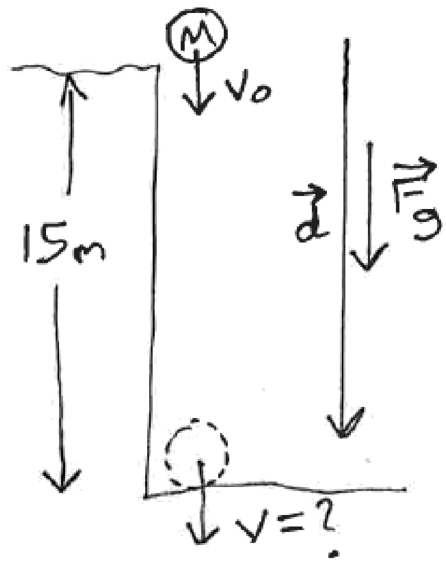
PH2213 Fox : Lecture 15
Chapter 7 : Work and Energy

Review : Key bits we have so far:

- Kinetic energy: $K = \frac{1}{2}mv^2$
- Work-K approach : $K_b = K_a + \sum W_i$ where:
- $W = \vec{F} \cdot \vec{d} = Fd\cos\phi$
  ϕ is the interior angle between $\vec{F}$ and $\vec{d}$ so $0 \leq \phi \leq 180$
  Compute the work that each force does individually.
  Note: $F = |\vec{F}|$ and $d = |\vec{d}|$ are **magnitudes** so are always positive (or zero). The **sign** of the work (positive or negative) comes from the $\cos\phi$ term.

Work Done by Gravity



A ball is thrown straight downward from the top of Hilbun with a speed of 20 m/s. How fast will the ball be moving when it hits the ground 15 m below?

$K = K_o + \sum W$ and here we only have a single force acting: gravity downward.
$K_o = \frac{1}{2}mv^2 = (0.5)(m)(20)^2 = 200m$ (we don't know the mass of the object here, so it'll just have to stick around as the symbol m for a while...)

$W_{F_g} = \vec{F}_g \cdot \vec{d} = F_g d\cos\phi$

The force of gravity has a magnitude of $F_g = mg$ (it's really important moving forward to remember that the symbol g is **positive**, on Earth representing a magnitude of 9.8 m/s^2). The displacement vector is straight down, with a magnitude of $d = 15\ m$ (that's meters, not mass now, so we'll just make sure everything is in standard metric units and quickly drop writing them to avoid confusion). As vectors, $\vec{F}_g$ and $\vec{d}$ are both pointing straight down, so they're in the same direction, making the angle between them $\phi = 0$. Finally then: $W_{F_g} = \vec{F}_g \cdot \vec{d} = F_g d\cos\phi = mgd\cos 0 = mgd = (m)(9.8)(15) = 147m$ (again, the m here is the mass of the object).

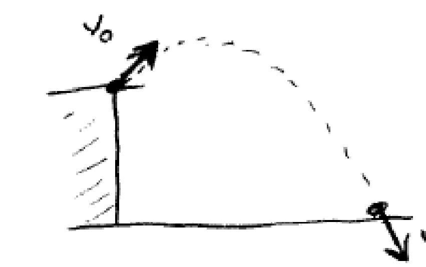
Putting everything together: $K = K_o + \sum W$ so $\boxed{\frac{1}{2}mv^2 = \frac{1}{2}m(20)^2 + 147m.}$

Every term in that equation has the same m (mass of object) in it, so we can divide the entire equation by m and eliminate it (evidently if we can throw a ball bearing or a Buick at 20 m/s initially downward, they'll both hit the ground at the same speed).

That leaves us with: $\frac{1}{2}v^2 = \frac{1}{2}(20)^2 + 147$ or multiplying the equation by 2: $v^2 = (20)^2 + (2)(147) = 400 + 294 = 694$ from which $v = \sqrt{694}$ or $|v| = 26.34\ m/s$. Note again this method tells us nothing about direction: the math doesn't tell us whether the ball is moving down, up, or sideways - the only thing we know is how fast it's going (its speed) at that point.

What if we kick the ball so that it flies off at the same 20 m/s but at a completely unknown launch angle? Can we still determine the speed when it strikes the ground 15 m below?

A ball is kicked from the top of Hilbun with a speed of 20 m/s. How fast will the ball be moving when it hits the ground 15 m below? (That's all we know here: the mass and launch angle are unknown.)



Technically, the displacement vector $\vec{d}$ is just a vector that points **directly** from the initial to the final location. Even though that's not the **trajectory** the object follows between those two points, let's compute the work done by gravity using $W = \vec{F} \cdot \vec{d}$ anyway and then we'll show why it still works.

$W_{F_g} = \vec{F}_g \cdot \vec{d} = mgd\cos\phi$ but if we arrange those terms this way: $W_{F_g} = mgd\cos\phi = (mg)(d\cos\phi)$ we see that $d\cos\phi$ is just the (magnitude) of the vertical change in position: the height of the building. So whatever d and ϕ are, that particular collection $d\cos\phi$ is just 15 m.

The work done by gravity on the ball then is $W_{F_g} = (m)(9.8)(15) = 147m$. (That m isn't units of meters, it's the variable representing the mass of the ball.)

Applying our work-kinetic-energy process then:

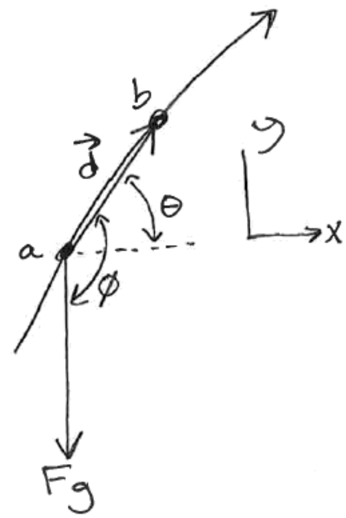
$K_b = K_a + \sum W$ becomes: $\frac{1}{2}mv_b^2 = \frac{1}{2}mv_a^2 + 147m$. Every term left here has the unknown mass m of the ball in it, so we can divide the whole equation by m and get rid of that variable. While we're at it, let's multiply the whole equation by 2 to get rid of those factors of 1/2, leaving us with: $v_b^2 = v_a^2 + (2)(147) = (20)^2 + 294 = 694$ so $|v_b| = 26.34\ m/s$.

That's exactly the same value we found when we threw the ball straight downward. In this version, we're launching the ball at the same 20 m/s but at any arbitrary angle and we still end up with the ball hitting the ground with the exact same speed of 26.34 m/s.

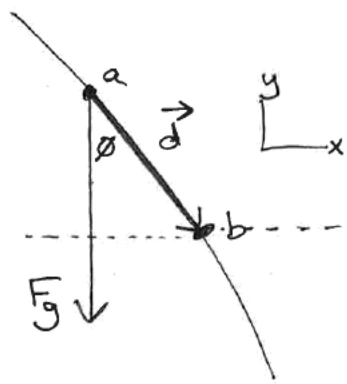
Let's be sure we can really do it this way though : i.e. we can ignore the actual path the ball took (the parabolic arc) and just directly use the force and the displacement vector (going in a straight line from the starting to ending points).

(See next page...)

Let's look at a part of the trajectory where the ball is flying to the right and upward, and focus on a tiny part of that path. The work done by gravity on this little segment will be $W = \vec{F}_g \cdot \vec{d} = F_g d \cos\phi$. The ball is travelling partly upward, so let θ be the angle above the horizontal. Then $\phi = 90 + \theta$ so $W = F_g d \cos(90 + \theta)$ but $\cos(90 + \theta) = -\sin\theta$ so our work expression becomes: $W = -F_g d \sin\theta$. Collecting terms, this is $W = -(mg)(d \sin\theta)$ but looking at the figure, $d \sin\theta$ is just the height change between points (a) and (b) on this little segment of the path. y_b is above y_a so we'd write that as $y_b - y_a$ which is just Δy.
Summary: along the part of the trajectory where the ball is flying partly upward, the work done by gravity is just $\boxed{W_{F_g} = -mg\Delta y}$.



This time, let's look at a part of the trajectory where the ball is flying to the right and downward, and focus on a tiny part of that path. The work done by gravity on this little segment will be $W = \vec{F}_g \cdot \vec{d} = F_g d \cos\phi$. The term $d \cos\phi$ is just the magnitude of the vertical part of the displacement, or $|\Delta y|$. Since $y_b < y_a$, we have to write this as $y_a - y_b$ to get a positive length, but that's $-\Delta y$ (since Δ means the later value minus the earlier value: $\Delta y = y_b - y_a$).
Putting this together, the work done by gravity as the object moved from (a) to (b) now would be written as $W_{F_g} = (mg)(-\Delta y)$ or $\boxed{W_{F_g} = -mg\Delta y}$ or exactly the same result!

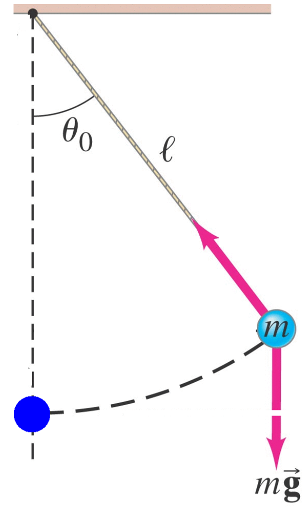


Adding up the work done over all those little segments then from the starting point all the way to the ending point (when the ball hits the ground), the sum of all those Δy's just gives us the overall change in height from the starting to the ending point. This gives us a very useful shortcut for dealing with the work done by gravity: it <u>only</u> depends on the **change** in the **vertical** coordinate of the object.

Note that wording again : $\boxed{\text{change in the \underline{VERTICAL} coordinate}}$. In many scenarios, we might find it convenient to have a rotated coordinate system, so our y axis may not actually be vertical. To emphasize this, I usually write the work done by gravity as:

$$W_{F_g} = -mg\Delta h$$

where Δh represents the change in the **height** coordinate: an 'axis' that is pointing vertically upward.



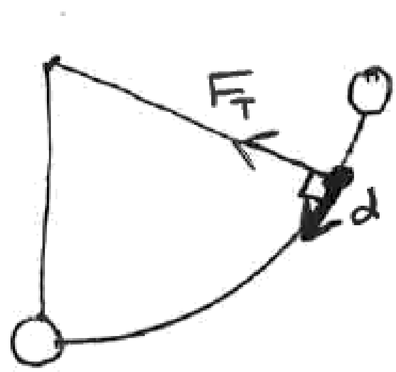
$\boxed{\textbf{Pendulum}}$: An object hanging on a string of length $L = 50\ cm$ is pulled 30 cm over to the side (i.e. horizontally) and released (at rest). How fast will the object be moving when it passes through the bottom of the arc?

What forces are acting on the object? How much work did each force do on the object as it moves from the initial position to the point at the bottom of the arc?

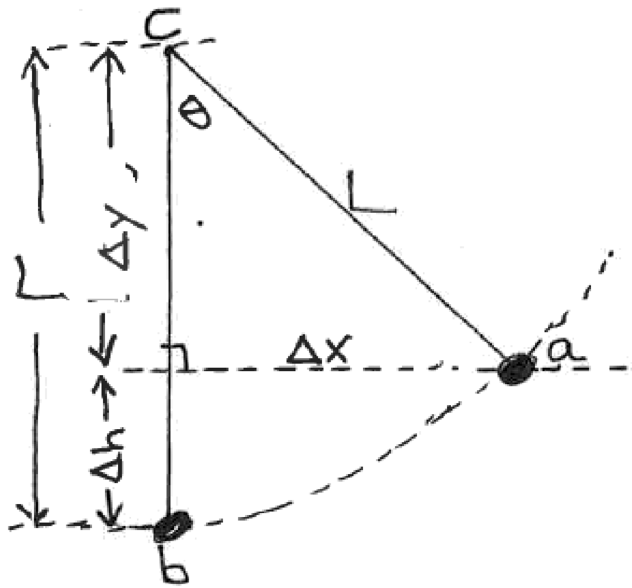
Back in chapter 5 we looked at a swingset stunt from Mythbusters, where we had an object swinging around in a complete circle. We found that the tension in the chain changed, so we already have a problem here. There are two forces acting on the object, $\vec{F}_T$ and $\vec{F}_g$ but the tension isn't constant (either in magnitude or direction), meaning $\sum \vec{F} = m\vec{a}$ won't be constant either. That in turns means we can't use any of our equations of motion to help here.

Let's attack the current version of the problem using Work and Kinetic energy methods instead. We'll use ⓐ to label the starting point, and ⓑ to label the point where the object is passing through the lowest point on the arc. Then: $\boxed{K_b = K_a + \sum W_i}$ where we have two works to compute: W_{F_T} and W_{F_g}.

$\boxed{\text{Work done by } F_T}$: The tension force is constantly changing magnitude and direction, but it's always perpendicular to the objects motion. If we break the trajectory into tiny steps, each tiny displacment d is tangent to the circle the object is travelling along, but F_T is always radial, so it's always perpendicular to that circle. Step by step, F_T is perpendicular to the displacements, so the angle between those vectors is always 90^o, meaning the dot product is always zero. In this scenario then $W_{F_T} = 0$.



Work done by F_g : we found last time that $W_{F_g} = -mg\Delta h$ meaning that the work done by gravity only depends on the **height change** of the object between the two points. Here it's negative since point ⓑ is lower than point ⓐ, but lower by how much?
In the figure, ⓒ designates the center of the circular path (the point about which the object is swinging). The distance from ⓒ to ⓐ is the length of the string $L = 50\ cm$. The distance labelled Δx in the figure was given to be $\Delta x = 30\ cm$. We have a right triangle there so $(\Delta x)^2 + (\Delta y)^2 = L^2$ so here $(30)^2 + (\Delta y)^2 = (50)^2$ from which $\Delta y = 40\ cm$.



The distance from ⓒ to ⓑ is also $L = 50\ cm$ though (we're assuming the string doesn't stretch as the object swings back and forth), so the height change Δh must be $50 - 40 = 10\ cm$.

Now those were all just magnitudes: physical lengths in the figure. We need to account for signs, so the **height change** here is actually $\Delta h = -10\ cm$ (point b is that much lower than point a).

Finally then, $W_{F_g} = -mg\Delta h = -(m)(9.8\ m/s^2)(-0.1\ m) = +0.98m$ (that m is the mass of the object).

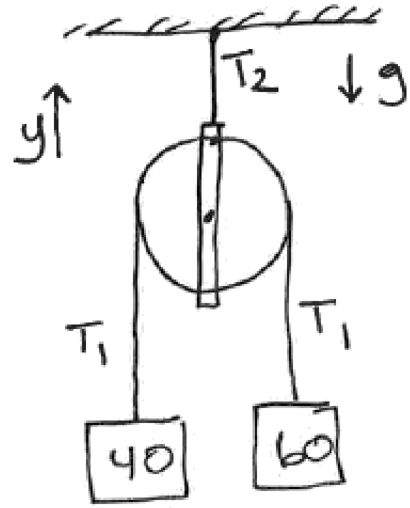
Determining the speed

Now we can use work-energy to find the speed of the object at point ⓑ. The object started at rest, so $K_a = \frac{1}{2}mv_a^2 = 0$.

$K_b = K_a + \sum W_i$ so here: $\frac{1}{2}mv_b^2 = 0 + W_{F_T} + W_{F_g} = 0 + 0 + 0.98m$ or $(0.5)mv_b^2 = 0.98m$. All the terms in this equation involve m, so we can divide the entire equation by m to cancel out that variable (fortunate since we don't know what the mass of the object is here). That leaves us with $0.5v_b^2 = 0.98$ or $v_b^2 = 1.96$ from which $|v_b| = 1.40\ m/s$.

The work-energy approach introduced in this chapter allows us to solve some problems that can't be solved using Newton's Laws and equations of motion. (Technically you could do a computer program to do it, breaking the problem down into tiny time steps and constantly recomputing the forces and accelerations, but here we have a direct method that gives the exact answer without any approximations.)

$\boxed{\textbf{Connected Objects (no friction)}}$:

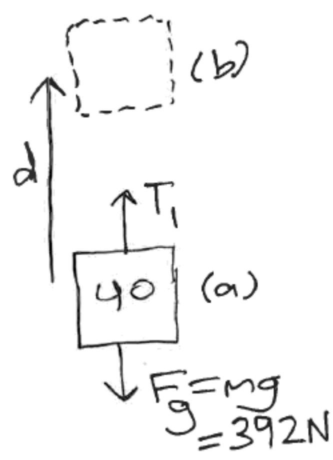


Let's redo the Atwood Machine problem we did in chapter 4, but using work and energy methods.
We have a 40 kg box on the left and a 60 kg box on the right, connected via a cable that runs over the pulley.
How fast will the boxes be moving when the heavier box has dropped by 50 cm?
In the original version of this problem, we used Newton's Laws and equations of motion to determine the acceleration of the boxes and the tension in the cable. This time, use the speed information we found via work-energy and infer the acceleration and tension from that.

Just like with Newton's Laws, when we have connected objects we need to apply the process separately to each object. There are situations where we can treat the entire system at once but there are also situations where we cannot, so for now **avoid the temptation to combine everything or 'unwrap' the objects so they appear to be moving in a line or anything like that.** Stick to the basics for now!

The objects are connected in such a way that whatever the heavier box is doing on the right, the lighter box is doing on the left, just in the opposite direction. For example at the point of interest, they'll both have the same speeds.

Work-Energy Applied to the box on the left : This box is starting at rest and has two forces acting on it: T_1 acting upward and F_g acting downward.

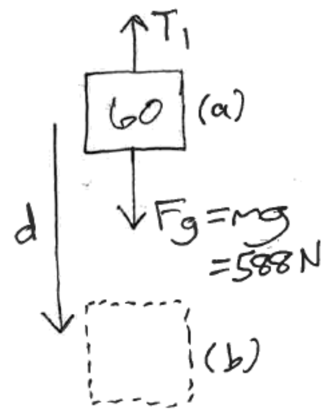


Work done by T_1 : this force is upward, and the displacement is upward so both vectors are in the same direction ($\phi = 0$) so: $W_{T_1} = \vec{T_1} \cdot \vec{d} = T_1 d \cos 0 = +0.5T_1$. (I've converted the 50 cm displacement into meters there.)

Work done by F_g : this force is downward, and the displacement is upward so these vectors are in exactly opposite direction ($\phi = 180^o$) so: $W_{F_g} = \vec{F_g} \cdot \vec{d} = mgd \cos 180^o = (40)(9.8)(0.5)(-1) = -196$.

$K_b = K_a + \sum W$ so here $\frac{1}{2}(40)(v_b)^2 = 0 + W_{T_1} + W_{F_g}$ or $\boxed{20v_b^2 = 0.5T_1 - 196}$.

Work-Energy Applied to the box on the right : This box is starting at rest and has two forces acting on it: T_1 acting upward and F_g acting downward.



Work done by T_1 : this force is upward, and the displacement is downward so the vectors are in the opposite direction ($\phi = 180^o$) so: $W_{T_1} = \vec{T_1} \cdot \vec{d} = T_1 d \cos 180^o = -0.5T_1$.

Work done by F_g : this force is downward, and the displacement is downward so these vectors are in the same direction ($\phi = 0$) so: $W_{F_g} = \vec{F_g} \cdot \vec{d} = mgd \cos 0 = (60)(9.8)(0.5)(1) = +294$.

$K_b = K_a + \sum W$ so here $\frac{1}{2}(60)(v_b)^2 = 0 + W_{T_1} + W_{F_g}$ or $\boxed{30v_b^2 = -0.5T_1 + 294}$.

We have two equations and two unknowns here:

$20v_b^2 = +0.5T_1 - 196$
$30v_b^2 = -0.5T_1 + 294$

We can solve for the speed here by simply adding the two equations together. The terms involving T_1 will cancel out, leaving us with:

$50v_b^2 = 294 - 196 = 98$ from which $|v_b| = 1.40\ m/s$.

$\boxed{\text{Use this information to determine the acceleration and tension.}}$

Tension : We already have equations relating v_b to the tension:

$20v_b^2 = +0.5T_1 - 196$ so $T_1 = (2)(20v_b^2 + 196)$ which yields $T_1 = 470.4\ N$
$30v_b^2 = -0.5T_1 + 294$ so $T_1 = (2)(30v_b^2 - 196)$ which yields $T_1 = 470.4\ N$ also.

Acceleration : We've done this problem before using Newton's Laws and equations of motion and found that $|a| = 1.96\ m/s^2$, so let's check the result we just found using work and energy.

Looking at the 40 kg block, using a coordinate system with +Y vertically upward, it's starting at rest and after moving 0.5 m they're travelling at a velocity of +1.40 m/s, so using the v^2 equation:

$v_y^2 = v_{oy}^2 + 2a_x \Delta y$ or $(1.40)^2 = (0)^2 + (2)(a)(0.5)$ from which $a = +1.96\ m/s^2$.

Looking at the 60 kg block, using a coordinate system with +Y vertically **downward**, it's starting at rest and after moving 0.5 m they're travelling at a velocity of +1.40 m/s, so using the v^2 equation:

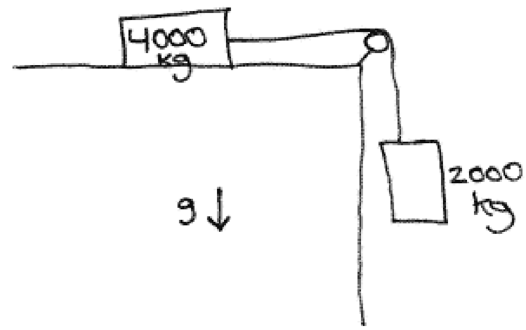
$v_y^2 = v_{oy}^2 + 2a_x \Delta y$ or $(1.40)^2 = (0)^2 + (2)(a)(0.5)$ from which $a = +1.96\ m/s^2$.

(Using a coordinate system with +Y vertically **upward**, we'd have $v_o = 0$, $v = -1.40\ m/s$, and $d = -0.5\ m$, leading to $a = -1.96\ m/s^2$ of course.)

Work-K Applied to our Jurassic Park example

Example: Connected Objects

Two objects are connected as shown in the figure. The object on the ground has a mass of 4000 kg, and the object hanging over the edge has a mass of 2000 kg. The friction coefficients for the object on the ground here are $\mu_s = 0.4$ and $\mu_k = 0.3$.



There isn't enough static friction to hold the objects in place, so assume they'll start moving and find:

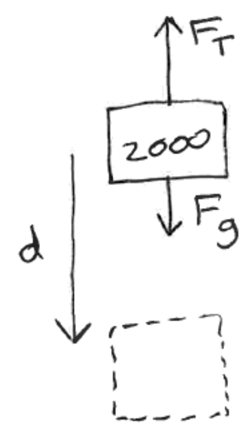
- How fast will the object be moving when the hanging object has dropped by 5 m?
- Use that information to determine the acceleration of the objects.

We did a similar problem back in chapter 5 using Newton's Laws and equations of motion. Here, we'll use work and kinetic energy to get the same results.

The objects are connected is such a way that whatever the hanging object is doing downward, the object on the ground is doing to the right. They'll both be moving together at the same speed.

The object on the ground has friction acting on it, so let's start with the simpler one: the object hanging over the side.

Work-Energy Applied to the hanging object : This object is starting at rest and has two forces acting on it, F_T acting upward and F_g acting downward, and it is displacing a distance of 5 m downward. $K_b = K_a + \sum W$ so $\frac{1}{2}(2000)(v_b)^2 = 0 + W_{F_g} + W_{F_T}$

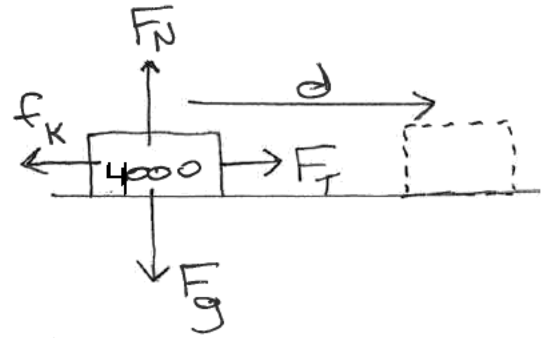


Work done by the force of gravity: $W_{F_g} = \vec{F_g} \cdot \vec{d} = F_g d \cos\phi = mgd\cos\phi$. The force is acting straight down, in the same direction as the displacement, so $\phi = 0^o$ here. $W_{F_g} = mgd\cos\phi = (2000)(9.8)(5)\cos(0^o) = +98000$.

Work done by the tension force: $W_{F_T} = \vec{F_T} \cdot \vec{d} = F_T d \cos\phi$. The tension force vector is upward and the displacement vector is downward, so $\phi = 180^o$ here. $W_{F_T} = F_T d\cos\phi = (F_T)(5)\cos(180^o) = -5F_T$.

Putting this together: $\frac{1}{2}(2000)(v_b)^2 = 0 + W_{F_g} + W_{F_T} = 98000 - 5F_T$ or finally $\boxed{1000v_b^2 = 98000 - 5F_T}$

Work-Energy Applied to the sliding object : This object is starting at rest and has **four** forces acting on it, F_g acting straight down, F_N acting straight up, F_T acting to the right, and f_k acting to the left. This object is displacing 5 m to the right. $K_b = K_a + \sum W$ so $\frac{1}{2}(4000)(v_b)^2 = 0 + W_{F_g} + W_{F_N} + W_{F_T} + W_{f_k}$



Work done by the force of gravity: This force vector is straight down and the displacement vector is to the right, so they're at right angles to one another. The angle between $\vec{F}$ and $\vec{d}$ is $\phi = 90^o$ so the $\cos\phi$ term in the definition of work means that this work will be ZERO.

Work done by the normal force: This force vector is straight up and the displacement vector is to the right, so they're at right angles to one another. The angle between $\vec{F}$ and $\vec{d}$ is $\phi = 90^o$ so the $\cos\phi$ term in the definition of work means that this work will be ZERO.

Work done by the tension force: This force is to the right and so is the displacement, so the angle between these vectors is $\phi = 0$. $W_{F_T} = \vec{F_T} \cdot \vec{d} = (F_T)(5)(\cos 0) = +5F_T$.

Work done by friction : This force is to the left as the object displaces to the right, so the angle between the vectors is now 180^o. $W_{F_k} = \vec{F_k} \cdot \vec{d} = (f_k)(5)(\cos(180^o)) = -5f_k$.

$f_k = \mu_k F_N$ though, so we'll need to determine how much normal force is present. If we look at the vertical components of all the forces acting on this object, we only have F_N upward and $F_g = mg = (4000)(9.8) = 39200\ N$ straight down so $\sum F_y = ma_y = 0$ on that object means that $F_N - 39200 = 0$ or $F_N = 39200\ N$. That makes $f_k = \mu_k F_N = (0.3)(39200) = 11760\ N$.

Finally, $W_{F_k} = -5f_k = -58800\ J$.

Determining the speed

Putting all this together: $K_b = K_a + \sum W$ so $\frac{1}{2}(4000)(v_b)^2 = 0 + W_{F_g} + W_{F_N} + W_{F_T} + W_{f_k}$

This yields: $\boxed{2000v_b^2 = 0 + 0 + 0 + 5F_T - 58800}$

We now have two equations with two unknowns:

$$1000v_b^2 = 98000 - 5F_T \text{ (from the falling object)}$$
$$2000v_b^2 = 5F_T - 58800 \text{ (from the sliding object)}$$

Adding these two equations together will eliminate the term involving the tension, yielding:

$3000v_b^2 = 98000 - 58800 = 39200$ from which $\boxed{|v_b| = 3.615\ m/s}$.

Determining the Acceleration

Just like with the previous example, we can use this information to infer the acceleration. Looking at the falling object, it's starting at rest and ends up moving 3.615 m/s after travelling 5 m, so $v^2 = v_o^2 + 2ad$ or $(3.615)^2 = (0)^2 + (2)(a)(5)$ from which $a = 1.3067\ m/s^2$ (matching what we got when we solved this via Newton's Laws).

We can also use the two equations the work-energy method yielded to determine the tension:

$1000v_b^2 = 98000 - 5F_T$ so $F_T = (98000 - 1000v_b^2)/5 = 16986\ N$
$2000v_b^2 = 5F_T - 58800$ so $F_T = (58800 + 2000v_b^2)/5 = 16987\ N$

(Close enough, since I rounded v off to 4 significant figures earlier, so there's no reason to expect these to agree to 5 significant figures...)