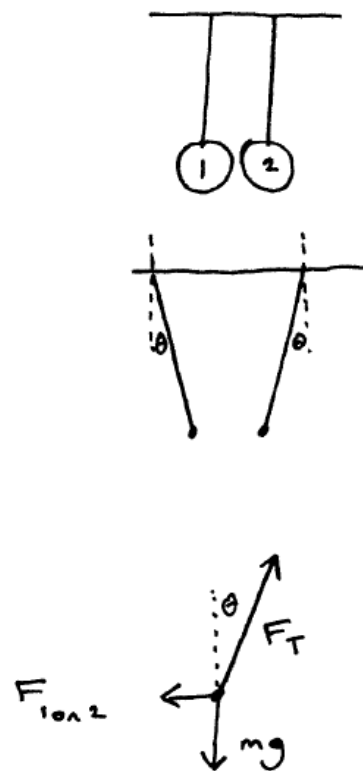
$\sum F_x = 0$ so $-F_{1 \text{ on } 2} + F_t \sin \theta = 0$ which we can write as $F_t \sin \theta = F_{1 \text{ on } 2}$.

$\sum F_y = 0$ so $-mg + F_t \cos \theta = 0$ which we can write as $F_t \cos \theta = mg$

Dividing the first equation by the second will let us cancel out the tension term:

$\frac{F_t \sin \theta}{F_t \cos \theta} = \frac{F_{1 \text{ on } 2}}{mg}$ or just $\tan \theta = F_{1 \text{ on } 2}/mg$.

How much force is ball 1 exerting on ball 2? $F = Gm_1m_2/r^2$ in general so here: $F = (6.67 \times 10^{-11})(1000)(1000)/(0.2)^2 = 1.668 \times 10^{-3}$ N.

The weight of the ball is $mg = (1000\ kg)(9.8\ m/s^2) = 9800\ N$ so:

$\tan \theta = (0.001668)/(9800) = 1.7 \times 10^{-7}$ from which $\theta = 9.75 \times 10^{-6}$ deg. (Which is close enough to vertical that the cables certainly look like they're hanging straight down, even if they're not quite...)

Example 3 : The space shuttle orbits the Earth at a height of 400 km above the surface of the earth. What is its period? (I.e. how long does it take to make one complete circle around the earth?)

The shuttle is moving in a circle with a radius r equal to 400 km plus the radius of the earth. Since it's moving in a circle, there is a radial acceleration of $a_r = v^2/r$. There must be a force creating this acceleration of $F = ma$ where m is the mass of the shuttle.

This force is being provided by the gravitational force between the Earth and the shuttle, so $F = GMm/r^2$ where M is the mass of the earth.

So: $\sum F = ma$ becomes $G\frac{Mm}{r^2} = m\frac{v^2}{r}$ and we can cancel some common terms here, leaving us with: $GM/r = v^2$. We're interested in the period, but $v = 2\pi r/T$ so making that substitution and rearranging to solve for T we get:

$$\boxed{T^2 = \frac{4\pi^2 r^3}{GM}}.$$

That's a generic result for any small body orbiting around a larger (much heavier) one that can be treated as not moving. It shows that T^2 is proportional to r^3, as noticed hundreds of years ago for the planets orbiting the sun, and for moons orbiting Jupiter. The constant of proportionality depends on the mass of the 'central object' - i.e. the thing about which the others are orbiting. (M would be the mass of the Sun, in the case of the planet's orbits; it would be the mass of Jupiter in the case of that planet's moons, etc).

For our shuttle situation, it's moving in a circular orbit but we have to be careful determining its radius. It's orbiting $400 \ km$ above the surface of the earth, which we'll assume is a sphere of radius $6380 \ km$ so the radius of the circular orbit the shuttle is in is actually the sum of these: $6780 \ km$ or $6,780,000 \ m$.

Looking up the mass of the Earth, $M = 5.98 \times 10^{24} \ kg$ so we have everything we need to compute the period now:

$T^2 = \frac{4\pi^2 (6,780,000)^3}{(6.67 \times 10^{-11})(5.98 \times 10^{24})} = 3.08 \times 10^7$ from which $T = 5554 \ sec$, which is about 93 minutes.

A good rule of thumb is that objects in 'near earth orbit' (NEO) are generally said to have orbital periods of about 90 minutes.
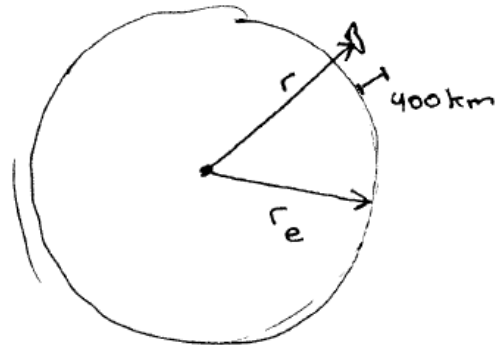
Example 4 : The earth rotates once on its axis every 24 hours (more or less). Where would we need to put a satellite so that it also takes 24 hours to rotate around the earth, thereby creating a **geostationary** satellite that is always in the same place in the sky (convenient for satellite-based TV transmission)?

This is the reverse of the previous problem. In this case we have a desired period for the satellite: 1 day, or $T = 86,400\ sec$ and want to determine the orbital radius r. Rearranging the boxed equation from the previous problem to solve for r:

$r^3 = \frac{GMT^2}{4\pi^2}$. Using the mass of the earth and the desired period of $T = 86,400\ sec$ we find that $r^3 = 7.54 \times 10^{22}$ or $r = 4.225 \times 10^7\ meters$ or about $42,250\ km$. The radius of the earth is about $6380\ km$ so this is $35,870\ km$ above the surface of the earth.

Note: light travels at a speed of about $3 \times 10^8\ m/s$ so it takes a signal about $0.12\ sec$ to get from the surface of the earth to this satellite, then another $0.12\ sec$ to get back. That's a total of $0.24\ sec$ or $240\ ms$ which makes satellite-based internet service very laggy as far as playing online computer games...

Example 5 : The earth takes 1 year to make a complete orbit around the Sun. The Earth doesn't move in **quite** a perfect circle, but assume it does, with an orbital radius of $149.6 \times 10^6\ km$. Compute the mass of the Sun.

We've already derived a generic equation relating orbital radius, period, and the mass of the central object so let's use that but rearrange to solve for M:

$$\boxed{M = \frac{4\pi^2 r^3}{GT^2}}$$

This is kind of useful because it lets us determine the mass of anything as long as we can spot something else orbitting it. Prior to the moon landings, satellites were sent to orbit the moon to get good pictures of potential landing sites, but also so that a very accurate value for the mass of the moon could be determined.

Getting back to the problem at hand, the orbital period of the Earth around the Sun is one year, or $T = 365.25636..\ days = 3.1558 \times 10^7\ sec$. The orbital radius is $r = 1.49 \times 10^{11}\ m$ so using those values we find that $M = 1.97 \times 10^{30}\ kg$ which is pretty close to the value given in the book. The orbit of the earth isn't quite a perfect circle, which introduced a small error.

Example 6 : Many stars are **binary** meaning there are two stars that rotate around one another. Assume we have two equal-mass stars that are separated by a constant distance of 7.6×10^{11} m and make one revolution every 14.0 yr. What must the mass of each of these stars be?

The figure shows this situation. The stars are moving in a circular path about a central point midway between them, so each is moving in a circle of radius $r = d/2$ where d is the given separation distance. This means there is a radial acceleration of $a_r = v^2/r$ which requires a force of $F = ma = mv^2/r$. This force is being provided by the gravitational attraction of the two bodies, which is $F = Gmm/d^2$ (we've used the fact that the masses of the two stars are the same in this problem).
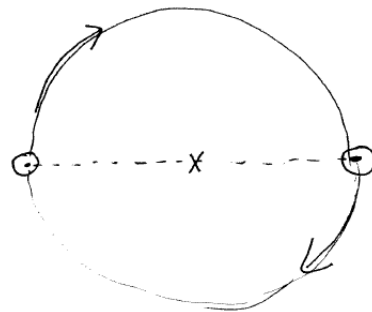
So: $G\frac{m^2}{d^2} = m\frac{v^2}{r}$ but $v = 2\pi r/T$ so $G\frac{m^2}{d^2} = m4\pi^2 r/T^2$ but $r = d/2$ so this becomes (after cancelling a common m from each side:

$G\frac{m}{d^2} = 2\pi^2/T^2$ or rearranging to solve for m:

$m = \frac{2\pi^2 d^3}{GT^2}$

Note how this is slightly different than the previous orbit problems we did. In those, we have a heavy, stationary 'central object' about which the other (very light) object (satellite, moon) was orbitting. In this case we have two things of the same mass circling around each other with the center of their orbits being midway between them.

Substituting in the values here, $d = 7.6 \times 10^{11}$ m, $T = 14.0$ $years$ or $T = (14 \; yr) \times \frac{3.16 \times 10^7 \; sec}{1 \; yr} = 4.424 \times 10^8$ sec, and $G = 6.67 \times 10^{-11} N \; m^2 /kg^2$ we arrive at $m = 6.64 \times 10^{29}$ kg (about 1/3 the mass of the Sun, in this scenario).

Example 7 : Consider the moon's orbit around the earth. It's tempting to draw it as a circle and use the same analysis we did with other earth-orbiting satellites. The moon's mass is quite large though, about 1/81 of the mass of the Earth, which is too large to ignore. The radial force the earth exerts on the moon causes it to move in a circular orbit, but the moon is exerting exactly that same amount of force on the Earth, causing **it** to move in a circular orbit as well! It's not as extreme as the binary star example, but there is some point about which **both** objects are rotating. Let's find where that is.
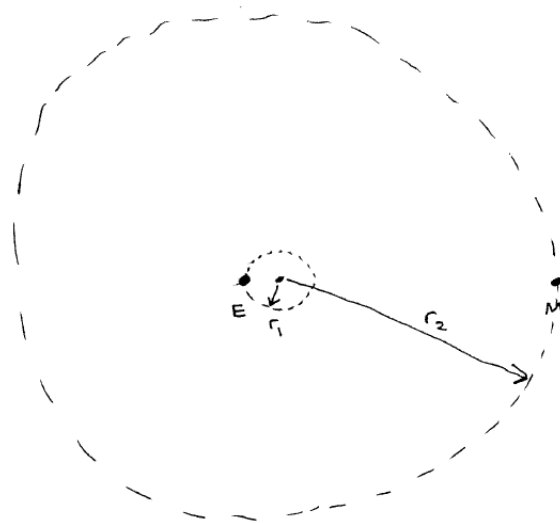
Remember that for spherically symmetric objects, we can pretend that all the mass it located at a point at its center, as far as Newton's gravity equation goes, so the figure denotes the Earth and Moon as points that are orbiting about some point. Let r_1 be the distance from the Earth (well, from the point-mass Earth, i.e. the center of the planet) to this point, and r_2 be the distance of the Moon from that point. The center of the Earth and the center of the Moon are separated by a distance of $d = 384,000 \ km$.

Let m_1 be the mass of the earth, and m_2 be that of the moon.

The magnitude of the force between these two is $F = G\frac{m_1 m_2}{d^2}$. The moon is orbiting about the central point at a distance of r_2 with some speed v_2 so there is a radial acceleration present of $a_r = v_2^2/r$ which requires a force of $\boxed{F = m_2 v_2^2/r_2}$ and the force providing this is the gravitational attraction between the Earth and the Moon.

Applying the same argument to the little circular orbit the Earth is making yields a required force of $\boxed{F = m_1 v_1^2/r_1}$.

The force between them is the same, by Newton's third law, so the right-hand sides of the two boxed equations must also be equal:

$m_2 \frac{v_2^2}{r_2} = m_1 \frac{v_1^2}{r_1}$

The speed of something moving in a circle is the circumference of that circle divided by the orbit period, so $v_2 = 2\pi r_2/T$ and $v_1 = 2\pi r_1/T$. Making these substitutions and cancelling common terms yields the particular simple relationship that: $m_1 r_1 = m_2 r_2$. (We'll see this again in later chapters when we talk about the center of mass of multiple objects and its impact on rotation of extended objects, not just the point-masses we're dealing with for now.)

That gives us our second equation since we already have $r_1 + r_2 = d$, the known distance between the (centers of the) earth and moon.

Let's determine r_1, the radius of the Earth's orbit about this common point. Combining these two equations yields:

$r_1 = (\frac{m_2}{m_1+m_2})d$. The Earth's mass is about 81 times that of the moon, so $m_1 = 81m_2$ and this becomes: $r_1 = \frac{1}{82}d = \frac{1}{82} \times (384,000 \ km) = 4683 \ km$.

The radius of the Earth is about 6380 km so the point that both the Earth and Moon are circling is located about 1/4 of radius of the Earth underground.

This results in the Earth 'wobbling' with a roughly 30 day period as it orbits the Sun.

You can apply the same process to other pairs of objects. Consider the Sun and Jupiter, the heaviest planet in the solar system. The Sun is roughly 1000 times more massive than Jupiter, but we find that the common point about which both are orbiting is about $800,000$ km out from the center of the Sun, which is just **outside** the surface of the Sun.

Observed from another star, the Sun would appear to wobble back and forth with a 12 year period due to the orbit of Jupiter. This sort of motion is detectable with current technology and is one of the ways planets have been detected around other stars (numbering in the thousands now...).