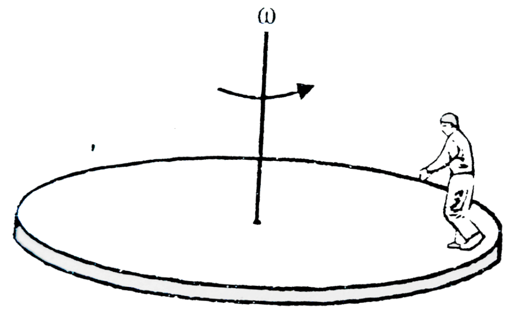
PH2213 Fox : Lecture 24
Chapter 10 : Rotational Motion, Angular Equations of Motion

A Merry-go-Round takes 6 seconds to spin up from rest to reach it's final speed, at which point it takes 4 seconds to make one complete rotation.

- What angular acceleration does this represent?
- How many rotations does the MGR go through from rest until it reaches its operating speed?



The Merry-go-Round (MGR) is starting at rest, so $\omega_o = 0$.

Once up to its operating speed, it's rotating once every **4** seconds, so $T = 4\ s$, making $\omega = 2\pi/T = 1.5708\ (rad)/s$.

It goes from rest to that (angular) speed in **6** seconds, so:

- $\omega = \omega_o + \alpha t$ or
- $1.5708\ (rad)/s = (0) + (\alpha)(6\ s)$ so:
- $\alpha = 0.2618\ (rad)/s^2$

(Note: I put 'radians' in parenthesis because it's not a real physical unit. The true units for angular velocity are just s^{-1} and for angular acceleration they'd be just s^{-2}.)

How 'far' (in angular terms) did the MGR rotate during this 6 second spin-up period?

- $\theta = \theta_o + \omega_o t + \frac{1}{2}\alpha t^2$ so:
- $\theta = 0 + (0)(6) + (0.5)(0.2618(rad)/s^2)(6\ s)^2 = 4.71\ rad$

Another path: we have the initial and 'final' angular velocities over this 6 second interval, and we also know the angular acceleration, so:

- $\omega^2 = \omega_o^2 + 2\alpha\Delta\theta$ or:
- $(1.5708)^2 = (0)^2 + (2)(0.2618)\Delta\theta$ which yields:
- $\Delta\theta = 4.71\ (rad)$ (same result)

Converting that to rotations: $(4.71\ rad) \times \frac{1\ rotation}{2\pi\ rad} = 0.75\ rotations$ Apparently it took 3/4 of a full turn to bring the MGR up to full speed.

Relating Angular and Linear Motion

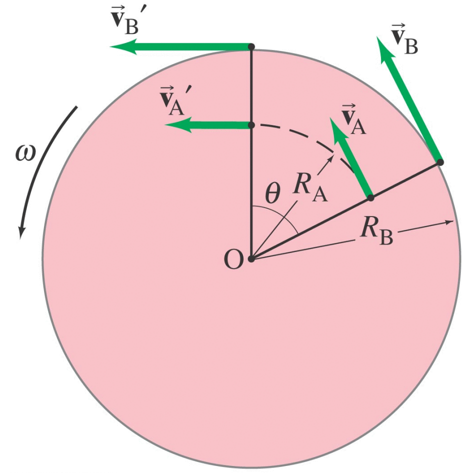
A point located some distance R from the axis of rotation traces out a circular path of length $l = R\theta$.

Differentiating that: $dl/dt = Rd\theta/dt$ or $\boxed{v = R\omega}$.

The **linear speed** of the object along it's circular path is equal to the **angular speed** of the object multiplied by its distance from the axis of rotation.

Note that a point farther out is moving proportionally faster.

An object moving in a circular of radius R at a speed v represents a **radial acceleration** of $a_r = v^2/R = (R\omega)^2/R = R\omega^2$ so $\boxed{a_r = v^2/R = \omega^2 R}$



> If the Merry-go-Round has a radius of $r = 4\ m$, how fast is a person on the outer edge moving when the ride is fully spun up?
> What is the person's radial acceleration at this point?

Recall we previously found that when running at full speed, the MGR has an angular speed of $1.5708\ (rad)/s$.

Speed: $v = R\omega = (4\ m)(1.5708\ (rad)/s) = 6.283\ m/s$ (and note we've discarded the non-unit units of radians from the final answer).

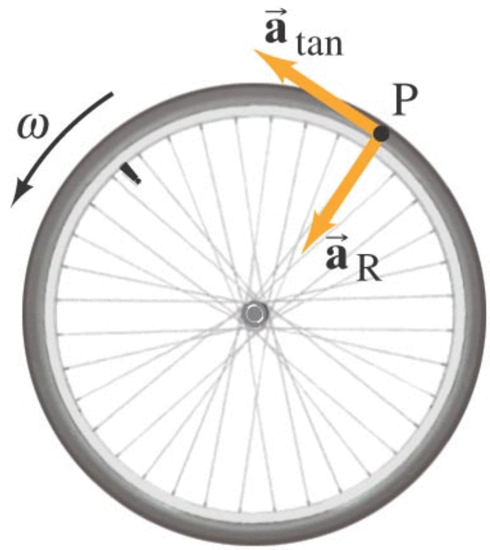
Radial Acceleration:

- Method 1 : $a_r = v^2/R = (6.283\ m/s)^2/(4\ m) = 9.87\ m/s^2$
- Method 2 : $a_r = \omega^2 R = (1.5708\ rad/s)^2(4\ m) = 9.87\ m/s^2$

Note in either case here we end up with a radial acceleration that's higher than g. It would take some effort for a person to remain standing in place with the MGR operating at this speed. Static friction almost certainly wouldn't be enough (see the MGR examples in examples10-core.pdf), so they better be holding onto something!

What if the object has an **angular acceleration** (like our MGR during its spin-up period)?
$l = r\theta$ so $dl/dt = rd\theta/dt$ or $v = r\omega$
Differentiating again: $dv/dt = rd\omega/dt = r\alpha$
NOTE: this is the linear acceleration of the object **along the path** : i.e. **tangent** to the circle it's tracing out. This is referred to as the **tangential** acceleration:

$$\boxed{a_t = r\alpha}$$



The point is rotating in a circle though, so it still **also** has a radial component to its acceleration:

$$\boxed{a_r = v^2/r = \omega^2 r}$$

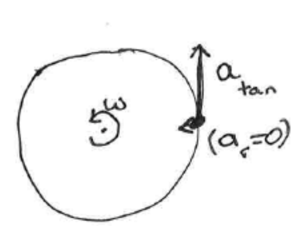
Find the tangential and radial components of the person on the outer edge of our Merry-go-Round:

(a) The instant after the MGR starts operating

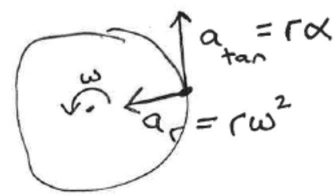
(b) The instant just before it reaches it's full operating speed

(c) The instant just after it's reached it's full operating speed and is no longer (angularly) accelerating

(a) At this point, the motion is starting from rest, so $v = 0$ and $\omega = 0$ at $t = 0$. The radial acceleration will be $a_r = v^2/r = \omega^2 r = 0$. The **angular** acceleration is $\alpha = 0.2681\ (rad)/s^2$ during this phase, so the **tangential** (physical) acceleration will be $a_t = R\alpha = (4\ m)(0.2681\ rad/s^2) = 1.05\ m/s^2$.
These two (perpendicular) components combine to yield the complete $\vec{a}$ acceleration at this instant, so the moment just after the motor is turned on, the person will feel an acceleration of magnitude $|a| = \sqrt{a_r^2 + a_t^2} = 1.05\ m/s^2$, and based on the picture, as a vector it will be pointing entirely tangent to their circular path.

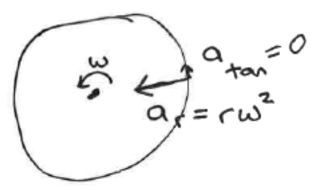


(b) At this point, the MGR is just about to reach it's full operating speed of $\omega = 1.5708\ rad/s$ and in the last phase of accelerating so $\alpha = 0.2681\ rad/s^2$ still. At this point, we have a radial acceleration of $a_r = \omega^2 R = (1.5708\ rad/s)^2(4\ m) = 9.87\ m/s^2$. It still has a tangential acceleration of $a_t = R\alpha = 1.05\ m/s^2$. The full acceleration vector has two components: radial and tangential, so the magnitude of the acceleration vector at this point is $|\vec{a}| = \sqrt{(a_t)^2 + (a_r)^2} = \sqrt{(1.05)^2 + (9.87)^2} = 9.93\ m/s^2$. This is the point where the overall acceleration reaches its maximum value and where the person will have to exert the most force to hold themselves in place.



(b)

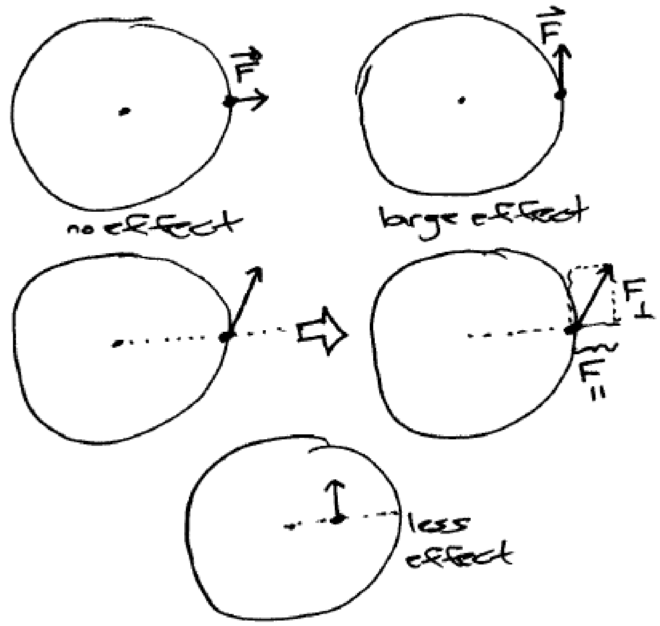
(c) At this point the angular acceleration has stopped and the MGR is just cruising around at a constant $\omega = 1.5708\ rad/s$. $\alpha = 0$ now, so $a_t = R\alpha = 0$: there's no more tangential acceleration. We do still have the usual radial acceleration though: $a_r = \omega^2 R = 9.87\ m/s^2$.
And again, the total acceleration magnitude has dropped slightly to just: $|\vec{a}| = \sqrt{(a_t)^2 + (a_r)^2} = \sqrt{(0)^2 + (9.87)^2} = 9.87\ m/s^2$.



Rotational Forces : Torque

Observation : Suppose we have bicycle wheel suspended on a stand so that the axle is fixed in place but the wheel can rotate about that point. We then apply the same magnitude of force to different points on the wheel, and at various angles.

- outer edge, radially (no effect)
- outer edge, tangent to wheel (considerable effect)
- outer edge but at an angle now: radial component has no effect; only the tangential component
- change where we apply the force now: move in close to the axis
- same force but closer in: smaller effect



The angular acceleration is proportional to r (how far out from the axis the force is applied), and is also proportional to the component of the force tangent to $\vec{r}$.

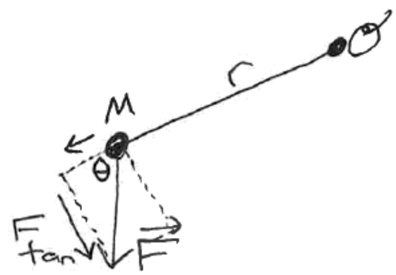
Define the **torque** to be $\tau = F_{tan}r$ (or equivalently $\tau = rF_{tan}$).

- Metric units: $N \cdot m$: $(newtons) \times (meters)$
- English units: $lb \cdot ft$ (sometimes seen as $ft \cdot lb$, but that's more often used to represent work and not torque...)

WARNING : $(force) \times (distance)$ is also units of work (energy, which has units of joules) so torque and energy *technically* represent the same fundamental units, but by convention, torque is **never** written as having units of Joules.

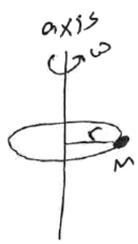
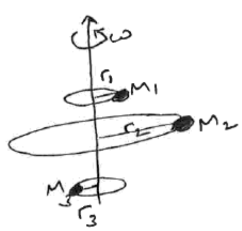
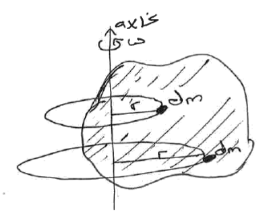
Consider a point-mass object on a thin (rigid) wire of length r. We apply a force $\vec{F}$ to that point. How does it move?
Start: $F_{tan} = ma_{tan}$
Morph into something involving rotational variables:

$F_{tan}r = ma_{tan}r$
$a_{tan} = r\alpha$ so:
$F_{tan}r = mr^2\alpha$ or:
$\tau = (mr^2)\alpha$



Follows the pattern of Newton's Laws ($F = ma$) but with rotational entities now (torque and angular acceleration).

The proportionality constant (m in Newton's laws) is replaced with something else (here mr^2 for the point mass on a wire). That entity is called the **moment of inertia**.

		
Point mass $I = mr^2$	Set of point masses (mobile) $I = \sum m_i r_i^2$	Continuous object $I = \int r^2 dm$

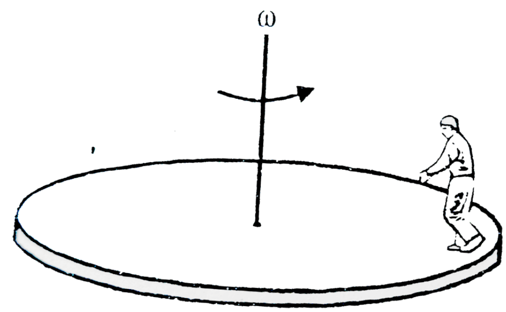
Rotational Version of Newton's Laws: $\boxed{\sum \tau = I\alpha}$

Note: All three of these are 'about a specified axis':

- torque about the axis
- angular acceleration about that axis
- moment of inertia about that axis

and remember that signs are taken to be positive for counter-clockwise and negative for clockwise angles, speeds, accelerations about that axis.

Example : Earlier, we had a Merry-go-Round that took 6 seconds to spin up from rest to its operating speed, where the final period was 4 seconds. We found the MGR had an angular acceleration of $\alpha = \frac{\pi}{12} = 0.2618\ s^{-2}$.
Suppose the MGR has a moment of inertia (for the given axis of rotation) of $I = 2500\ kg\ m^2$. (I looked up some actual MGR dimensions and this would be reasonable.)



What torque must the motor driving the MGR be generating?

Compare this to typical car engine torques, which range from 100 to 400 **lb ft**. (The engines in 18-wheelers generally have torques in the 1000 to 2000 $lb\ ft$ range.)

$\tau = I\alpha$ so here $\tau = (2500\ kg\ m^2)(0.2618\ s^{-2}) = 654.5\ N\ m$

Converting units:

$\tau = (654.5\ N\ m) \times \frac{1\ lb}{4.4448\ N} \times \frac{3.281\ ft}{1\ m} = 483\ lb\ ft$

That's at the high end for most passenger cars, but well below the motor torque in an 18-wheeler, so this might be plausible. A smaller engine would just mean less torque, resulting in a longer spin-up time for the ride.