PH2213 Fox : Lecture 16
Chapter 7 : Work and Energy

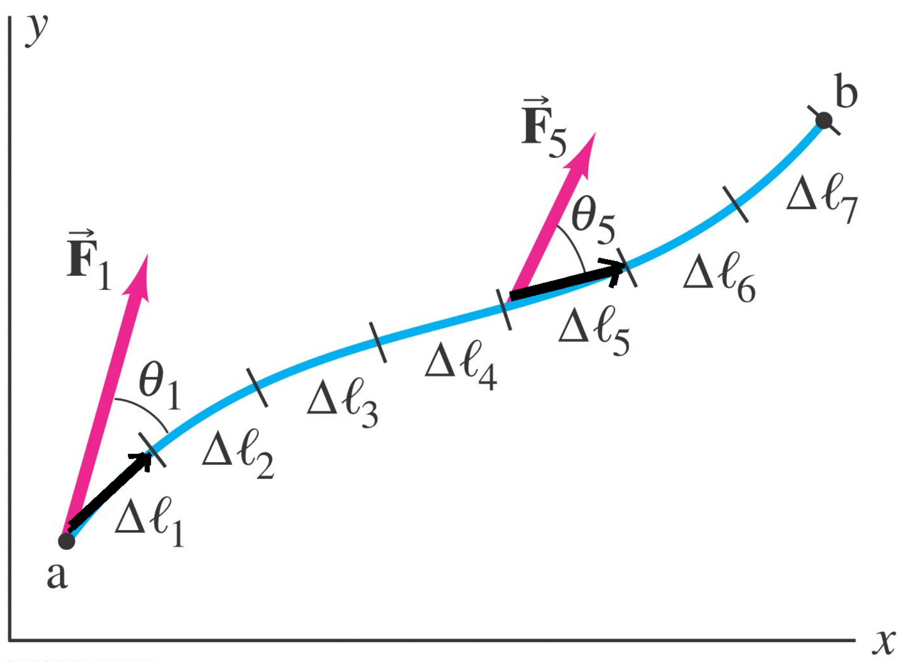
Work Done by a Varying (i.e. non-constant) Force

Suppose as the object moves from (a) to (b) we have some force that isn't constant. Perhaps the object is an airplane flying through air currents (wind) that is changing magnitude and direction (or a boat travelling through ocean currents that are changing, or a space satellite travelling near Jupiter where the dozens of moons are exerting varying amounts of gravitational force as the satellite travels through the system...)

We can't use Newton's Laws easily since the force isn't constant, meaning that the acceleration isn't constant and we can't use our equations of motion.

How can we use work and energy to deal with this situation?

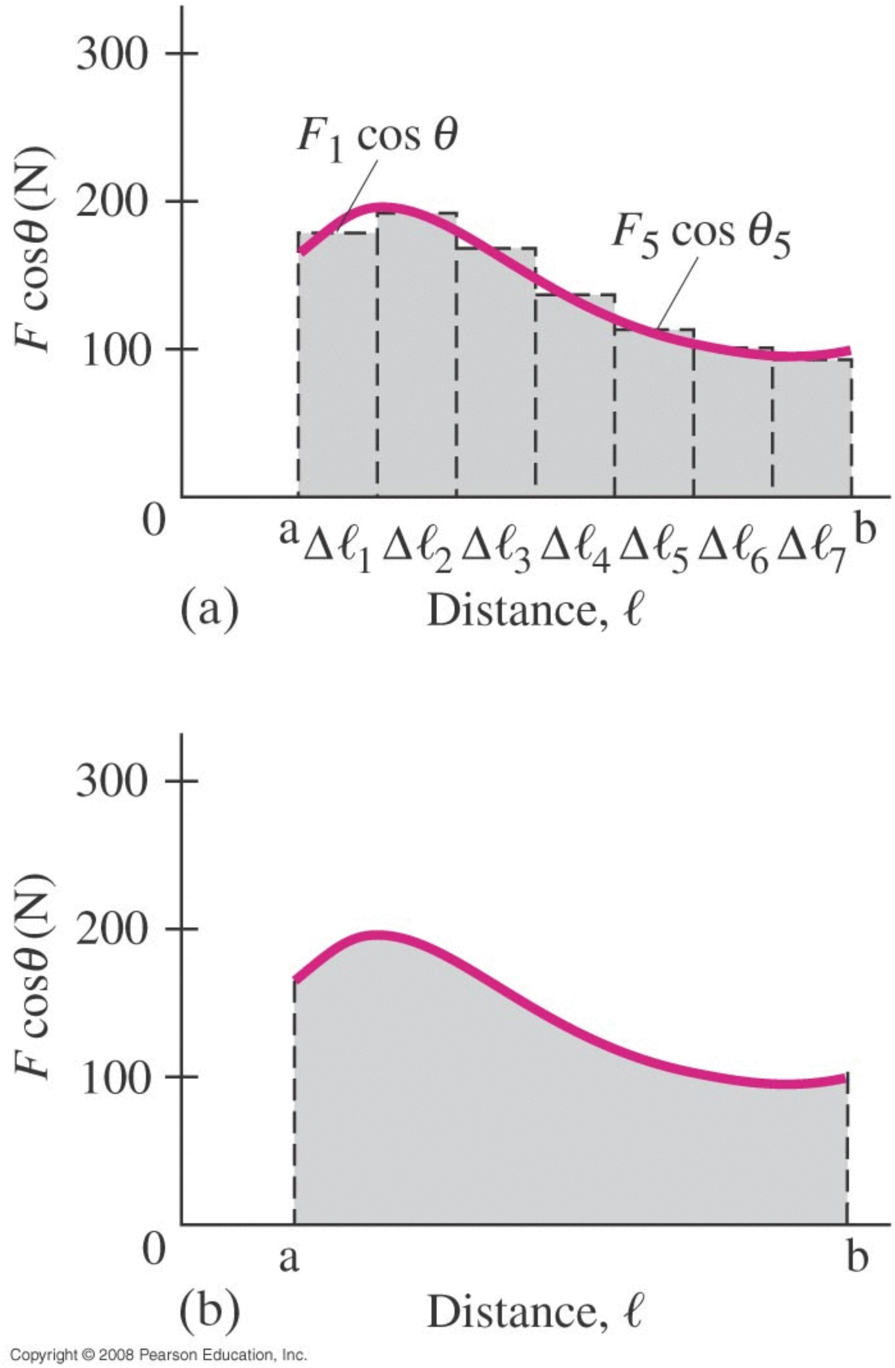
The pendulum example from the last lecture provides a hint: we can break the path into tiny steps and analyze the work done along the way.



Breaking the path into small steps, we see that on the first interval the force will do work on the object of: $\Delta W = \vec{F} \cdot \vec{d} = (F_1 \cos\theta_1)(\Delta l_1)$.
On the second segment: $\Delta W = \vec{F} \cdot \vec{d} = (F_2 \cos\theta_2)(\Delta l_2)$, and so on.

Adding up all the steps from (a) to (b), the total work done will be:
$W = \sum \Delta W = \sum F_i \Delta l_i \cos(\theta_i)$

If we plot the along-path component of the force as a function of position along the path, we see that adding all those $(F\cos\theta)(\Delta l)$ terms in effect is computing the area under this curve:



Reducing the Δl into infinitesimal calculus steps dl, we arrive at the final result:

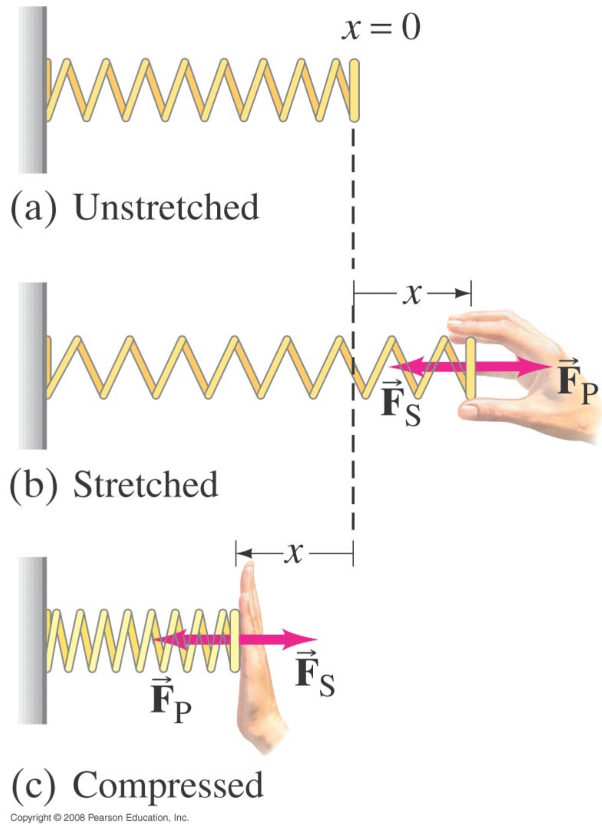
$$\boxed{\text{Varying forces: } W = \int_a^b \vec{F} \cdot d\vec{l}}$$

Now, actually doing that integral version of W for non-constant forces can be non-trivial, to say the least, but fortunately there is a useful subset of forces that encompass a large swath of situations.

Linear Restoring Force

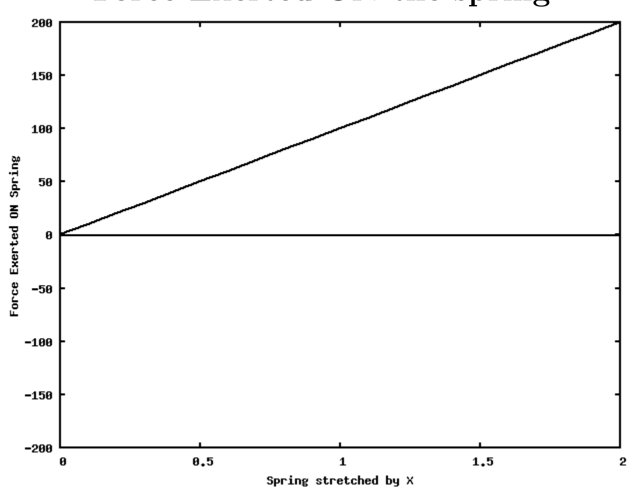
These are forces that are in the same (or directly opposite) direction as the displacement of the object, and vary only with position - and do so in a nice linear way. **Springs** are a good example of this type of force.
Hooke's Law : For reasonable displacements, the restoring force (here due to the spring) can be written as $F_s = -kx$ where k is called the **spring constant** (with units of Newtons/meter).
Other situations:

- spring
- deflection of the end of the meter stick over edge of table
- pendulum (not quite, but sometimes an acceptable approximation)
- pressure waves in air (sound)
- structure response to wind (buildings, bridges)

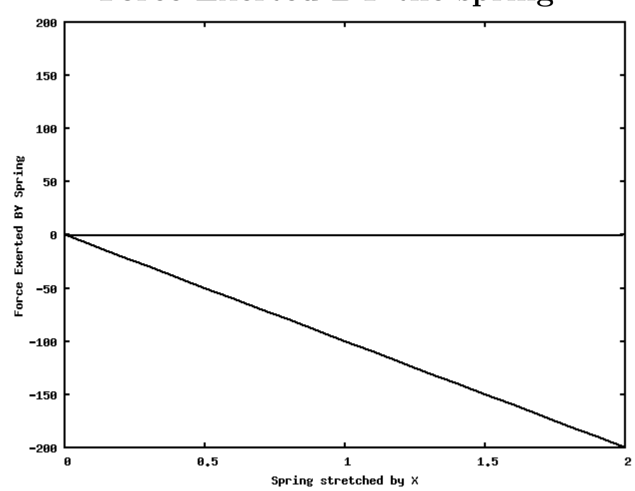


Graphing the pulling force and the spring force as a function of x, we have:

Force Exerted ON the spring



Force Exerted BY the spring



Work done when springs (linear restoring forces) are involved.

Since the integral is just the area under the $F(x)$ vs x curve, we can bypass actually doing it. It's basically just the area of a triangle of base x and height $\pm kx$ and for a triangle $Area = \frac{1}{2} \times Base \times Height$ so these integrals yield $W = \pm\frac{1}{2}kx^2$.

If the spring force is in the <u>same direction</u> as the object's displacement, the spring will be adding energy to the object (positive work). If the spring force is in the <u>opposite direction</u> as the object's displacement, the spring will be removing energy from the object (negative work).

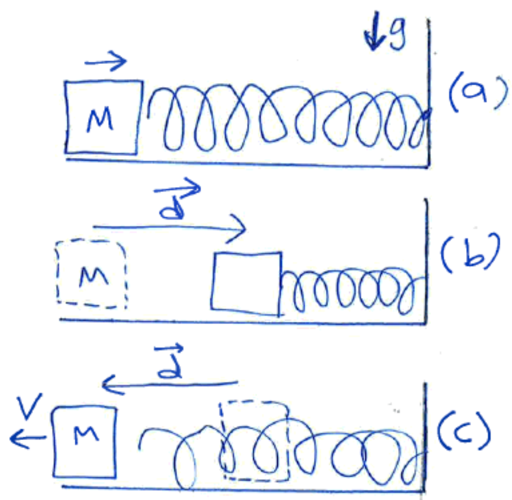
Let's digest this using an example:

EXAMPLE 1 : A 10 kg box slides along a horizontal frictionless surface to the right at $v_a = 2\ /ms$. It encounters a spring with $k = 200\ N/m$ that has one end attached to a wall.

(a) How far will the spring compress before bringing the box to a stop?

(b) What are the forces on the box at that point?

(c) How fast will the box be moving when it bounces off the spring?



(a) At position a the object has just touched the unstretched spring. The more it slides to the right, the more the spring gets compressed, and the stronger and stronger the spring force F_S to the left becomes. (The spring force is to the left as the object displaces to the right, so they're in opposite directions: the spring is doing negative work here.) At position b the object has (momentarily) come to a stop.

Applying Work-Energy between those two points:

$K_b = K_a + \sum W$ or here: $0 = \frac{1}{2}(10\ kg)(2\ m/s)^2 + W_{F_g} + W_{F_N} + W_{F_S}$

$\vec{F}_g$ is vertically downward and $\vec{d}$ is to the right, so there's a $\phi = 90^o$ angle between them. $W_{F_g} = \vec{F}_g \cdot \vec{d} = F_g d \cos(90^o) = 0$. Gravity didn't do any work on the object.

Same argument for the work done by the normal force: it's also perpendicular to the displacement.

The spring force on the object is to the left, as it displaces to the right, so $W_{F_S} = -\frac{1}{2}kd^2$.

Putting all this together, $K_b = K_a + \sum W$ becomes $0 = 20 + 0 + 0 - \frac{1}{2}(200)d^2$ or $0 = 20 - 100d^2$ from which $d = \sqrt{0.2} = \pm 0.4472\ m$. (Again with the sign ambiguities, but we see from the situation that the box has come to a stop after pushing the spring in by $0.4472\ m$.)

(b) What forces are acting on the box at this point? We have gravity downward, the normal force upward, and the compressed spring will be exerting a force of $F_S = -kd = -(200\ N/m)(0.4472\ m) = -89.44\ N$. (I.e., 89.44 N to the left.) That's the only force with a horizontal component, so the object will immediately start accelerating off to the left (in effect 'bouncing off' the spring).

(c) How fast will the box be moving when it bounces off the spring? (I.e. when it's moved far enough to the left that it's no longer in contact with the spring.)

Applying Work-K again from point ⓑ to ⓒ we have:

$K_c = K_b + \sum W$ or $\frac{1}{2}mv_c^2 = 0 + W_{F_g} + W_{F_N} + W_{F_S}$

Again, the vector force of gravity and the vector normal force are both perpendicular to the vector displacement, so neither of those two forces are doing any work on the box.

In the case of the spring, the compressed spring will be exerting a force to the left as the box displaces to the left: those vectors are in the same direction, so the spring is doing positive work on the box: $W_S = +\frac{1}{2}kd^2$ now.

Work-K becomes: $\frac{1}{2}(10\ kg)(v_c)^2 = 0 + 0 + 0 + \frac{1}{2}(200\ N/m)(0.4472\ m)^2$ or $5v_c^2 = 20$ from which $|v_c| = 2\ m/s$.

The algebra doesn't tell us the direction, but since the spring force is entirely to the left here the box (starting at rest) will be accelerating to the left, building up a velocity to the left.

EXAMPLE 2 : How far would the box slide into the spring if we also had friction present? Rework part (a) above assuming we have $\mu_k = 0.1$ between the box and the floor (but just under where the spring is located).

Applying Work-Energy between those two points, we just add another term: the work that friction is doing on the box.

$K_b = K_a + \sum W$ or here: $0 = \frac{1}{2}(10\ kg)(2\ m/s)^2 + W_{F_g} + W_{F_N} + W_{F_S} + W_{friction}$

All the same arguments from above apply, so gravity and the normal force aren't doing any work here, and the spring is doing $W_{F_S} = -\frac{1}{2}kd^2$.

The work that friction does will be $W_{friction} = \vec{f_k} \cdot \vec{d}$. The object is displacing to the right, but kinetic friction will be a vector pointing directly left. Those vectors are in opposite directions, so $W_{friction} = \vec{f_k} \cdot \vec{d} = f_k d \cos 180 = -f_k d$.

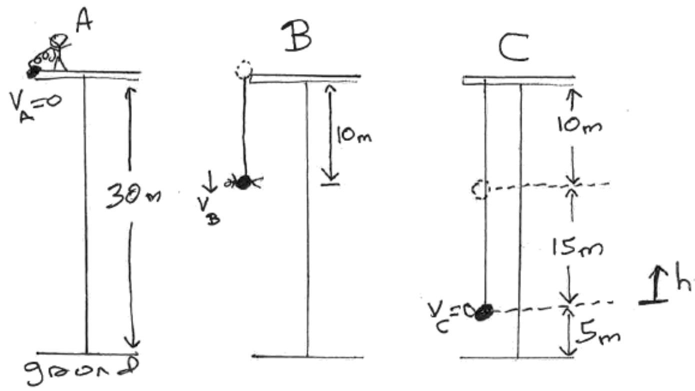
$f_k = \mu_k F_N$, so we'll need to find the normal force here. In the vertical direction, we have F_g straight down and F_N straight up and no other forces with vertical components so $F_N = F_g = mg = (10\ kg)(9.8\ m/s^2) = 98\ N$. That means $f_k = \mu_k F_N = (0.1)(98\ N) = 9.8\ N$ And finally, $W_{friction} = -f_k d = -9.8d$.

Putting all the parts together then: $K_b = K_a + \sum W$ and here we have: $0 = \frac{1}{2}(10\ kg)(2\ m/s)^2 + W_{F_g} + W_{F_N} + W_{F_S} + W_{friction}$ so $0 = 20 + 0 + 0 - 100d^2 - 9.8d$.

Rearranging a bit: $100d^2 + 9.8d - 20 = 0$, which has two solutions, $d = +0.40\ m$ and $d = -0.499\ m$. We know from reality that the box will slide to the right, so the positive solution is the one we're looking for.

EXAMPLE 3 : Bungee Cord Jumper
A 100 kg person steps off the stop of a ledge. Their bungee cord has a length of 10 m and they want to stop before hitting the ground below, so at this location they need to come to a stop after the cord has stretched by 15 m.

(a) What spring constant does the bungee cord need to have?



Let's use Work-K from ⓐ to ⓒ here. At both those points, the object isn't moving (although it won't stay stationary at point (c) very long, as we'll see), so $K_a = K_c = 0$. There are two forces doing work here:

- Gravity is acting straight <u>downward</u> and does so for the entire 25 m <u>downward</u> displacement over this interval. $W_{F_g} = \vec{F}_g \cdot \vec{d} = F_g d \cos 0 = mgd = (100\ kg)(9.8\ m/s^2)(25\ m) = +24500\ J$. Gravity alone has added that much to the person's kinetic energy. (Now the object isn't actually moving when it gets to that point, so some other force must be doing negative work and removing all that energy and we do have such a force present here: the bungee cord 'spring'.)

- The bungee cord acts like a spring, but only if it's being stretched. The cord is 10 m long, so for the first 10 meters of the fall, the cord hasn't been stretched yet. That doesn't start until the person has dropped 10 meters: any further displacement and the cord will now start to stretch. In this scenario, when the person reaches point (c), the cord has been stretched by 15 m. The cord is now acting like a stretched spring, so it's exerting it's force <u>upward</u> as the person displaces <u>downward</u> so here the 'spring' is doing negative work: $W_{F_S} = -\frac{1}{2}kd^2 = -(0.5)(k)(15)^2 = -112.5k$.

Putting all this together:

$K_c = K_a + \sum W$ or $0 = 0 + W_{F_g} + W_{F_S}$ or $0 = 0 + 24500 - 112.5k$ from which $k = 217.8\ N/m$.

What forces are acting on the person at point (c)?

We have a force of gravity downward of $F_g = mg = (100\ kg)(9.8\ m/s^2) = 980\ N$.

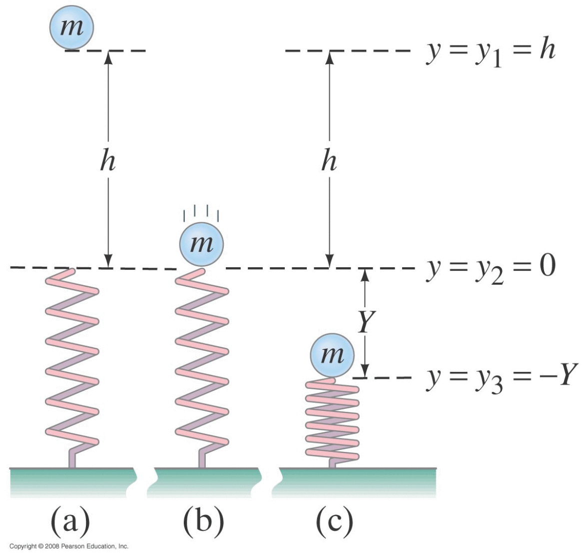
The 'spring', having been stretched by 15 m from it's rest length, is exerting a force upward of $F_S = kd = (217.8\ N/m)(15\ m) = 3267.7\ N$.

That means the person will be feeling a net force upward of $3267.7 - 980 = 2286.7\ N$ which implies an instantaneous acceleration from $F = ma$ of $a = F/m = (2286.7)/(100) = 22.87\ m/s^2$ (about 2.3 g's) upward. (They won't maintain that acceleration since the spring force keeps changing depending on the length of the spring.)

EXAMPLE 4 : Object Dropped on Spring

A 2 kg object is dropped (from rest) onto a spring $h = 50$ cm below. How much will the spring compress if the spring has a force constant of $k = 2000$ N/m?

This starts off looking similar to the bungee problem, but here we know the spring constant and are looking for how far the spring will compress before bringing the object to a (momentary) stop.



We'll treat Y as the **positive distance the object will travel from when it first touches the spring in figure (b), to the point where it's come to a stop as position (c)**. For example, $Y = +10$ cm would mean that the object dropped 50 cm then another 10 cm for 60 cm in total, with the spring compressing by 10 cm.

$K_c = K_a + \sum W$ and here the object isn't moving at either (a) or (c) so both K's are zero. We have gravity and the spring doing work here so:

Work done by gravity : $\vec{F_g}$ is constant and straight down, and the object is displacing downward also, so gravity is doing positive work. The total displacement from (a) to (c) will be $0.5 + Y$ (meters). The work done by gravity then will be $W_{F_g} = F_g d \cos\phi = (mg)(0.5 + Y)\cos 0$ or $W_{F_g} = (2)(9.8)(0.5 + Y)$ which we can write as: $\boxed{W_{F_g} = 9.8 + 19.6Y}$.

Work done by the spring : $\vec{F_S}$ will be upward but will only exist between points (b) and (c). The object is displacing downward there, so the spring will be doing negative work: $W_{F_S} = -\frac{1}{2}kd^2$ but the d here will just be what we're calling our Y variable. $W_{F_S} = -(0.5)(2000)(Y)^2$ or $\boxed{W_{F_S} = -1000Y^2}$.

Putting this all together now: $K_c = K_a + \sum W$ becomes: $0 = 0 + 9.8 + 19.6Y - 1000Y^2$

Rearranging that into the usual quadratic form: $1000Y^2 - 19.6Y - 9.8 = 0$ which has two solutions: $Y = +0.1093$ m and $Y = -0.0897$ m. (Remember we're treating Y as the positive distance below $y = 0$ so $Y = +0.1093$ m puts the object at a (lowercase) y coordinate of $y = -0.1093$ m.)

Which one is right? From how we defined Y, the positive solution has to be the right one since the object does stop somewhere below point (b) in the figure. It travels 0.5 m from (a) to (b) and then some additional amount Y (which we treated as positive being that far below that point).

(Note: The other solution represents the unphysical scenario of the object becoming attached to the spring and then bouncing back upward until it (momentarily) comes to a stop 8.97 cm above point (b).)

Addendum : Bungee Cord Jumper

Suppose a different person, with a different **mass** wants to jump?

We still want their motion to stop 5 meters before the ground.

Going back to our original Work-K equation (and leaving k as an unknown to find):

$K_c = K_a + \sum W$ or $0 = 0 + W_{F_g} + W_{F_S}$

$W_{F_g} = \vec{F}_g \cdot \vec{d} = F_g d \cos 0 = (mg)(d) = (M)(9.8)(25)$

$W_{F_s} = -\frac{1}{2}kd^2 = -\frac{1}{2}(k)(15)^2$ so collecting terms:

$K_c = K_a + \sum W$ becomes $0 = 0 + (M)(9.8)(25) - \frac{1}{2}(k)(15)^2$ and rearranging and evaluating the numeric parts, we end up with: $245M = 112.5k$ or $k = 2.18M$ (with M in kilograms, and k in N/m). A lighter person would need to use a bungee cord with a smaller spring constant.

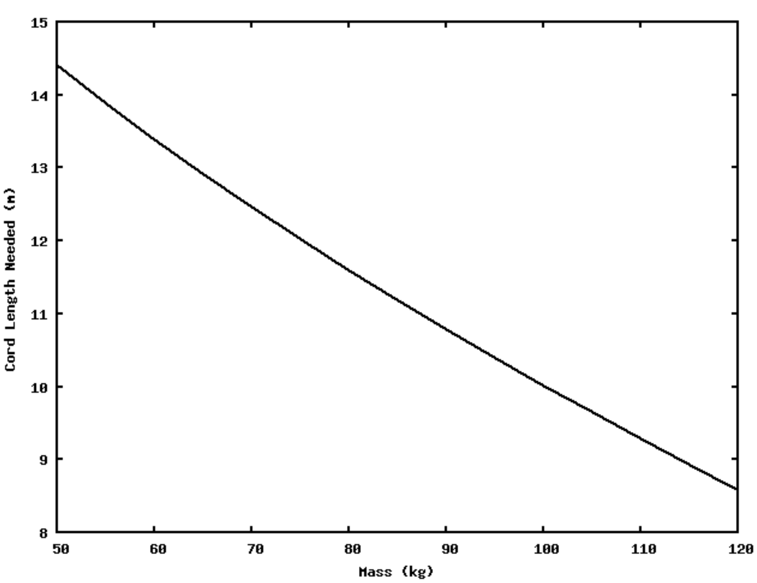
That's not terribly practical since we'd potentially need to haul around a large number of different cords. What is another approach?

Here gravity is adding energy of $(M)(9.8)(25)$ joules, so with a lighter person, gravity is doing less work. So we need the cord to do less work to slow them down to rest at the desired point (5 meters above the ground). The work done by the cord is $-\frac{1}{2}kd^2$ where k is the spring constant for the cord, and d is how much the cord stretches (was 15 meters above).

We do have two variables we can play with here though: the spring constant, and how much the cord has to stretch. Suppose we keep k fixed (always use the same cord) but adjust the **length** of the cord? Let's say the cord has a length of L. The person free-falls for the first L meters, then the cord needs to stretch the rest of the way, which will be a distance of $25 - L$.

Our Work-K equation with L as the variable becomes:

$K_c = K_a + \sum W$ or $0 = 0 + W_{F_g} + W_{F_S}$ so $0 = 0 + (M)(9.8)(25) - \frac{1}{2}(218)(25 - L)^2$.
Evaluating and rearranging:
$109(25 - L)^2 = 245M$ or $(25 - L)^2 = 2.248M$ or $25 - L = \sqrt{2.248M}$ or finally $\boxed{L = 25 - \sqrt{2.248M}}$.
As a check, if we plug in $M = 100$ kg (the original person's mass in the example) we get $L = 10.01$ (close enough to the original 10 m length).
A child with $M = 50$ kg would need a cord of length $L = 14.4$ m (they free-fall for longer before the cord comes into play). A heavier adult with $M = 120$ kg would need a cord of length $L = 8.58$ m (so they free fall for a shorter distance, then the cord stretches for a longer amount).



Cord Length Needed vs Mass

That gives us another option then. We keep one longer cord, but tie it off to (in effect) adjust it's length depending on the mass of the person.

Addendum : Linear Restoring Forces and DFQ's

Linear restoring forces appear frequently. Let's see what happens if that's the only force present on an object. Then Newton's laws $\sum F = ma$ become: $-kx = ma$.

The acceleration is the derivative of velocity, which in turn is the derivative of position, so this can be written as: $-kx = m\frac{d^2x}{dt^2}$ or:

$$\boxed{\frac{d^2x}{dt^2} = -\frac{k}{m}x}$$

This is a second order, linear differential equation. Basically, there's a differential equation hiding in any scenario where forces aren't constant.

This is one of the early second-order equations you deal with in that class but you actually already know the solution. What function $x(t)$ has the property that if you differentiate it, then differentiate it again, you basically get back to the original function (with a flip in the sign and some constants possibly coming out)? There are only two functions that do this: **sines and cosines**.

That means that whenever we have a **linear restoring force**, we have **periodic motion** as a solution.

- If you hang an object from a spring and let it go, it starts bouncing up and down on the spring.
- If you deflect the end of a ruler held in place on the other end, the free end will vibrate.
- A string on a musical instrument displaced to the side will vibrate back and force (rapidly enough to produce sounds).
- Bridges and buildings can flex or oscillate (sometimes to the point of collapse).

Any physical situation where linear restoring forces are involved has oscillations or vibrations as a solution (and vice versa: if we see that behaviour, there's most likely a linear restoring force hiding in the system somewhere).