PH2213 Fox : Lecture 20
Chapter 9 : Linear Momentum

Real Version of Newton's Laws

$$\boxed{\sum \vec{F} = \frac{d\vec{p}}{dt} \text{ where } \vec{p} = m\vec{v}}$$

$\vec{p} = m\vec{v}$ is called the **momentum** of an object, and has units of $kg\ m/s^2$ (there's no special symbol or name for that).

If the mass is constant, $d(m\vec{v})/dt = md\vec{v}/dt = m\vec{a}$ so this 'real' version reduces to what we've been using so far, as long as the mass is constant.

On the other hand, this 'real' version of Newton's Laws also handles cases where the mass of the object isn't constant. Examples are rockets, where most of the mass of the rocket is actually it's fuel, which is being consumed while the rocket engine is running. It also handles things like an open crate traveling along a conveyor as items are being added to it (thus increasing it's mass).

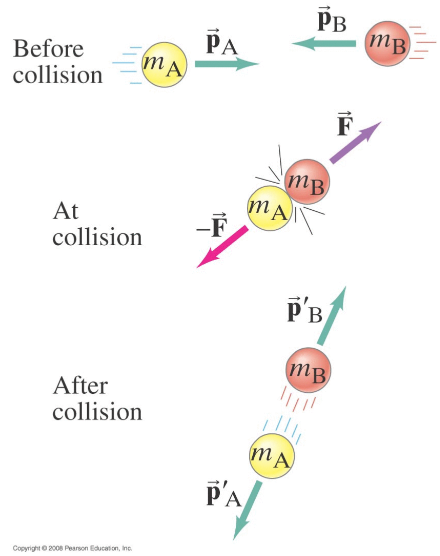
Implications

- $\vec{F}_{avg} = \frac{\Delta\vec{p}}{\Delta t}$
  If we see an object change its momentum over some interval Δt, we can determine the average force that must have been acting on it during that interval.
- Rearranging: $\Delta\vec{p} = \int \vec{F}(t)dt$ which we can write as
  $v(t) = v_o + \frac{1}{m}\int \vec{F}(t)dt$
  giving us a way to handle time-varying forces
- Conservation of momentum during interactions (see next page)

We will find, as we work through the examples, that **mechanical energy**, our particular subset of energies in the form of $(K + U_g + U_s)$, is usually NOT conserved during collisions.

We will also find that the forces involved during the brief time intervals involved in most collisions are tiny compared to the force of the collision itself, so we can usually ignore any other forces acting on the object(s) during that tiny time interval.

Conservation of Momentum



Consider two objects that are interacting via some force (electrical, gravitational, elastic, contact F_N, etc...)

The force that A exerts on B causes B's acceleration:

$$\vec{F}_{A\ on\ B} = m_B \vec{a}_B = m_B \frac{d\vec{v}_B}{dt}$$

The force that B exerts on A causes A's acceleration:

$$\vec{F}_{B\ on\ A} = m_A \vec{a}_A = m_A \frac{d\vec{v}_A}{dt}$$

BUT Newton's third law: $\vec{F}_{B\ on\ A} = -\vec{F}_{A\ on\ B}$
Implies: $m_B \frac{d\vec{v}_B}{dt} = -m_A \frac{d\vec{v}_A}{dt}$

Rearranging: $m_A \frac{d\vec{v}_A}{dt} + m_B \frac{d\vec{v}_B}{dt} = 0$

Or in terms of momentum $\vec{p} = m\vec{v}$:

$d\vec{p}_A/dt + d\vec{p}_B/dt = 0$ or $\frac{d}{dt}(\vec{p}_A + \vec{p}_B) = 0$ or $\boxed{\vec{p}_A + \vec{p}_B = constant}$

We only considered two interacting objects here, but this idea holds in general for any number of objects that are interacting with one another.

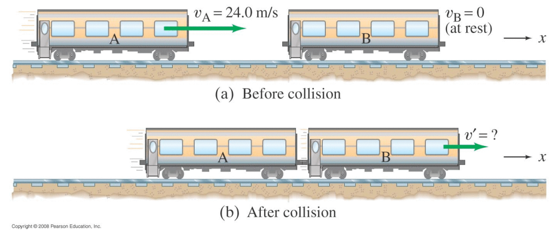
When objects interact, the total momentum of the system remains constant.

The total momentum $\sum \vec{p}$ before, during, and after the collision remains the same, unless some other force comes into the picture to change it.

Note: this book uses a capital letter P to represent the total momentum $\vec{P} = \sum \vec{p}_i$ of a collection of interacting objects. I tend to avoid it since too many people confuse it with power, which also uses a capital P.

Example: 1-D Collision

A 10,000 kg railroad car moving down the track at 24 m/s collides with a 15,000 kg car that is initially at rest.



- If the cars lock together, what is their common speed after the collision?
- Look at the mechanical energy before and after the collision. Is it conserved?
- A high speed video of this collision shows that the interaction took 20 ms. What force did A exert on B? B on A?

(a) Common speed after collision :

Momentum is conserved as a vector, but here we're starting off with momentum only in the X direction (horizontal direction) so let's look at just that direction:

Before the collision, we have a 10,000 kg object moving at 24 m/s in the +X direction, and another 10,000 kg object at rest, so $(\sum p)_{before} = \sum(m_i v_i) = (10000)(24) + (15000)(0) = 240000\ kg\ m/s$.

(Luckily, although momentum and collisions is an important situation, there aren't any new unit symbols that come into play here. The units of momentum are just mass times velocity or $kg\ m/s$.)

Right after the collision, the two objects are connected and are moving at the same velocity, so $(\sum p)_{after} = \sum(m_i v_i) = (10000)(v) + (15000)(v) = 25000v$

Conservation of momentum then: $(\sum p)_{after} = (\sum p)_{before}$ so: $25000v = 240000$ or $v = 9.6\ m/s$. The two railcars move off together at 9.6 m/s to the right after the collision.

(b) Mechanical Energy before and after collision :

Nothing is changing height here, and we don't have any springs, so we can ignore the U_g and U_s terms in E and just focus on the kinetic energies before and after.

Before: 10,000 kg object moving at 24 m/s : $K = \frac{1}{2}mv^2 = 2,880,000\ J$.

After: 10,000 kg object moving at 9.6 m/s and the 15,000 kg object also moving at 9.6 m/s : $K = \frac{1}{2}(10000)(9.6)^2 + \frac{1}{2}(15000)(9.6)^2 = 1,152,000\ J$.

Notice that **more than half** of the mechanical energy of these two objects 'vanished' as a result of the collision. Energy overall will be conserved, but apparently over half the energy must have gone into 'other work' (friction, heat, deformation or destruction of objects, and so on). 24 m/s is about 54 $miles/hr$ and there's likely be quite a bit of destruction involved with a 10,000 kg railcar traveling that fast plowing into another railcar at rest...

continued...

(c) Forces involved in the collision : Let's focus on the railcar that was initially at rest. Initially it has a momentum of $p = mv = 0$, but right after the collision it has a momentum of $p = mv = (15000)(9.6) = 144,000\ kg\ m/s$. The force on that railcar can be estimated via $F_{avg} = \frac{\Delta p}{\Delta t} = \frac{p_{after} - p_{before}}{\Delta t} = \frac{144000-0}{0.02} = +7,200,000\ N$. (Note we need everything in standard units: kg, m, ¡b¿seconds¡/b¿, etc so the 20 ms needed to be converted into 0.02 s.)

That railcar 'felt' a force of 7.2 million Newtons to the right, causing it to accelerate from rest to 9.6 m/s in 0.02 s (for an acceleration of $a = +480\ m/s^2$ or about 49 g's).

Let's look at the railcar on the left now. Initially it had a momentum of $p_{before} = mv = (10000)(24) = 240000\ kg\ m/s$. After the collision, it has a momentum of $p_{after} = mv = (10000)(9.6) = 96000\ kg\ m/s$.

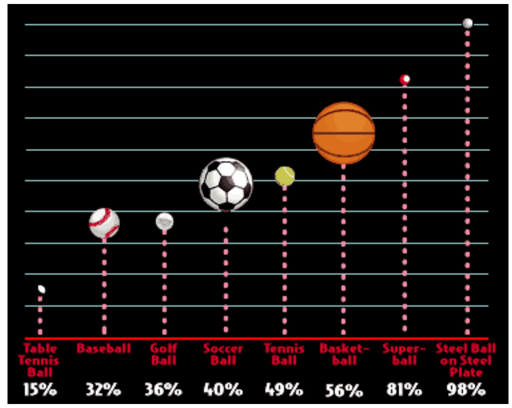
$F_{avg} = \frac{\Delta p}{\Delta t} = \frac{p_{after} - p_{before}}{\Delta t} = \frac{96000-240000}{0.02} = -7,200,000\ N$.

The left railcar 'felt' a force of 7.2 million Newtons to the left, causing it to slow down from 24 m/s to 9.6 m/s in 0.02 s (for an acceleration of $a = -720\ m/s^2$ or about -73 g's).

- **The cars are exerting the same force on each other** - this is an interaction force - so the magnitude of the force has to be the same each way, it's just in the opposite direction on each object. (Newton's third law...)
- **The accelerations each object felt are not the same though** since $a = F/m$ depends on the mass of each object.

Example: Bouncing Ball

Suppose a 0.1 kg ball is dropped from a height of 1 meter. It lands on a hard, horizontal table and bounces back up to a height of 80 cm. How much force did the table exert on the ball if the two were in contact for $\Delta t = 0.001\ s$? How much energy was lost in the collision? (What fraction of the initial energy was lost?)



The ball starts with some initial $U_g = mgh$ of gravitational potential energy, which is converted into $K = \frac{1}{2}mv^2$ of kinetic energy as it falls down towards the table. Once bouncing back up, that kinetic energy gets converted back into potential energy, but we see here that the ball doesn't make it all the way back up: it must have lost some (mechnical) energy during it's collision with the table.

The figure on the right shows how balls of various types bounce. A ping-pong ball will apparently lose about 85% of it's initial energy during a bounce. A basketball loses about half it's energy when it interacts with the floor. The one that comes closest to retaining all it's energy is the steel ball bearing bouncing off a heavy steel plate.

Initial Drop Phase : Ball starts at rest at a height of 1 meter above the table. How fast will it be moving when it hits the table?

Lots of options here. Sticking with energy: $(K + U_g + U_s)_b = (K + U_g + U_s)_a + W_{other}$, where (a) labels the initial location and (b) denotes the point when the ball it just about to hit the table.

Here then: $\frac{1}{2}Mv^2 + 0 + 0 = 0 + Mgh + 0 + 0$ or after some algebra: $v = \sqrt{2gh}$.

So: if we release the ball 1 meter above the table, it will be traveling downward at $v = \sqrt{(2)(9.8)(1)} = 4.4272\ m/s$ the instant just before it collides with the table.

Upward Phase : Once the ball bounces back up, it's moving upward at some velocity so has some initial kinetic energy, which is converted into U_g until the ball finally comes to a stop 80 cm above the table. Using the same CoE process, that means the ball must have been moving upward at $v = \sqrt{(2)(9.8)(9.8)} = 3.9598\ m/s$ at the instant just after its collision with the table.

Momentum is a vector, but the ball is traveling vertically here so let's use a coordinate system with $+Y$ upward.
The Y component of the collision force will be:

$F_{avg} = \frac{\Delta p}{\Delta t} = \frac{p_{after} - p_{before}}{\Delta t}$

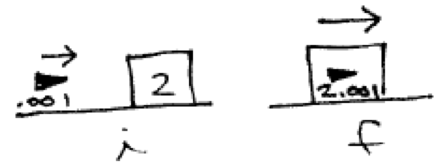
Here then: $F_{avg} = \frac{(0.1\ kg)(3.9598\ m/s) - (0.1\ kg)(-4.4272\ m/s)}{0.001\ s} = 838.7\ N$.

(Note we're dealing with vectors here, so had to use the correct **signs** for each velocity.)

Compare that to the force of gravity acting on the ball: $F_g = mg = (0.1\ kg)(9.8\ m/s^2) = 0.98\ N$. The 'collision force' (basically the contact or normal force between the ball and the table) is almost a **thousand** times stronger. (We'll see this routinely in our collision examples: the force involved in the collision usually overwhelms all other forces that may be acting on the objects.)

Example: 1-D Collision

A 5 gram bullet traveling (horizontally) at 300 m/s strikes a 2 kg block of wood sitting on a frictionless surface, initially at rest. The bullet embeds itself in the wood.



- How fast will the (combined) object be moving immediately after this collision?
- How much energy was lost in the collision?
- If the collision took $\Delta t = 0.1\ ms = 1 \times 10^{-4}\ s$, what force was involved?
- Compare that force to other forces present in the problem (gravity, F_N, and static friction, assuming $\mu_s = 1$ here).

This is a 1-D problem again, so we can focus on just the X (horizontal) direction.

(a) Speed of combined object after collision :

Before the collision: block of wood at rest, and a 0.005 kg object with $v = 300\ m/s$.

After the collision: both objects moving off at some velocity v, so:

Conservation of momentum: $(\sum p)_{before} = (\sum p)_{after}$ so:

$(2)(0) + (0.005)(300) = 2v + 0.005v = 2.005v$ or $2.005v = 1.5$ so $v = 0.7481\ m/s$.

(b) Energy lost :

Nothing is changing height, and no springs, so again we'll focus on just the kinetic energy before and after:

$K_{before} = \sum K = \frac{1}{2}(2\ kg)(0)^2 + \frac{1}{2}(0.005\ kg)(300\ m/s)^2 = 225\ J$.

$K_{after} = \sum K = \frac{1}{2}(2.005\ kg)(0.7481\ m/s)^2 = 0.56\ J$.

(Notice I treated the block plus the bullet as a single object of mass 2.005 kg since they've merged and are now moving together as a single object after the collision.)

Well - we didn't lose just half the energy here, we lost 224.44 J out of 225 J or 99.75%!

(c) Force involved : this is an interaction force, so the force the bullet exerts on the block is the same as the force the block exerted on the bullet (just in the opposite direction), so we don't technically need to calculate both, but we'll do so anyway as a check:

Force of bullet on block: the block has a momentum before the collision of zero, and a momentum after the collision of $p_{after} = mv = (2\ kg)(0.7481\ m/s) = 1.4962\ kg\ m/s$.

$F_{avg} = \frac{\Delta p}{\Delta t} = \frac{p_{after} - p_{before}}{\Delta t} = \frac{1.4962 - 0}{0.0001} = +14962\ N$.

Force of block on bullet: before the collision the bullet has $p_{before} = (0.005\ kg)(300\ m/s) = 1.5\ kg\ m/s$. After the collision, it has a momentum of $p_{after} = (0.005\ kg)(0.7481\ m/s) = 0.0037405\ kg\ m/s$. $F_{avg} = \frac{\Delta p}{\Delta t} = \frac{p_{after} - p_{before}}{\Delta t} = \frac{0.0037405 - 1.5}{0.0001} = -14962\ N$.

That's about $15,000\ N$ of force, which is nearly 3400 *pounds*, comparable to the weight of an SUV.

It's worth pausing for a moment to highlight a couple of things we've learned when collisions are involved:

When Collisions (or explosions) are Involved:

- Mechanical energy is (usually) NOT conserved. There is 'other' work that is converting some (most?) of the initial energy into heat and other forms.
- The forces involved in the collision itself normally OVERWHELM any other forces present in the system.

This leads to a good rule of thumb to use if a collision (or explosion) occurs somewhere in a scenario:

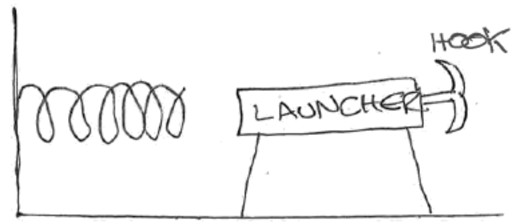
When collisions (or explosions) are involved: focus on that interaction via conservation of momentum only.

NOTE: it's a common mistake when collisions are involved to try and use conservation of energy to solve the whole problem, but as we've seen, some energy (and sometimes nearly all the energy) ends up vanishing into 'other work' as a result of the collision.

We can use all our existing machinery (Newton's laws, equations of motion, work, energy) right up to the point of the collision, and again right after the collision, but **during** the collision itself, we can **only** rely on conservation of momentum.

During the brief Δt of the collision/explosion, the collision force is typically orders of magnitude larger than every other force present in the situation, so we can ignore everything else and focus solely on conservation of momentum to get us through that 'event' in the scenario.

Explosion : In an old Mythbusters episode, they test-fired a grappling-hook launcher. The 'launcher' (essentially a little cannon) was placed on a table and an explosive charge inside the barrel was set off, sending the grappling hook horizontally (to the right) at a speed of 40 m/s. As a result, the launcher itself recoiled backwards (to the left), where a shock absorber (a spring) kept it from hitting the wall. (Ignore friction here.)



The grappling hook (alone) has a mass of 10 kg, the (unloaded) launcher has a mass of 80 kg, and the shock-absorbing spring had a spring constant of $k = 50,000\ N/m$. Determine:

(a) How far did the launcher push the spring in? ________ m

(b) How much energy was ADDED to the system by the explosion? ________ J

(c) How much force did the **launcher** 'feel' if the explosion took 2 ms? $|F_{avg}| =$ ________ N

There's a collision or explosion present here, so let's focus on that part, where we can ONLY count on conservation of momentum happening.

CoM in the X direction (with +X to the right here): $\sum_{before}(mv_x) = \sum_{after}(mv_x)$ so:

$(10)(0)+(80)(0) = (10\ kg)(40\ m/s)+(80\ kg)(v_x)$ so $0 = 400+80v_x$ or $\boxed{v_x = -5\ m/s}$. The launcher must have recoiled to the left at 5 m/s right after the explosion went off.

Now we can look at the launcher interacting with the spring. We have a 40 kg object moving at 5 m/s (to the left), running into the spring. The spring will compress until all that kinetic energy has been 'stored' in the spring.

Basically the $K = \frac{1}{2}mv^2 = (0.5)(80\ kg)(5\ m/s)^2 = 1000\ J$ is converted into $U_s = \frac{1}{2}kd^2$ of potential energy: $1000 = (0.5)(50000)(d)^2$ yields $d = 0.2000\ m$. The spring compresses 20 cm in bringing the launcher to a (momentary) stop.

How much energy did the explosion ADD to the system?

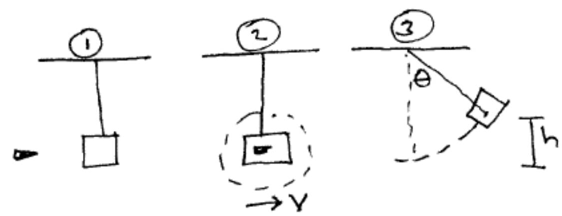
Before the explosion, nothing is moving. Right after, we have a 80 kg object moving at 5 m/s and a 10 kg object moving at 40 m/s. Adding the kinetic energies of those: $E = \frac{1}{2}(80)(5)^2 + \frac{1}{2}(10)(40)^2 = 1000J + 8000J = 9000\ J$.

The explosion basically did 9000 J of 'other work' to the system. It probably released more energy than that (sound, heat, the light flash, etc) but that's how much went into the mechanical energy of the two moving objects.

Force on LAUNCHER during the explosion: before the explosion, the launcher was not moving, to $p_{before} = mv_x = 0$. Right after, $p_{after} = (80\ kg)(-5\ m/s) = -400\ kg\ m/s$. $F_{avg} = \Delta p/\Delta t = \frac{(-400)-(0)}{0.002} = -200,000\ N$.

Force on HOOK during the explosion: $p_{before} = 0$ and $p_{after} = (10\ kg) * (40\ m/s) = +400\ kg\ m/s$, so $F_{avg} = \Delta p/\Delta t = \frac{(400)-(0)}{0.002} = +200,000\ N$.

Ballistic Pendulum
Suppose we hang a 1 kg block of wood from the ceiling on a 1 m long string. A 5 gm (0.005 kg) bullet comes in from the left at 300 m/s and embeds itself into the block. What happens? Conservation of momentum occurs during the collision, so immediately after the collision the combined block and bullet 'object' will be moving to the right with some speed. That means it has some kinetic energy at that point, and we earlier looked at the pendulum problem: if it's moving at some speed at the bottom, how does that relate to the angle it will swing out to?



It might be tempting to take the kinetic energy of the bullet and convert that into U_g of the pendulum to find it's maximum height, but **we have a collision** here, so it's almost certain that some (maybe most) of the mechanical **energy will be lost** at that point. (Not really 'lost' of course - energy is conserved, but a lot will be converted into heat during the collision, so it's no longer part of our 'mechanical energy' collection of terms.)

Collision : We have a 1-D collision here, with the bullet travelling to the right striking the block of wood, embedding itself into the wood, and then the instant after the collision the combined object will be moving to the right with some velocity: a velocity we can find via Conservation of Momentum. So:

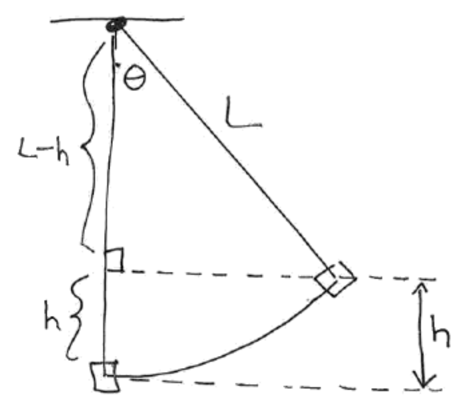
X direction : $\sum\limits_{before} mv = \sum\limits_{after} mv$ so: $(0.005\ kg)(300\ m/s) + (1.00\ kg)(0\ m/s) = (1.005\ kg)(v)$

or $1.5 = 1.005v$ from which $\boxed{v = 1.4925\ m/s}$. The combined object is moving that fast (to the right) in the instant just after the collision. (Verify this, but before the collision we have 225 J of mechanical energy; after the collision we only have 1.12 J remaining, so we definitely can't use CoE alone to solve this!)

The Swing : We've done this problem before a couple of times. We have a pendulum, with a mass on the end of a 1 m long string. It's moving to the right with $v = 1.4925\ m/s$. The pendulum will swing over to the right, gradually converting K into U_g until it reaches a point where it stops moving. At that point, all the $\frac{1}{2}mv^2$ of kinetic energy has been converted into mgh of gravitational potential energy. (Remember, there's no 'other work' here being done by F_T since that force is always perpendicular to the displacement as the block moves along the circular path.)

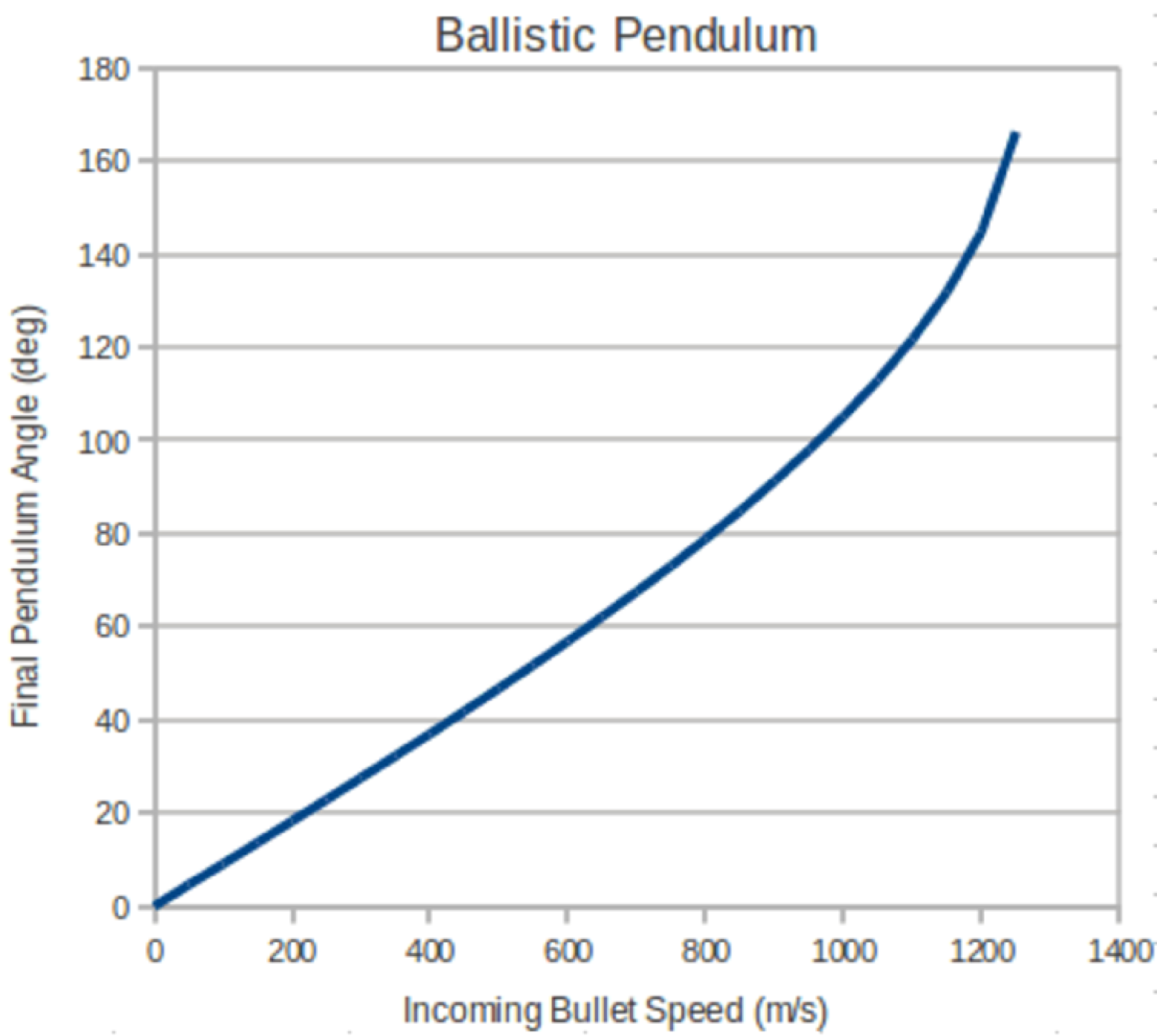
$\frac{1}{2}mv^2 = mgh$ so $h = v^2/(2g) = (1.4925)^2/(19.6) = 0.11366\ m$.

We can convert that into an angle as we've done before also.
From the figure, we see that $\cos\theta = \frac{L-h}{L} = \frac{1-0.11366}{1} = 0.88634$ from which $\theta = 27.6^o$.
For a given bullet mass, we could go through this process for various reference speeds and come up with a table or scale that maps incoming speed into the final angle and use that to estimate the bullet speed without requiring fancy (expensive) high-speed cameras.



ADDENDUM : Pendulum Maximum Angle vs Incoming Bullet Speed

With a 5 *gram* bullet striking (and embedding into) a 1 *kg* block of wood,
how does the deflection angle of the pendulum vary with the incoming bullet velocity?



The steps involved in this calculation involved the (linear) Conservation of Momentum (CoM) step, which relates the incoming bullet velocity to the resulting velocity of the combined object. Then we squared that velocity in a Conservation of Energy (CoE) step to determine the maximum height the combined object will reach, then finally an inverse cosine step to compute the angle.

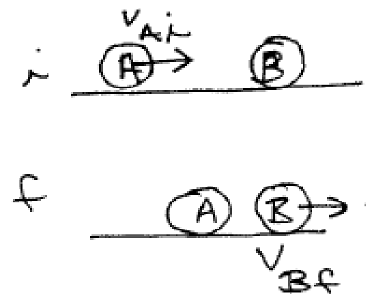
Despite the two nonlinear steps in this process, the resulting deflection angle varies in a surprisingly linear way until the incoming bullet velocity gets very large.

Characterizing Collisions

Elastic Collisions (energy conserved also) ← RARE

Note : these pool-balls are special frictionless versions that are not rotating - they're just idealized point masses that have mass and velocity. We'll get to real rotating objects in the next chapter. (So think of these as collisions happening out in space...)

Ball Ⓐ moving at 10 m/s directly strikes ball Ⓑ which was at rest. After the collision, we observe that Ⓐ is at rest. Find v_B after the collision and examine the energy before and after the collision.
Assume $m_A = m_B = 0.17\ kg$



Momentum is conserved as a vector. In this collision, everything is happening in a single direction, which we'll call X.

Before collision: $\sum\limits_{before} m_i v_i = (0.17\ kg)(10\ m/s) + (0.17\ kg)(0) = 1.7\ kg\ m/s$

After collision: $\sum\limits_{after} m_i v_i = (0.17\ kg)(0\ m/s) + (0.17\ kg)(V)$.

Momentum is conserved, so $1.7 = 0.17V$ or $V = 10\ m/s$. Apparently ball B must be moving off with the same speed as the incoming cue ball had.

Let's look at the total mechanical energy before and after this collision:

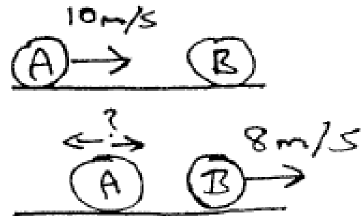
$\sum\limits_{before} K = \frac{1}{2}(0.17)(10)^2 + 0 = 8.5\ J$
$\sum\limits_{after} K = 0 + \frac{1}{2}(0.17)(10)^2 = 8.5\ J$ also.

> These types of collisions are referred to as **ELASTIC COLLISIONS** : momentum is conserved (like always) **and** energy is conserved. This sometimes happens with collisions between very solid objects that do not deform, or when something slides into a spring. Elastic collisions are **not** the normal situation for most real-world colliding objects. Some energy is almost always lost, so unless specifically noted we can't normally assume that energy will be conserved in collision problems.

Actual pool balls are pretty solid objects, so their collisions with one another are pretty close to being elastic.

Inelastic Collision : Some mechanical energy lost $\leftarrow$ COMMON

Suppose we actually observe ball Ⓑ moving off at 8 m/s instead of 10 m/s. Determine what happened to Ⓐ and look at the mechanical energy before and after the collision.



Before collision: $\sum_{before} m_i v_i = (0.17)(10) + (0.17)(0) = 1.7\ kg\ m/s$

Momentum must be be conserved and Ⓑ is moving more slowly now, so Ⓐ must be moving as well at some X velocity V:

After collision: $\sum_{after} m_i v_i = (0.17)(V) + (0.17)(8) = 0.17V + 1.36$

Conservation of momentum: $1.7 = .17V + 1.36$ so $V = 2\ m/s$ (positive so A must be moving to the right, following B but at a much slower speed).

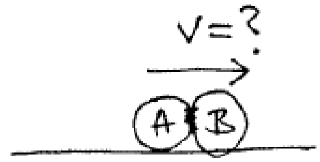
$$\sum_{before} K = \frac{1}{2}(0.17)(10)^2 + 0 = 8.5\ J$$
$$\sum_{after} K = \frac{1}{2}(0.17)(2)^2 + \frac{1}{2}(.17)(8)^2 = 0.34J + 5.44J = 5.78J$$

In this collision, 32% of the initial mechanical energy was 'lost' (and would have gone into heat probably).

> We apparently lost some energy in this collision. This is the more usual case with collisions, and these are called **INELASTIC COLLISIONS** : momentum is conserved (as always), but some mechanical energy is lost (often being converted into heat). (The train-car collision and bullet+block examples from the previous lecture are inelastic collisions.)

Totally Inelastic Collision : Maximum loss of mechanical energy

Let's modify our previous examples so that the pool balls are covered with Velcro or super-glue causing them to stick together during the collision, and move off as a single object. How fast does the combined object move after the collision? How much energy is lost now?



Before collision: $\sum_{before} m_i v_i = (0.17)(10) + (0.17)(0) = 1.7\ kg\ m/s$

After collision: now we have a single blob of mass $0.17 + 0.17 = 0.34\ kg$ moving at some velocity V: $\sum_{after} m_i v_i = (0.34\ kg)(V)$.

Momentum is conserved, so $1.7\ kg\ m/s = (0.34\ kg)(V)$ or $V = 5\ m/s$.

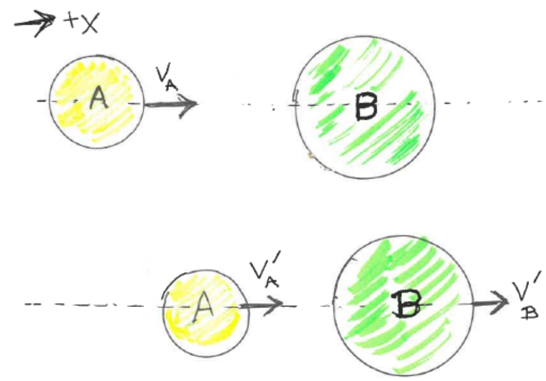
Let's look at the total mechanical energy before and after this collision.
$\sum_{before} K = \frac{1}{2}(0.17)(10)^2 + 0 = 8.5\ J$
$\sum_{after} K = \frac{1}{2}(0.34)(5)^2 = 4.25J$

Compared to the previous examples, in this collision, 50% of the initial mechanical energy was 'lost' (and would have gone into heat probably).

It turns out this 'sticking together' scenario is the one where the largest possible amount of energy gets lost, and it is referred to as a **TOTALLY INELASTIC COLLISION**.

$\boxed{\textbf{Example : 1-D Elastic Collision}}$

(Note: this is basically HW09-28 in the Test 3 sample problems, just with different numbers.)
A 2 kg hockey puck, moving in the +X direction with a speed of 8 m/s has a head-on collision with a 4 kg push that is initially at rest.
Assuming an **elastic** collision occurs here, what will the **velocities** of the two objects be just after the collision?



Conservation of momentum in the X direction: $\sum\limits_{before}(mv_x) = \sum\limits_{after}(mv_x)$
so here we have: $(2)(8) + (4)(0) = (2)v_A' + (4)v_B'$ or $16 = 2v_A' + 4v_B'$ or dividing everything by 2: $\boxed{8 = v_A' + 2v_B'}$.

If this is an elastic collision, the mechanical energy right before and right after the collision will remain the same. There aren't any springs and nothing changes height during the brief collision event, so in effect we just need to look at the kinetic energy here.

Total kinetic energy before: $\sum K_i = \frac{1}{2}(2)(8)^2 + \frac{1}{2}(4)(0)^2 = 64\ J$.

Total kinetic energy right after: $\sum K_i = \frac{1}{2}(2)(v_A')^2 + \frac{1}{2}(4)(v_B')^2$.

Setting those equal, we have $\boxed{64 = (v_A')^2 + 2(v_B')^2}$.

We have two equations and two unknowns, but one of the equations involves the squares of the variables so how do we combine these? One approach is to rearrange the first equation into the form: $v_A' = 8 - 2v_B'$ and replace the v_A' variable in that second equation:

$64 = (8 - 2v_B')^2 + 2(v_B')^2$

Expanding out that first term on the RHS of the equation leaves us with:

$64 = 64 - 32v_B' + 4(v_B')^2 + 2(v_B')^2$

Subtracting 64 from each side and combining like terms:

$0 = 6(v_B')^2 - 32v_B'$ which we can factor into: $0 = 2v_B'(3v_B' - 16)$ which immediately has solutions of either:

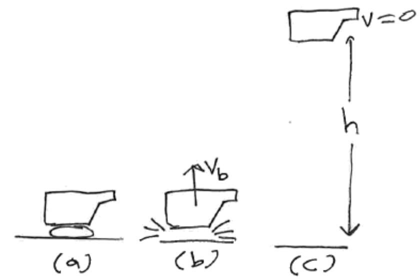
- $v_B' = 0$ which means that $v_A' = 8 - 2(0) = 8\ m/s$, or:
- $v_B' = 5.333\ m/s$, which means that $v_A' = 8 - 2(5.333) = -2.666..\ m/s$

The first 'solution' represents the objects just retaining their original velocities as if nothing happened: they missed each other, or A just passed right through B. Neither is the case we're looking for though, so the actual solution must be the second one. The heavier puck (initially at rest) moves off to the right, and the lighter one bounces back at the given velocity.

Additional Example: Anvil Toss

In the anvil-tossing example we did back in the 1-D and 2-D motion chapters, a 90 kg anvil was launched via an explosive and ended up reaching an apogee height of 78.4 m. (Assume vertically straight up for this problem.)

- How much energy was released in the explosion? (C-4 has a chemical energy density of about 6 MJ/kg, so how much was needed?)
- If the detonation took $\Delta t = 0.1\ ms$, what force did the explosion exert on the anvil?



HINT: focus on CoE outside of the tiny explosion window. What is U_g at the apogee point? What is K at launch? (Use to determine v_o, which we'll need in the momentum calculation.)

At point (a), we have the anvil sitting at rest with a blob of explosive between it and the ground.

At point (b), the explosion has gone off, launching the anvil vertically upward. In a CoE sense, $(K + U_g + U_s)_b = (K + U_g + U_s)_a + W_{other}$ where it's the explosion being that W_{other} term: in a *very* brief time interval, a bunch of energy was converted from chemical energy in the explosive into mechanical energy (really just kinetic energy) of the anvil.

From (b) to (c), that initial kinetic energy is getting converted into gravitational potential energy and when the object reaches its maximum height, it's all now in the form of $U_g = mgh = (90\ kg)(9.8\ m/s^2)(78.4\ m) = 69,149\ J$.

Now we can work backwards. That $69,149\ J$ of U_g at the apogee point (c) must be the anvil's kinetic energy at point (b), meaning it was launched with a speed of $\frac{1}{2}mv^2 = (0.5)(90)(v)^2 = 69149$ or $v_b = 39.2\ m/s$.

That $69,149\ J$ of initial kinetic energy came from the chemical energy stored in the explosive, so we can work out how much C-4 was used here:

$69149\ J \times \frac{1\ kg}{6000000\ J} = 0.0115\ kg$ or about 11.5 *grams*. (Probably more though since some fraction of the initial chemical energy also gets converted into sound, light, heat, and other types of energy during the explosion.)

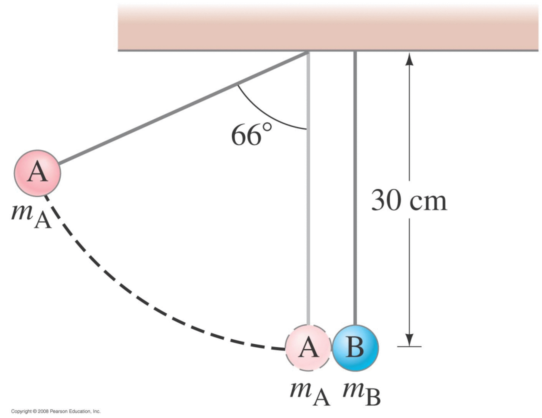
We can also estimate the force involved by looking at the change in momentum of the anvil. Before the detonation, it's at rest so $p_{before} = 0$. Just after the explosion, it's momentum is $p_{after} = (90\ kg)(39.2\ m/s) = 3528\ kg\ m/s$.

$F_{avg} = \frac{\Delta p}{\Delta t} = \frac{3528-0}{0.0001} = 35,280,000\ N$ (about 6 times larger than the force involved in the colliding railcars we started off with!)

Additional Example : Pendulum with 1-D Elastic Collision

(HW09-86) : Two balls of masses $m_A = 40\ g$ and $m_B = 60\ g$ are suspended on $30\ cm$ long strings as shown in the figure. The lighter ball is pulled away to a 66^o angle and released at rest. Assume that the balls are made of a material that allows this to be an **elastic** collision.

(Note that the magic words **elastic collision** appear here, which means that during the collision both energy and momentum will be conserved.)



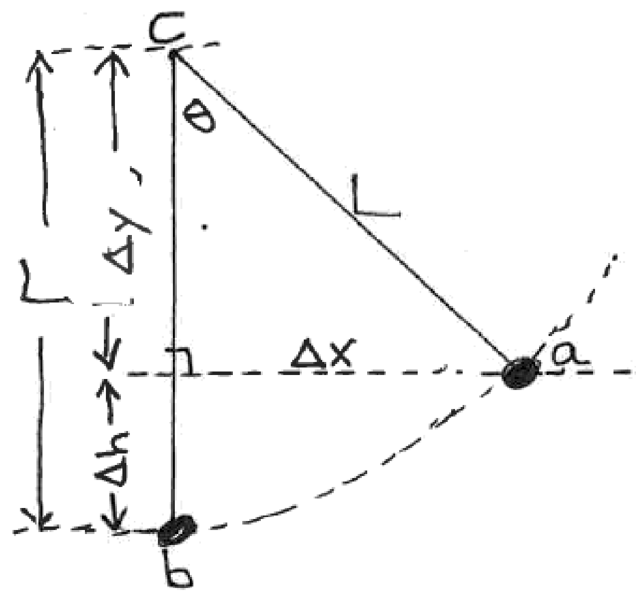
(a) What is the velocity of the lighter ball just before impact?

(b) What is the velocity of each ball just after the elastic collision?

(c) What will be the maximum height of each ball after the elastic collision?

Basically object A will swing down and collide with object B. After the collision, each object will have some velocity that will get converted back into gravitational potential energy, resulting in each object swinging out to some height above the bottom of the arc (probably different heights..)

We can use CoE to determine how fast object A is moving just before it strikes object B. We've done that before back in chapter 7 and 8 with pendulums, so I'll pull in a figure from back then. Between the starting location for object A and the point just before striking object B, conservation of energy tells us that: $(K + U_g + U_s)_2 = (K + U_g + U_s)_1 + \sum_{other} W$. We'll measure our heights (for the $U_g = mgh$ term) relative to the lowest point on the circle. The only 'other' force acting on object A is the tension in the string but in the pendulum geometry we've already argued that since the tension force is radial and the displacement is tangent to the circle, the angle between the vector tension force and the vector displacement is 90^o all along the path, so $\vec{F}_T$ does no work in this scenario. CoE becomes: $\frac{1}{2}mv^2 + 0 + 0 = 0 + mgh + 0 + 0$ or $v = \sqrt{2gh}$ where here $h = L - L\cos\theta = 0.3 - 0.3\cos(66^o) = 0.178\ m$ so $|v| = \sqrt{(2)(9.8)(0.178)} = 1.868\ m/s$. Object A strikes object B at that speed.



If we look in the horizontal direction, conservation of momentum is $\sum(m_i v_i)_{before} = \sum(m_i v_i)_{after}$ so $m_A v_A = m_A v'_A + m_B v'_B$. (We're using that 'prime' symbol to label the velocities after the collision.)

We have **two** unknowns on the right hand side now, so conservation of momentum alone won't be enough to solve this problem.

We were told that this was one of those ultra rate ELASTIC collisions where mechanical energy is also conserved. During the tiny Δt of the collision, the objects won't change height significantly (or at all) and we don't have any springs here, so basically it's kinetic energy that must be being conserved here. That gives us our second equation: the total kinetic energy just before the collision is the same just after the collision (AGAIN this is VERY RARE and is NOT TRUE for most collisions).

In this special case though: $\sum K_{before} = \sum K_{after}$ so: $\frac{1}{2}m_A v_A^2 = \frac{1}{2}m_A(v_A')^2 + \frac{1}{2}m_B(v_B')^2$

We now have two equations and two unknowns. After a bit of algebra, we find that:

- $v_A' = v_A(\frac{m_A - m_B}{m_A + m_B})$
- $v_B' = v_A(\frac{2m_A}{m_A + m_B})$

- **WARNING** : Very specialized equations
- ONLY VALID for 1-D elastic collisions where the second object was initially at rest

I have to repeat again here that these are VERY specialized equations that only apply for 1-D ELASTIC collisions where the 2nd object is initially at rest. They CANNOT be used in any other situation. (They're so frequently mis-used on tests that I didn't include them on the equation sheet.)

Since we were told this was an ELASTIC collision though, we can use them, resulting in:

- $v_A' = (1.868\ m/s)(\frac{0.04-0.06}{0.04+0.06}) = -0.3726\ m/s$
- $v_B' = (1.868\ m/s)(\frac{(2)(0.04)}{0.04+0.06}) = +1.4944\ m/s$

(Note that v_A' is negative, meaning that ball A bounced back to the left.)

We showed earlier we can connect the speed at this point to the height where the object momentarily comes to a stop is: $v = \sqrt{2gh}$ so $h = v^2/(2g)$ so we can find the heights each object returns to:

- Object A : $h = v^2/(2g) = (-0.3726)^2/19.6 = 0.00708\ m$
- Object B : $h = v^2/(2g) = (+1.4944)^2/19.6 = 0.11394\ m$

And if we needed the angles each object swings out to, we can convert h into θ since $h = L - L\cos\theta$. Rearranging that, we find that $\cos\theta = 1 - \frac{h}{L}$.

Ball A will swing back to the left, reaching $\cos\theta = 1 - (0.00708)/(0.30) = 0.9764$ from which $\theta = 12.5^o$.

Ball B will swing out to the right, reaching $\cos\theta = 1 - (0.11394)/(0.30) = 0.6202$ from which $\theta = 51.7^o$.