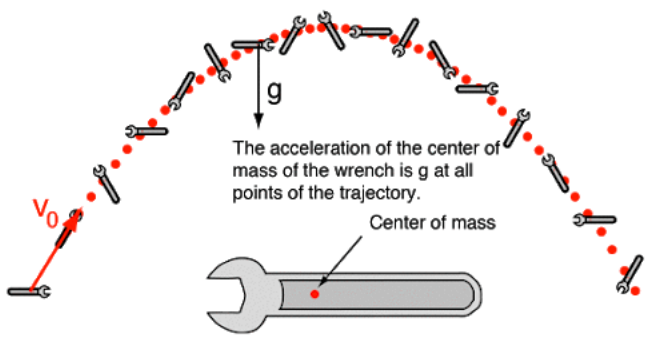
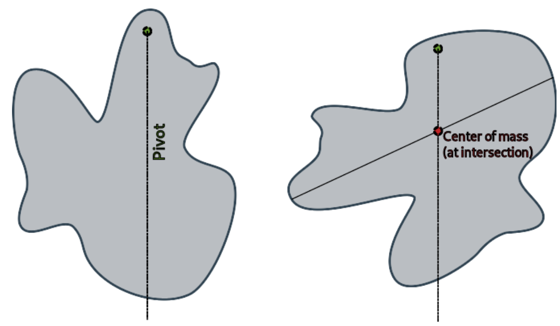
PH2213 Fox : Lecture 22
Chapter 9 : Linear Momentum

Center of Mass

Real objects aren't point masses. A real object will react according to Newton's Laws but may also rotate.

Observations





Center of Mass

: mass-weighted average coordinate

Point Masses

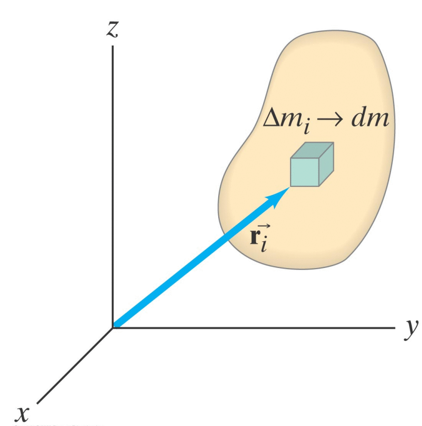
$$\vec{r}_{cm} = \frac{\sum_{i=1}^{N} m_i \vec{r}_i}{\sum_{i=1}^{N} m_i}$$

$$\vec{r}_{cm} = \frac{\sum_{i=1}^{N} m_i \vec{r}_i}{M}$$

$$x_{cm} = \frac{\sum_{i=1}^{N} m_i x_i}{M} \qquad y_{cm} = \frac{\sum_{i=1}^{N} m_i y_i}{M} \qquad z_{cm} = \frac{\sum_{i=1}^{N} m_i z_i}{M}$$

Solid

$$\vec{r}_{cm} = \frac{\int \vec{r} dm}{M}$$



Technically finding the CM for a solid object involves doing an integral, we'll show how we can use symmetry arguments to deal with common geometric shapes, and then show how we can find the CM of a more complicated object by breaking it down into simpler components.

Why is the CM concept important?

Start: $\vec{r}_{cm} = \frac{1}{M}\sum m_i\vec{r}_i$ so rearrange into: $M\vec{r}_{cm} = \sum m_i\vec{r}_i$.

Differentiate with respect to time : $M\vec{v}_{cm} = \sum m_i\vec{v}_i$

Differentiate with respect to time again: $M\vec{a}_{cm} = \sum m_i\vec{a}_i$.

Right-hand side: $m_1\vec{a}_1 = \sum \vec{F}_1$ (sum of all the forces acting on object 1 (or part 1, or molecule 1,...)

The RHS thus is basically:
    the sum of all forces acting on object 1, PLUS
    the sum of all forces acting on object 2, PLUS
    ....etc...

BUT: the vast majority of those forces are INTERNAL. Somewhere in that sum will be a term where object N is exerting some force on object M **and** a term where object M is exerting an equal and opposite force on object N. **All those pairs will cancel due to Newton's third law.**

This leaves the RHS only containing any **external** forces still acting on the object (like gravity):

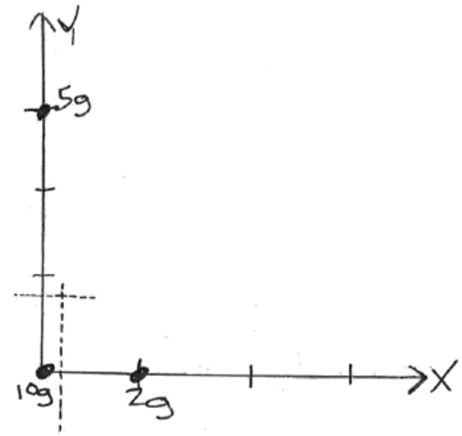
$$\boxed{M\vec{a}_{cm} = \sum_{external}\vec{F}.}$$

This is a BIG DEAL: if a composite object (like a book or the wrench on the previous page, made of gazillions of atoms for example) is tossed into the air, it's CM will follow the same path a point object of the same mass would have.

The object can (and almost certainly will) ROTATE about that center of mass point though, and that's what the remainder of the course will focus on: **angular** motion, and the forces (**torques**) that cause that rotational motion.

A mobile is constructed with three steel balls connected with thin (massless) rods. Determine the center of mass of the mobile.

- $M_1 = 10\ g$ at the origin: $x = 0, y = 0$
- $M_2 = 2\ g$ at $x = 1\ m, y = 0$
- $M_3 = 5\ g$ at $x = 0, y = 3\ m$



- $X_{cm} = \frac{\sum(m_i x_i)}{\sum m_i}$
- $Y_{cm} = \frac{\sum(m_i y_i)}{\sum m_i}$

Note that in the CM equations, the masses appear both in the numerator and denominator, so we can just leave all the masses in grams here: the 'mass units' will end up cancelling out as long as we use the same units for mass throughout the problem.

It can be convenient to organize the CM calculation in a table where we start off with each object's mass and location, then add columns for $m_i x_i$ and $m_i y_i$ (and potentially $m_i z_i$) terms that are involved in the sums:

Object i	Mass (grams)	X coordinate (meters)	Y coordinate (meters)	$m_i x_i$	$m_i y_i$
1	10	0	0	0	0
2	2	1	0	2	0
3	5	0	3	0	15
Sum	**17**	−−	−−	**2**	**15**

Collecting what we need:

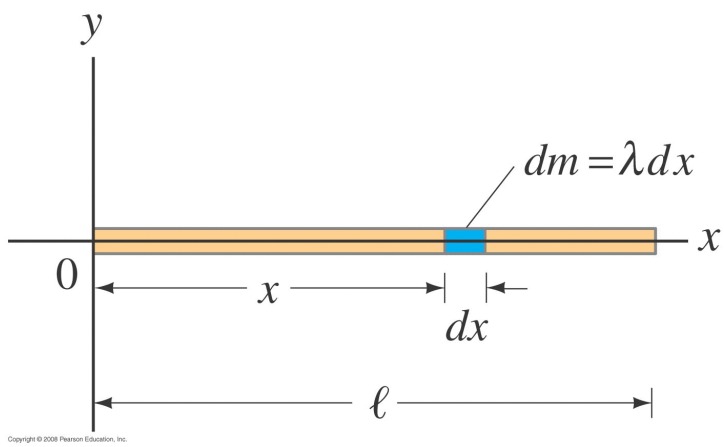
- $X_{cm} = \frac{\sum(m_i x_i)}{\sum m_i} = \frac{2\ grams-meters}{17\ grams} = 0.1176\ m$
- $Y_{cm} = \frac{\sum(m_i y_i)}{\sum m_i} = \frac{15\ grams-meters}{17\ grams} = 0.8824\ m$

The dotted lines in the figure represent $x = 0.1176\ m$ and $y = 0.8824\ m$ and the point where those intersect is the location of the center of mass of this mobile. If we toss the mobile into the air, that point will follow the usual point-mass parabolic equations of motion, with the little masses making up the mobile rotating around that point.

Determine the CM of a thin rod of mass M and length L.

Common **mass-density symbols** you may encounter:

- Mass/volume : ρ (what we typically think of as 'density')
- Mass/area : σ (areal density: often used for objects that are thin sheets of material)
- Mass/length : λ (linear density: often used for long thin objects like string, wire, cable, etc)



Let's actually do this as an integral first. This object is basically a thin wire of mass M and length L. We can find the X center of mass by breaking it up into tiny fragments of mass dm and summing (integrating) the result.

We'll be integrating along the X axis from $x = 0$ to $x = L$ so we need to convert dm into dx but the mass of a little fragment of length dx will be the mass/length of the wire times the little length we're dealing with: $dm = (M/L)(dx)$ or $dm = \lambda dx$

The integral for the X center of mass then becomes:

$X_{cm} = \frac{1}{M} \int x dm = \frac{1}{M} \int x(\lambda dx) = \frac{\lambda}{M} \int_o^L x dx$

Let's replace λ with M/L, and by now you've seen how to do simple polynomial integrals: $\int x dx = \frac{1}{2}x^2$ evaluated at the limits:

$X_{cm} = \frac{1}{M}(\frac{M}{L})(\frac{1}{2}L^2 - \frac{1}{2}(0)^2)$ so

$X_{cm} = \frac{1}{L}(\frac{1}{2}L^2)$.

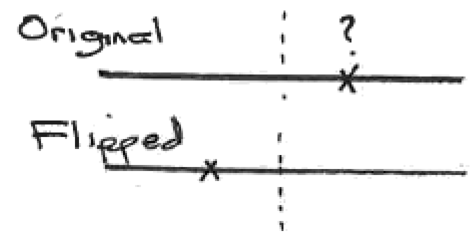
Or finally: $X_{cm} = L/2$

For a uniform straight rod (wire, string, cable, ...) of length L, the center of mass will be right at it's midpoint : that is, right at it's **geometric center**.

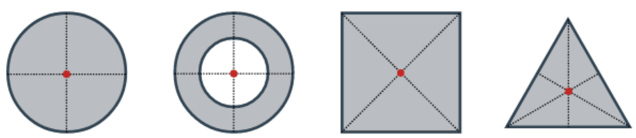
Let's approach this in a very different way now that skips all the math.

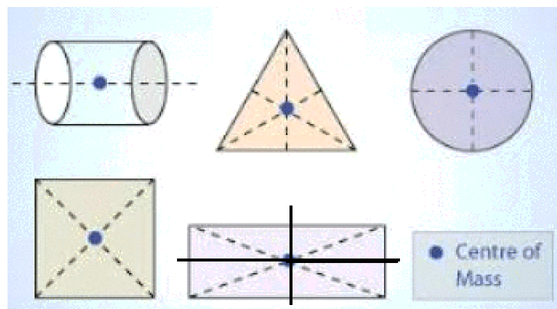
Center of Mass from Symmetry arguments

Let's take our wire segment and flip it around it's midpoint. If we claim that the CM is just to the right of the center in the original orientation, it should now be to the left of center in the flipped copy of the wire. If the wire has the same density everywhere though, we should get the same result for the CM location when we do the integral for either one of these copies of the wire. Thus the CM can't be where I marked it. The only place it can be is right at the midpoint.



We can make the same sort of argument for many simple geometric shapes by looking for point, lines, or planes about which the object is symmetric.



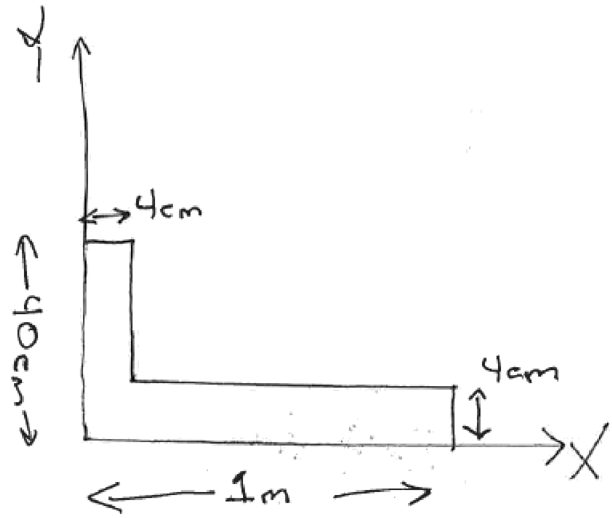


In the figure on the right, for example, the top left object is a cylinder and we've drawn a dotted line along it's axis. We can spin the cylinder about that axis and nothing changes: it's the identical cylinder, so the CM has to lie somewhere along that line. (If we move it radially outward, then the CM would move as we rotate the cylinder about that axis and mathematically it can't since the original and rotated cylinders are identical.) Then we can use the same symmetry argument we used for the wire so say the CM also has to lie midway between the two ends. The intersection of that plane and the dotted line along the axis is a single point: the center of mass.

We won't dwell too much on this now, but you'll see much more of this in your statics and dynamics classes where it's used to drastically simplify some of the calculations involved.

CM of Composite Objects

Determine the CM of the thin L-shaped object shown (called a 'carpenter's square'). Assume the density/area σ is constant throughout the material.



This L-shaped object has no lines or planes of symmetry. The trick here is to recognize that this L-shaped object is really made up of two simple rectangular sections, and the CM of each of those separately is trivial since each will be at their respective geometric centers. Let's see how we can use that idea to find the CM of the overall object.

We should do this derivation as an integral, but it's probably easier to follow if we treat the object as an infinite number of infinitesimal point masses that we are summing over to find the center of mass. In the X direction, we would do:

$x_{cm} = \frac{\sum(m_i x_i)}{M}$ so: $\sum(m_i x_i) = M_{total} x_{cm}$

That sum involves adding up the contributions of every point mass in the entire object, so let's split the sum up into two parts: the points that are in part A and those that are in part B:

$\sum(m_i x_i) = \sum\limits_A (m_i x_i) + \sum\limits_B (m_i x_i)$

BUT: $\sum\limits_A (m_i x_i) = M_A x_{cm,A}$ and $\sum\limits_B (m_i x_i) = M_B x_{cm,B}$

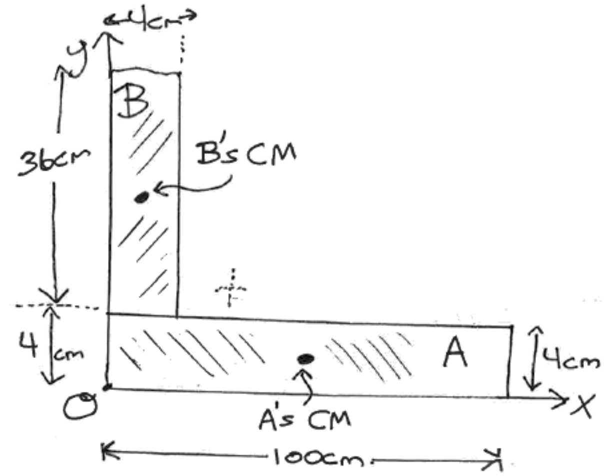
Thus: $M_{total} x_{cm} = M_A x_{cm,A} + M_B x_{cm,B}$ or finally:

$\boxed{x_{cm} = \frac{M_A x_{cm,A} + M_B x_{cm,B}}{M_{total}}}$

What does that mean in words? If we have a 'composite' object that is made up of simpler geometric parts, we can find the CM of each part, then treat those as point objects to determine the overall CM of the object.

Let's apply this process to our L-shaped object now, which we can divide into two rectangular segments.

Part A here is a $(100\ cm) \times (4\ cm)$ rectangle, so its CM will be at its geometric center: $x = 50\ cm$ and $y = 2\ cm$. Its mass will be its area times it's area-density, so $M_A = (100)(4)\sigma = 400\sigma$



Part B here is a $(4\ cm) \times (36\ cm)$ rectangle (only 36 cm since the bottom 4 cm is in part A). Its midpoint will be at $x = 2$ and in the Y direction it'll be at the Y midpoint of just that piece, so will be $36/2 = 18\ cm$ in from either end of B. That puts the point 18 cm above the bottom of B, which itself is 4 cm above the X axis, so the actual Y coordinate of that point will be $Y = 4 + 18 = 22\ cm$. And the mass of this part will be $(area) \times (arealdensity) = (4)(36)\sigma = 144\sigma$.

At this point we've effectively replaced the solid L-shape with just two points:

- Point A : $x = 50\ cm$, $y = 2\ cm$, and $M = 400\sigma$
- Point B : $x = 2\ cm$, $y = 22\ cm$, and $M = 144\sigma$

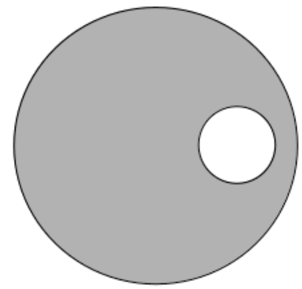
What is the center of mass of those two points, with those masses?

- $X_{cm} = \frac{\sum m_i x_i}{\sum m_i} = \frac{(400\sigma)(50\ cm) + (144\sigma)(2\ cm)}{400\sigma + 144\sigma} = \frac{20000\sigma + 288\sigma}{544\sigma} = 37.29\ cm$
- $Y_{cm} = \frac{\sum m_i y_i}{\sum m_i} = \frac{(400\sigma)(2\ cm) + (144\sigma)(22\ cm)}{400\sigma + 144\sigma} = \frac{800\sigma + 3168\sigma}{544\sigma} = 7.29\ cm$

(That is the point where the two little dotted lines intersect in the figure.)

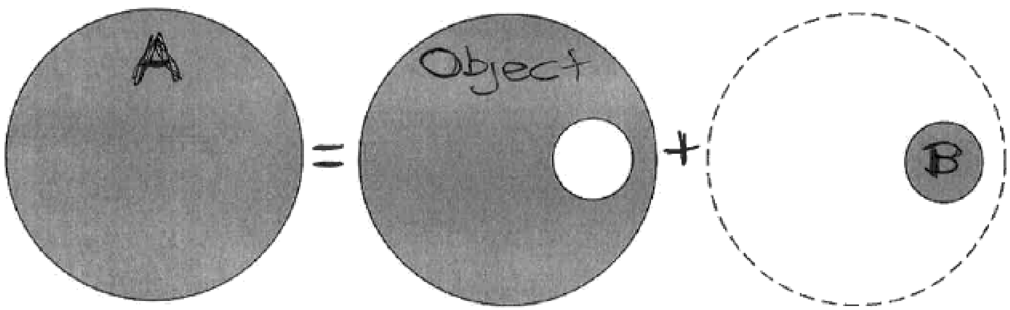
CM of Composite Objects (II)

Suppose we want to determine the CM of an object like that shown on the right: a thin flat metal disk with a hole punched in it.



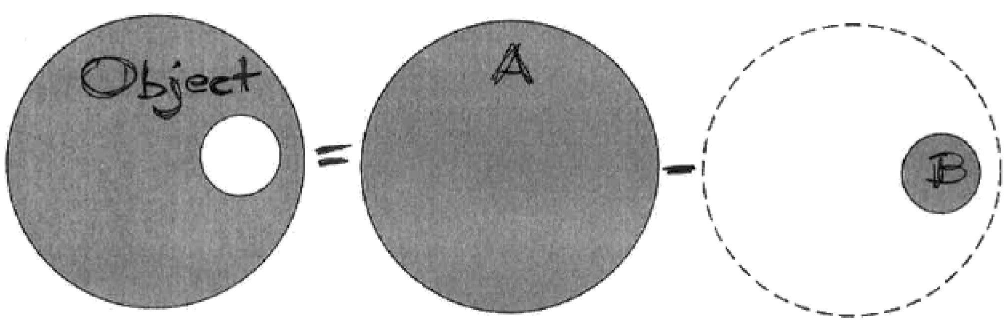
Unlike the previous example, where we could create an object from two simpler sections (with easy to find CM's), here we have a nice simple disk, with another simple shape cut out of it.

The trick here is to note that we can create a complete disk by taking our object and 'adding' a little disk of metal that fills in the missing part. Let's call the 'filled' disk object **A**, and the little disk we had to add will be called object **B** (with the thing we're trying to solve for just called 'Object').



$$\sum_A m_i\vec{r}_i \quad = \quad \sum_{object} m_i\vec{r}_i \quad + \quad \sum_B m_i\vec{r}_i$$

Calculating the CM involves $\sum m_i\vec{r}_i$ (then dividing by the mass) so like before we can break the sum for the full object (A) into two partial sums: a sum over the actual object (with the hole in it) plus a sum of the patch (object B).



$$\sum_{object} m_i\vec{r}_i \quad = \quad \sum_A m_i\vec{r}_i \quad - \quad \sum_B m_i\vec{r}_i$$

Rearranging the sums (and the pictures) gives us what we need.

$\vec{r}_{CM,object} = \frac{M_A\vec{r}_{CM,A} - M_B\vec{R}_{CM,B}}{M_A - M_B}$

In effect, it's the 'point mass' version of the CM calculation, but where we're treating the patch we had to add as if it had a negative mass.