PH2213 Fox : Lecture 14
Chapter 7 : Work and Energy

This chapter switches gears and introduces a different approach for ‘motion’ problems based on energy.

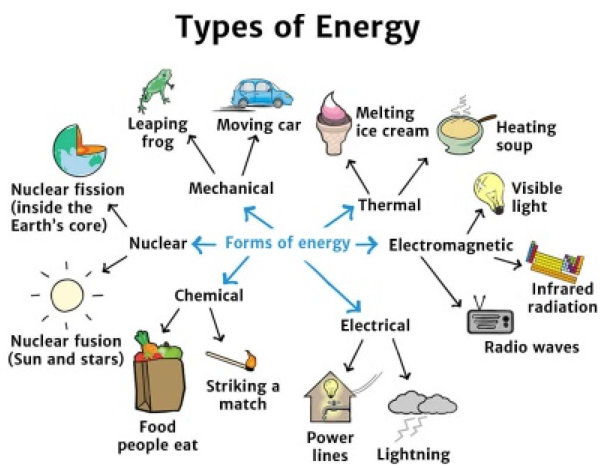
Chapters 1 through 6 : $\boxed{\text{micro}}$-management of an object’s motion

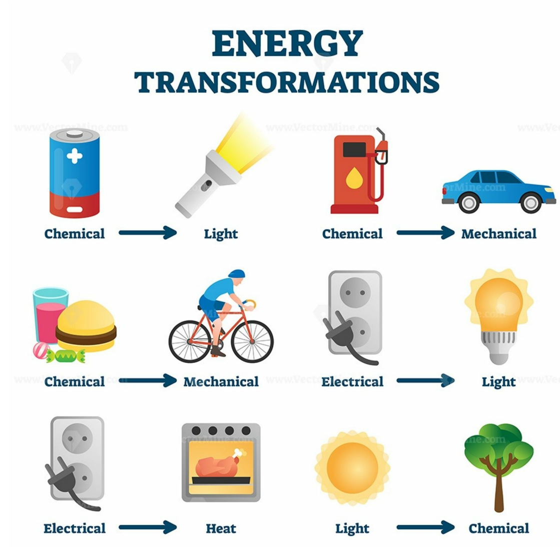
- $\sum \vec{F} = m\vec{a}$ PLUS (many) equations of motion
- simulations
- computer models
- detailed moment-by-moment trajectories
- movie/videogame ‘frame-by-frame’ scene creation
- ‘simple’ only if forces are constant

Chapters 7, 8 and 9 : $\boxed{\text{macro}}$-management of an object’s motion

- energy approach
- relates initial and final conditions directly
- (can still often use to reverse engineer intermediate details if needed)
- provides method for dealing with <u>non-constant</u> forces
- handle abrupt events (explosions, collisions)

Energy can appear in many forms, and there are often methods for converting one type into another:





Over the next few chapters, we'll focus on a subset of these called **mechanical energy**, the first of which is **kinetic energy**:

Kinetic Energy

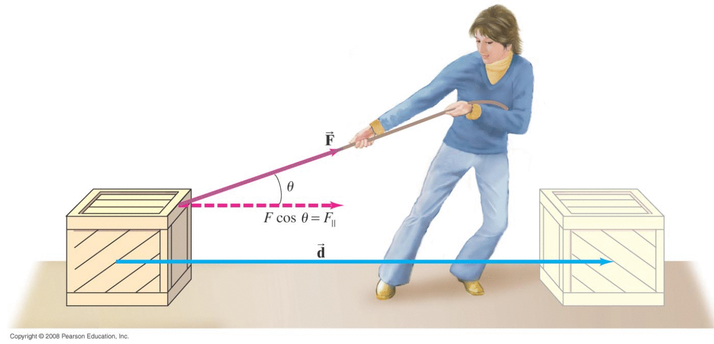
- Object of some mass m travelling at some <u>speed</u> v
- $K = \frac{1}{2}mv^2$
- Metric units (MKS) : $kg\ m^2/s^2$ or Joules (J)
- Metric units (CGS) : $g\ cm^2/s^2$ or ergs
- English: calories : $1\ cal = 4.184\ J$ so $1\ J = 0.239005736\ c$
- English: Calories : $1\ Cal = 1000\ cal = 4184\ J$
- English: 1 BTU (British Thermal Unit) is about 1055 J (about 252 cal)

Examples

- $100\ kg$ person walking at $2\ m/s$: $K = \frac{1}{2}(100\ kg)(2\ m/s)^2 = 200\ J$
- $2\ g$ $(0.002\ kg)$ bullet at $300\ m/s$: $K = \frac{1}{2}(0.002)(300)^2 = 180\ J$
- $1000\ kg$ car at $60\ mph$ $(26.82\ m/s)$: $K = 359,660\ J$

Other Forms of Energy		
chemical	1 snickers candy bar	$271\ C = 271,000\ c \times \frac{4.184\ J}{1\ c} = 1,130,000\ J$
chemical	6 inch Subway club sandwich	$1,600,000\ J$ $(1.6\ MJ)$
chemical	gasoline	$47.2\ MJ/kg$
nuclear	$U - 235$	$79,500,000\ MJ/kg$
electrical	supercapacitor	$1\ MJ/kg$
electrochemical	Lithium-ion battery	$1\ MJ/kg$
electrochemical	lead-acid car battery	$0.1\ MJ/kg$
nuclear	U-235	$79,500,000\ MJ/kg$
$E = mc^2$	anti-matter	$90,000,000,000\ MJ/kg$

A 50 kg crate is being pulled with a force of $F = 100\ N$ at a 37^o angle as shown. If it is initially moving to the right at 1 m/s, how fast will it be moving after displacing 2 m to the right? (Assume no friction here.)



Let's do this symbolically and morph the usual Newton+Equations-of-Motion approach into an energy-based one.

Coordinate system : let's use a coordinate system with an origin where the crate is initially located, with +X pointing to the right in the figure, and +Y pointing vertically upward.

Applying Newton's Laws then, we have $\sum F_x = ma_x$ so here $F\cos\theta = ma_x$ which we can rearrange into:

$$a_x = F\cos(\theta)/m$$

The speed of the crate after it's displaced to the right is related to the initial speed via one of our equations of motion:

$$v^2 = v_o^2 + 2a_x\Delta x$$

Substituting in the expression we found for a_x, we have:

$$v^2 = v_o^2 + 2(\frac{F\cos\theta}{m})\Delta x$$

Multiplying all the terms in this equation by $\frac{m}{2}$, and replacing the distance displaced by $\Delta x = d$ we have:

$$\frac{1}{2}mv^2 = \frac{1}{2}mv_o^2 + Fd\cos\theta$$

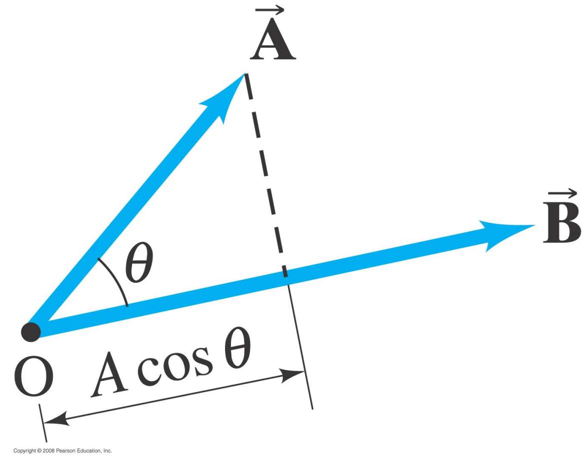
In words: The object's final kinetic energy is equal to it's initial kinetic energy plus that term on the right. The other terms in the equation are energies, so that one must be as well. It represents the amount of **energy** the pulling force added to the crate's kinetic energy as it moved from the initial to the final position.

That energy added (or removed) by a force acting on an object is called the **WORK** that force did on the object, and (in most physics textbooks anyway) is referred to using the symbol W. (There's some symbol overload there, since W or w are sometimes also used to represent the weight (a force) of an object.)

The **work** done by a force $\vec{F}$ on an object as it displaces by some vector displacement $\vec{d}$ is $W = Fd\cos\phi$ where ϕ is the angle between the force and the displacement vectors (basically it's picking up just the component of the force that is inline with the motion of the object). Mathematically, we are taking two vectors and creating a scalar using this special type of multiplication that is called the **dot product** or **scalar product** of the two vectors: $\vec{A}\cdot\vec{B} = |\vec{A}|\ |\vec{B}|\ \cos\phi = AB\cos\phi$

Scalar (Dot) Product of Two Vectors

The **scalar** or **dot** product of two vectors is a one of several forms of vector multiplication that turns out to be useful. (We'll see another, the cross-product, that gets involved in rotational motion and torques when we get near the end of the course.)
The scalar product of two vectors $\vec{A}$ and $\vec{B}$ is $AB\cos\theta$ and can be interpreted as the magnitude of one vector times the projection of the other vector onto it. **NOTE** I usually use ϕ to represent the angle between the two vectors to differentiate it from other angles in a problem, where θ might be used to represent the slope of a ramp for example.



The figure shows $\vec{A}$ being projected onto $\vec{B}$ creating the product $(A\cos\theta)(B)$ but if we instead project $\vec{B}$ onto $\vec{A}$, we end up with $(A)(B\cos\theta)$ which is the same value, so the dot product is **commutative**: $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$. (We're used to that with 'normal' multiplication since $2\times 3 = 3\times 2$ but there are other forms of vector multiplication that do not commute and we'll run into one (the cross product) later in the semester.)

Section 7-2 in the book covers the definition of the dot product and a couple of examples of using it. See also the homework solutions pdf for problems 7.18, 7.20 and 7.22 for other examples.

Example : Suppose we have an object moving up an incline, displacing by $\vec{d} = (3\hat{\imath} + 2\hat{\jmath})\ m$. (We're using a coordinate system where X is horizontally to the right and Y is vertically upward.)
Multiple forces may be acting on this object, but we want to determine the work that one particular force did. Suppose that force happens to be $\vec{F} = (-4\hat{\imath} + 3\hat{\jmath})\ N$.

(a) Compute the **work done** to the object by this force as it displaced the given amount.

(b) Use the dot product definition to determine the angle between those two vectors.

(If you sketch this out, this force is actually pulling in a way that will somewhat slow the object down.)

$W = \vec{F}\cdot\vec{d} = (-4\hat{\imath} + 3\hat{\jmath})\cdot(3\hat{\imath} + 2\hat{\jmath})$

Expanding this out (the usual FOIL method still works):

$W = (-4)(3)\hat{\imath}\cdot\hat{\imath} + (-4)(2)\hat{\imath}\cdot\hat{\jmath} + (3)(3)\hat{\jmath}\cdot\hat{\jmath} + (3)(2)\hat{\jmath}\cdot\hat{\jmath}.$

The scalar parts of the multiplication we can still do, so that leaves us with:

$W = -12\hat{\imath}\cdot\hat{\imath} - 8\hat{\imath}\cdot\hat{\jmath} + 9\hat{\jmath}\cdot\hat{\jmath} + 6\hat{\jmath}\cdot\hat{\jmath}.$

What do those remaining unit-vector dot products mean now?

If we go back to the basic definition of dot produce: $\vec{A} \cdot \vec{B} = AB\cos\phi$.

- $\hat{\imath} \cdot \hat{\imath}$: the dot product is the magnitude of the first vector times the magnitude of the second vector, times the cosine of the angle between them.
- These are unit vectors, so $|\hat{\imath}| = 1$
- The vectors are both pointing in the same direction, so the angle between them is 0^o
- $\hat{\imath} \cdot \hat{\imath} = (1)(1)\cos 0^o = 1$

The same will occur for any unit vectors when we're taking the dot product of that unit vector with itself: **the dot product of any unit vector with itself is 1**.

What if we do the dot product between two **different** unit vectors (where the unit vectors represent the coordinate axes like x, y and z)? $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ are pointing along the coordinate axes, which are mutually perpendicular, so the angle ϕ will be 90^o in these cases. But cosine of 90 degrees is zero, so: **the dot product of any two DIFFERENT unit vectors is 0**.

Now we can go back and finish up our calculation. We ended up with:

$W = -12\hat{\imath} \cdot \hat{\imath} - 8\hat{\imath} \cdot \hat{\jmath} + 9\hat{\jmath} \cdot \hat{\jmath} + 6\hat{\jmath} \cdot \hat{\jmath}$.

But we can replace all those dot-product pairs now:

$W = -12(1) - 8(0) + 9(0) + 6(1)$ which leads to $W = -6\ joules$.

As the object slid up the incline, this particular force would have **removed** $6\ J$ of energy from it.

What is the angle between the two vectors here?

We could probably draw this out on some coordinate system and go through a series of trig steps to determine that, but the dot product gives us a shortcut:

$$\boxed{\vec{A} \cdot \vec{B} = AB\cos\phi}$$

Here, our two vectors were $\vec{F} = -4\hat{\imath} + 3\hat{\jmath}$ (the force in newtons in this example) and $\vec{d} = 3\hat{\imath} + 2\hat{\jmath}$ (the displacement in meters) and we know that their dot product was -6.

Using the definition of the dot product, $\vec{F} \cdot \vec{d} = Fd\cos\phi$ where ϕ is the 'angle between the two vectors', which is what we're trying to determine. We know the left hand side of that equation is -6. We also know that $F = |\vec{F}| = \sqrt{(-4)^2 + (3)^2} = 5$ and $d = |\vec{d}| = \sqrt{(3)^2 + (2)^2} = \sqrt{13}$ so $\vec{F} \cdot \vec{d} = Fd\cos\phi$ becomes: $-6 = 5\sqrt{13}\cos\phi$ from which $\cos\phi = \frac{-6}{5\sqrt{13}} = -0.33282...$ and finally $\phi = 109.4^o$.

NOTE: the 'angle between the vectors' is always in the range from 0^o to 180^o, which exactly matches the angle range your calculator's inverse cosine function will return.

Work-Kinetic Energy Theorem

We usually have multiple forces acting on an object and each one is potentially adding or removing energy from the object ('doing work on the object'). In each case, the work done involves just the component of the force that's in the direction of the objects motion so each force is doing work that involves a dot product between that force and the displacement of the object.

Combining all these works with the object's original kinetic energy yields what the object's final kinetic energy must be:

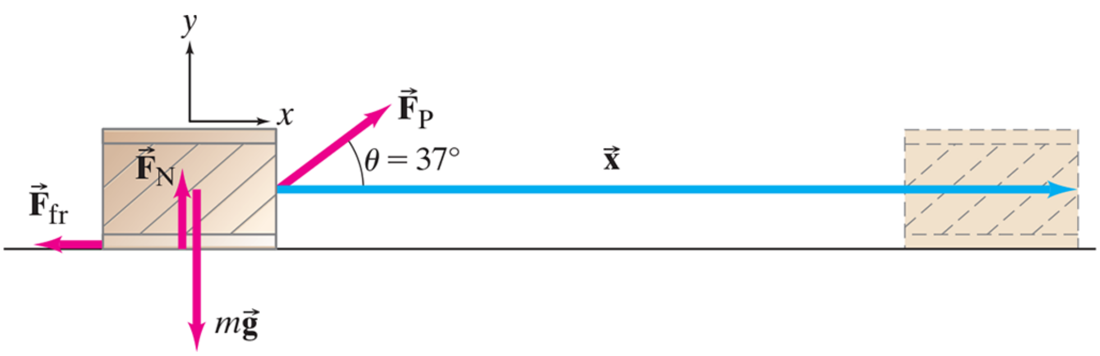
$$\boxed{K_{final} = K_{initial} + \sum W_i}$$

where for each force $\vec{F_i}$ acting on the object we compute the work done by that force by taking the dot product of that force and the displacement of the object:

$$\boxed{W_i = \vec{F_i} \cdot \vec{d} = F_i d \cos\phi_i}$$

Let's revisit the crate being pulled, and this time we'll include **friction**, with $\mu_k = 0.2$.

A 50 kg crate is being pulled with a force of $F = 100\ N$ at a 37^o angle as shown. If it is initially moving to the right at 1 m/s, how fast will it be moving after displacing 2 m to the right? Assume the coefficient of kinetic friction between the crate and the floor is $\mu_k = 0.2$.



What are all the forces acting on the object?

- The pulling force at the angle shown.
- Gravity F_g straight down
- Normal force F_N straight up
- Kinetic friction

The object is displacing 2 m to the right here. The book figure uses $\vec{x}$, but I'll call that $\vec{d}$ ('d' for 'displacement').

$\boxed{\text{Compute the work done by each force}}$ where $W = \vec{F} \cdot \vec{d} = Fd\cos\phi$

Remember here that F and d are the magnitudes of the corresponding vectors, so they are always positive (or zero, but never negative). **The sign of the work W is determined by the cosine term.**

- Work done by F_g : this force is acting straight down (the negative Y direction), and the displacement vector is in the positive X direction, so there's a 90^o angle between those two vectors. $W_{F_g} = F_g d\cos(90^o) = 0$ since the cosine of 90 degrees is zero. Gravity is not doing any work on the crate (i.e. not adding or removing any energy from it).
- Work done by F_N : the normal force is acting straight up (the positive Y direction), and the displacement vector is in the positive X direction, so there's a 90^o angle between those two vectors. $W_{F_N} = F_N d\cos(90^o) = 0$ since the cosine of 90 degrees is zero. The normal force is not doing any work on the crate (i.e. not adding or removing any energy from it).
- Work done by the F_{pull} : $W_F = \vec{F} \cdot \vec{d} = Fd\cos\phi = (100\ N)(2\ m)\cos(37^o) = +159.73\ J$. This force did do work on the crate, adding that much energy to it.
- Work done by friction (see below)

Work done by friction : for this one, we'll have to go back to the previous chapter since $f_k = \mu_k F_N$ which means we need to determine the normal force present on the crate.

$W_{f_k} = \vec{f_k} \cdot \vec{d} = f_k d\cos\phi$ where

- f_k is the magnitude of the force of friction here so $f_k = \mu_k F_N$
- d is the magnitude of the displacement (here $d = 2\ m$)
- ϕ will be the angle between those two vectors. $\vec{f_k}$ is directly to the left and $\vec{d}$ is directly to the right, so $\phi = 180^o$
- That means that $W_{f_k} = f_k d\cos\phi = (\mu_k F_N)(d)(\cos 180^o)$ or $W_{f_k} = -\mu_k F_N d$

What is the normal force here? Looking in the Y direction, we have F_N vertically upward, F_g vertically downward, and we have a Y component of the pulling force upward.

$\sum F_y = 0$ becomes $F_N - mg + 100\sin(37^o) = 0$ or $F_N = mg - 100\sin(37^o) = (50)(9.8) - 60.2 = 429.8\ N$.

Finally then, the work done by friction will be $W_{f_k} = -\mu_k F_N d = -(0.2)(429.8\ N)(2\ m) = -171.92\ J$.

Finally we can put this all together.

$K = K_o + \sum W = K_o + W_{F_g} + W_{F_N} + W_F + \mathbf{W_{f_k}} = 25 + 0 + 0 + 159.73 - 171.92 = 12.81\ J$.

Converting that back into speed:

$K = \frac{1}{2}mv^2$ so here $12.81 = (0.5)(50\ kg)(v)^2$ from which $|v| = 0.716\ m/s$.

Note that the crate is **slowing down** here.

What if the question asked how fast the crate was moving after displacing $\boxed{5\ m}$ to the right? (Turns out this is a BAD QUESTION because the crate can't actually travel that far.)

We don't have to redo much of the previous work. The work done by gravity and the normal force are still zero, so we're left with:

- Work done by the pulling force : $W_F = \vec{F} \cdot \vec{d} = Fd\cos\phi = (100\ N)(\mathbf{5}\ m)\cos(37^o) = +399.32\ J$.
- Work done by friction: $W_{f_k} = -\mu_k F_N d = -(0.2)(429.8\ N)(\mathbf{5}\ m) = -429.8\ J$.

Our Work-Kinetic Energy equation then becomes:

$K = K_o + \sum W = K_o + W_{F_g} + W_{F_N} + W_F + W_{f_k} = 25 + 0 + 0 + 399.32 - 429.8 = -5.48\ J$.

BUT $K = \frac{1}{2}mv^2$ can never be negative. m is always positive, and v^2 will be positive also.

$\boxed{\text{A NEGATIVE kinetic energy is IMPOSSIBLE}}$, so either we did something wrong here (which we didn't), or the original problem statement must be wrong. It's not possible to pull the crate the full 5 meters. Friction is removing energy faster than the pulling force is adding energy, and eventually the 25 J of kinetic energy we started with will be gone. $K = 0$ means $v = 0$, so the crate will come to a stop.

> Any time you work a problem and end up finding a NEGATIVE value for K, there's a 100% chance that either the problem is wrong (unlikely with a HW or test problem), or you did something wrong that lead to that result. **Don't just ignore the sign and take the square root anyway** - it's a useful RED FLAG that denotes an error to find. (Frequently this happens when students use a negative value for g, or for F or d in the equation $W = Fd\cos\phi$. The F and d are magnitudes of those vectors (always positive); the SIGN for the work comes from taking the cosine of the angle ϕ.

Determine how far the crate will move before coming to a stop.

'Coming to a stop' means $v = 0$, so sticking with our energy approach that means our final $K = 0$, and now we're looking for the d where that will happen.

Leaving d as a symbol in our work terms:

- Work done by the pulling force : $W_F = \vec{F} \cdot \vec{d} = Fd\cos\phi = (100\ N)(\mathbf{d})\cos(37^o) = +79.86d$.
- Work done by friction: $W_{f_k} = -\mu_k F_N d = -(0.2)(429.8\ N)(\mathbf{d}) = -85.96d$.

Our Work-K equation: $K = K_o + \sum W$ becomes: $0 = 25 + 0 + 0 + 79.86d - 85.96d$ or $0 = 25 - 6.1d$ from which $d \approx 4.1\ m$