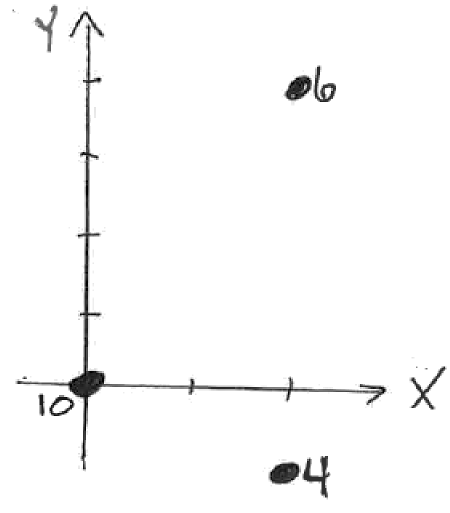
PH2213 Fox : Lecture 25
Chapter 10 : Rotational Motion, Angular Equations of Motion

(a) Compute the moment(s) of inertia about the X and Z axes for the collection of points shown in the figure.

$10\ kg$ at origin
$4\ kg$ at $x = 2\ m, y = -1\ m$
$6\ kg$ at $x = 2\ m, y = 4\ m$

(b) Find the center-of-mass of this mobile and find it's moment of inertia about an axis that passes through the CM and comes up out of the page (i.e. parallel to Z).



Moment of inertia about X axis : $I = \sum mr^2$

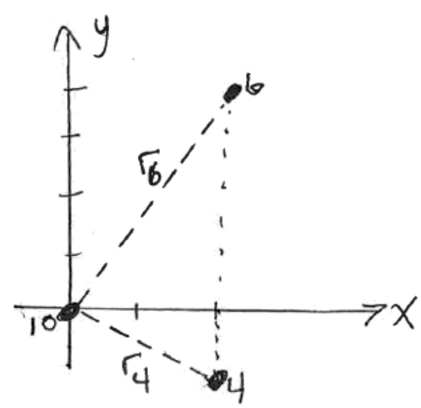
The $10\ kg$ object is right on the X axis, so $r = 0$ for that object.
The $4\ kg$ object is $1\ m$ from the X axis, so $r = 1$ for that object.
The $6\ kg$ object is $4\ m$ from the X axis, so $r = 4\ m$ for that object.

$I_x = \sum m_i r_i^2 = (10\ kg)(0\ m)^2 + (4\ kg)(1\ m)^2 + (6\ kg)(4\ m)^2 = 0 + 4 + 96 = 100\ kg\ m^2$.

Moment of inertia about Z axis : $I = \sum mr^2$

Our axis now is coming up out of the page through the origin. r is the distance from the axis to each point mass. The $10\ kg$ object is right on the Z axis, so $r = 0$ for that object.

How far is the $4\ kg$ object from the Z axis? The dotted line labelled r_4 is the direct, perpendicular distance from this object to the axis. It's the radius of the circle that this point will travel with we rotate the mobile about Z. So: $r_4 = \sqrt{(1\ m)^2 + (2\ m)^2} = \sqrt{5}$



Similarly, how far is the $6\ kg$ object from the Z axis? $r_6 = \sqrt{(2)^2 + (4)^2} = \sqrt{20}$.

Finally then: $I_z = \sum m_i r_i^2 = (10)(0) + (4)(\sqrt{5})^2 + (6)(\sqrt{20})^2 = 0 + 20 + 120 = 140\ kg\ m^2$.

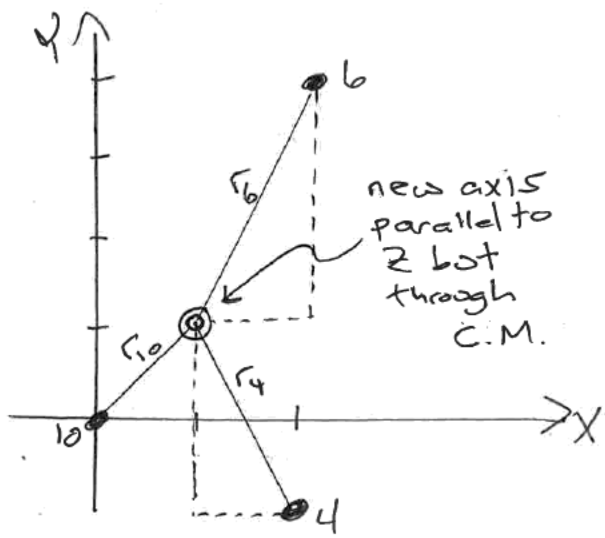
(continued...)

Center of Mass

Suppose we want to rotate the object around an axis that's parallel to Z but offset so that it passes directly through the center-of-mass of the mobile?

First, where is the center of mass of this object?

$X_{cm} = \frac{\sum m_i x_i}{\sum m_i} = \frac{(10)(0)+(4)(2)+(6)(2)}{10+4+6} = 1.0\ m \qquad Y_{cm} = \frac{\sum m_i y_i}{\sum m_i} = \frac{(10)(0)+(4)(-1)+(6)(4)}{10+4+6} = 1.0\ m$



How far is each point from this new rotation axis? The axis is coming up out of the page passing through the CM point marked on the figure. How far is each point from that axis?

- 10 kg object: $r_{10} = \sqrt{(1)^2 + (1)^2} = \sqrt{2}\ m$
- 4 kg object: $r_4 = \sqrt{(1)^2 + (2)^2} = \sqrt{5}\ m$
- 6 kg object: $r_6 = \sqrt{(1)^2 + (3)^2} = \sqrt{10}\ m$

Moment of inertia about this new axis: $I_{cm,z} = \sum m_i r_i^2 = (10\ kg)(\sqrt{2}\ m)^2 + (4\ kg)(\sqrt{5}\ m)^2 + (6\ kg)(\sqrt{10}\ m)^2 = 20 + 20 + 60 = 100\ kg\ m^2$.

Note that this I is smaller than when we rotated the object about the Z axis. Shifting the axis over a bit decreased the object's moment of inertia. Since $\tau = I\alpha$ that would mean we'd need less torque to accelerate the object about this new axis, versus accelerating it about the Z axis.

Parallel Axis Theorem : If we know the moment of inertia of an object about some axis that passes through it's center of mass, but need to find it's moment of inertia about some other axis that's parallel but offset some distance d : $\boxed{I = I_{cm} + Md^2}$.

The table on the next page shows the moment of inertia for various extended objects with common geometrical shapes and most of them are about axes that pass through the CM of the object.

For our mobile, we computed $I_{cm,z}$: the moment of inertia for an axis that is parallel to Z but passes through the objects center of mass, and found that was 100 $kg\ m^2$.

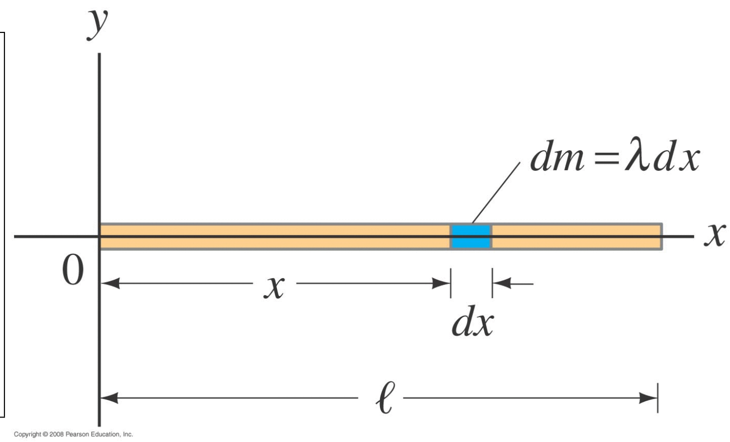
What if we want to know the moment of inertia for rotations about the Z axis? We could compute it directly (like we did already), but these two axes are parallel and offset from one another by a distance $d = \sqrt{2}\ meters$ so using the parallel axis theorem:

$I = I_{cm} + Md^2$ becomes: $I = (100\ kg\ m^2) + (20\ kg)(\sqrt{2})^2 = 100 + 40 = 140\ kg\ m^2$.

NOTE that the moment of inertia is actually minimized if we rotate the object about it's center of mass. If the rotation axis is anywhere offset from the center of mass, I has that $+Md^2$ term adding to it's I value.

Moment of Inertia for Simple Solid Object

Determine the moment of inertia of a thin rod of mass M and length L rotating about an axis that passes through one end of the rod, perpendicular to it (the Y axis in the figure).



$I = \int r^2 dm$ and we're going to want to integrate over some physical variable like r. If we orient the rod as shown in the figure, then $dr = dx$ and we'll be integrating from $x = 0$ to $x = L$. Each little dx element will have a mass of $dm = \lambda dx$ where $\lambda = (mass)/(length) = M/L$.

Our integral then becomes: $I = \int \lambda x^2 dx = \lambda \int x^2 dx = \lambda(\frac{x^2}{3}|_0^L) = \lambda \frac{L^3}{3}$

Now $\lambda = M/L$ so we can write this as $I = (\frac{M}{L})(\frac{L^3}{3}) = \frac{1}{3}ML^2$.

What if we slide the axis of rotation over so it passes through the center of mass (i.e. the axis will be parallel to Y but pass through the point $x = L/2$)? What will the new moment of inertia be?

We could do this via an integral again, putting the origin at the midpoint of the rod and then integrating from $x = -L/2$ to $x = +L/2$, but we can also short-cut this via the parallel axis theorem: $I = I_{cm} + Md^2$

Here I_{cm} is the thing we're looking for. We just calculated the moment of inertia about the end of the rod and that axis is parallel to the axis we're interested in but offset by $d = L/2$ so:

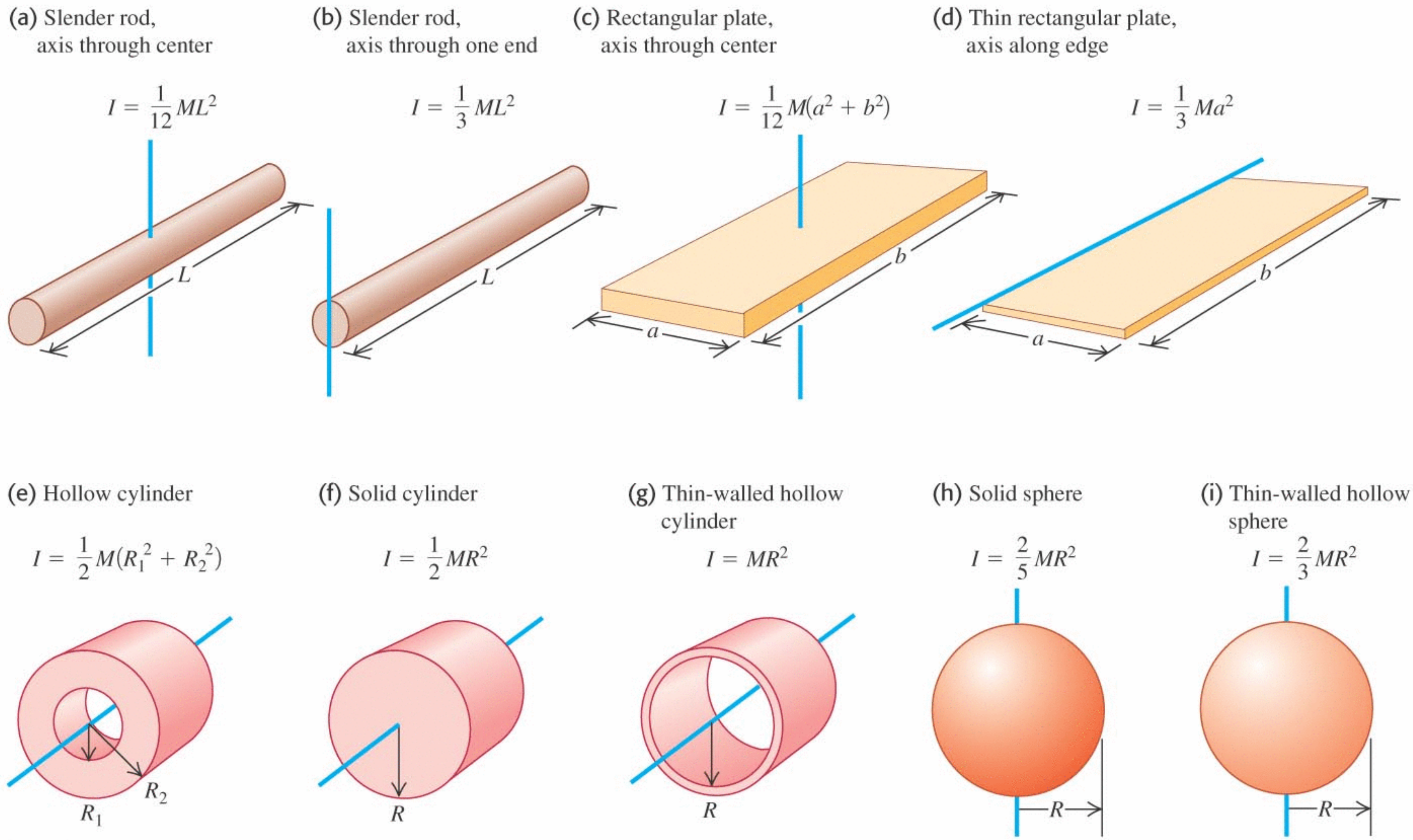
$\frac{1}{3}ML^2 = I_{cm} + M(\frac{L}{2})^2$ or here $\frac{1}{3}ML^2 = I_{cm} + \frac{1}{4}ML^2$.

Rearranging: $I_{cm} = (\frac{1}{3} - \frac{1}{4})ML^2 = \frac{1}{12}ML^2$.

The chart on the next page shows the moments of inertia for several common geometric shapes, with most of them shown for rotations about an axis that passes through the CM of the object. We can use the parallel axis theorem to find I about some other axis that's parallel to the axis through the CM.

Wikipedia has a much longer table of moments of inertia here:

`https://en.wikipedia.org/wiki/List_of_moments_of_inertia`


(a) Slender rod, axis through center
$I = \frac{1}{12}ML^2$
L
(b) Slender rod, axis through one end
$I = \frac{1}{3}ML^2$
L
(c) Rectangular plate, axis through center
$I = \frac{1}{12}M(a^2 + b^2)$
a
b
(d) Thin rectangular plate, axis along edge
$I = \frac{1}{3}Ma^2$
a
b
(e) Hollow cylinder
$I = \frac{1}{2}M(R_1^2 + R_2^2)$
R_1
R_2
(f) Solid cylinder
$I = \frac{1}{2}MR^2$
R
(g) Thin-walled hollow cylinder
$I = MR^2$
R
(h) Solid sphere
$I = \frac{2}{5}MR^2$
R
(i) Thin-walled hollow sphere
$I = \frac{2}{3}MR^2$
R

Moment of Inertia of Composite Objects

Sometimes we can construct objects from simple geometric parts (disks, rods, spheres, etc).

Back in chapter 9 we found a way to determine the center-of-mass of composite objects that was somewhat tedious, but finding the moment of inertia of a composite object is almost trivial.

Suppose our 'object' is the Thor's Hammer geometry shown in the example below. We can construct this from a thin rod (the handle) plus a cylinder (the head). Those are objects we can find in the moment of inertia table on the previous page. How can we use that information to find 'I' for the overall object?

Take the definition of I and break the integral into integrals over each part:

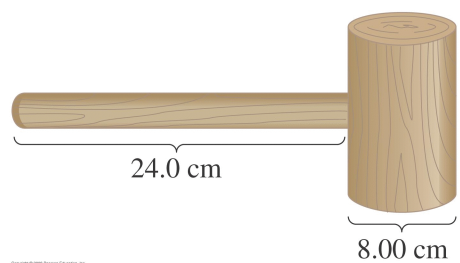
$$I = \int r^2 dm = \int_{part\ 1} r^2 dm + \int_{part\ 2} r^2 dm + \cdots$$

But each of those integrals gives us I for that part, so: $\boxed{I = I_{part\ 1} + I_{part\ 2} + \cdots}$.

We do have to account for the fact that the 'I' values given in the table are for rotations about a particular axis (usually through the CM) and the axis involved in our scenario will be offset from that, but the parallel axis theorem lets us make that adjustment: $I = I_{cm} + Md^2$

The hammer in the figure consists of two parts:

- A thin handle with a mass of 5 kg.
- A cylinder with a mass of 10 kg.
- The relevant sizes of the objects are given in the figure.



Suppose we're holding onto the hammer and swinging it around an axis that's 40 cm to the left of the left end of the handle. Also, suppose this axis is a 'vertical line' (a line running from the top of the page to the bottom).

Handle : According to the table (upper left figure), if this object were rotating around an axis through it's CM we'd have $I_1 = \frac{1}{12}ML^2 = (5\ kg)(0.24\ m)^2/12 = 0.024\ kg\ m^2$. The actual axis involved here is 40 cm to the left of the left end of the 'rod', so it's $40 + 12 = 52\ cm = 0.52\ m$ to the left of the axis given in the table. Making this adjustment:

$I_1 = I_{cm} + Md^2 = (0.024\ kg\ m^2) + (5\ kg)(0.52\ m)^2 = 1.376\ kg\ m^2$.

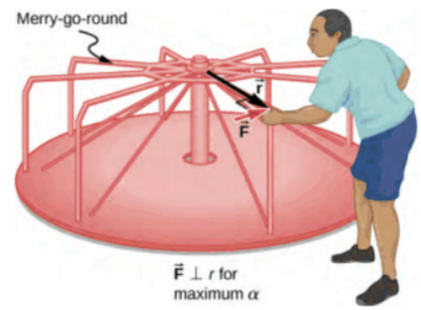
Head : According to the table (2nd figure on 2nd row), object were rotating around an axis through it's CM we'd have $I_2 = \frac{1}{1}MR^2 = (0.5)(10\ kg)(0.04\ m)^2 = 0.008\ kg\ m^2$. The actual axis involved here is 40 cm to the left of the left end of the handle, which is $40 + 24 + 4 = 68\ cm = 0.68\ m$ to the left of the axis used in the table though, so making that adjustment:

$I_1 = I_{cm} + Md^2 = (0.008\ kg\ m^2) + (10\ kg)(0.68\ m)^2 = 4.632\ kg\ m^2$.

Finally, $I = \sum I_i = 1.376 + 4.632 = 6.008\ kg\ m^2$.

(And ultimately we could use how much torque our muscles we can create about our shoulder joint to determine the angular acceleration we could give the hammer and find how fast it's moving after swinging it through 90 degrees, say, or the reverse: time how long it takes and use that to determine the torque our shoulder muscles put out.)

Example : Consider the father pushing a playground merry-go-round as shown in the figure. He exerts a force of 250 N tangent to the edge of the 200.0 kg merry-go-round, which has a 1.50 m radius. Consider the merry-go-round itself to be a uniform disk, and ignore friction.
NOTE: a uniform solid disk has a moment of inertia of $I = \frac{1}{2}MR^2$.



(a) How much torque is the person providing? (In $N \cdot m$ and in the usual English units of torque, which are $ft \cdot pounds$.)

(b) Calculate the angular acceleration produced when no one is on the merry-go-round.

(c) How about when an 18.0 kg child sits 1.25 m away from the center? (Treat the child as a point-mass being added at the given location. How does that change the moment of inertia? α?)

(d) If the force is only applied for a quarter-turn (after which the force is removed), what will the angular speed of the MGR be?

(a) $\tau = rF_{tan}$ and the 250 N of force is being applied tangent to the outer edge of the MGR, so here $F_{tan} = 250\ N$ and $r = 1.50\ m$ so $\tau = (250\ N)(1.5\ m) = 375\ N\ m$. (Hm. Converting units, that's $(375\ N\ m) \times \frac{1\ lb}{4.448\ N} \times \frac{3.281\ ft}{1\ m} = 277\ lb\ ft$ which is in the range that car engines put out (100 to 400 foot-pounds). Seems unlikely, but we'll press on...)

(b) $\tau = I\alpha$ so we'll need the moment of inertia. We're told to model this as a uniform solid disk, so $I = \frac{1}{2}MR^2 = (0.5)(200\ kg)(1.5\ m)^2 = 225\ kg\ m^2$.
$\alpha = \tau/I = (375\ N\ m)/(225\ kg\ m^2) = 1.67\ rad/s^2$.

(c) Here we're adding a 18 kg point mass to a point 1.25 m from the axis. Note that from the definition of I, we can just add the two moments of inertia:

$$I = \sum_{object} mr^2 = \sum_{part\ A} mr^2 + \sum_{part\ b} mr^2 = I_A + I_B$$

We know the disk has moment of inertia of $I_A = 225\ kg\ m^2$. Part B here is a point mass located the given distance from the axis, so $I_B = mr^2 = (18\ kg)(1.25\ m)^2 = 28.125\ kg\ m^2$ so the overall moment of inertia of the child plus the MGR is $I = 225 + 28.125 = 253.125\ kg\ m^2$.

(d) What will the new α be? $\tau = I\alpha$ so $\alpha = \tau/I = (375\ N\ m)/(253.125\ kg\ m^2) = 1.48148\ rad/s^2$.

How fast will the thing be going around after a quarter turn? That represents an angular change of $\frac{1}{4}(2\pi) = \pi/2\ radians$. Using one of our rotational equations of motion:
$\omega^2 = \omega_o^2 + 2\alpha\Delta\theta$ so here $\omega^2 = (0)^2 + (2)(1.48148)(\pi/2) = 4.654$ so $\omega = 2.157\ rad/s$.

How fast would the child be moving at that point? $v = r\omega = (1.25\ m)(2.157\ rad/s) = 2.797\ m/s$ (about 6 miles/hour).

What tangential acceleration would the child have felt? $a_{tan} = r\alpha = (1.25\ m)(1.48148\ rad/s^2) = 1.85\ m/s^2$ (about 0.2 $g's$, so they probably won't be thrown off...)