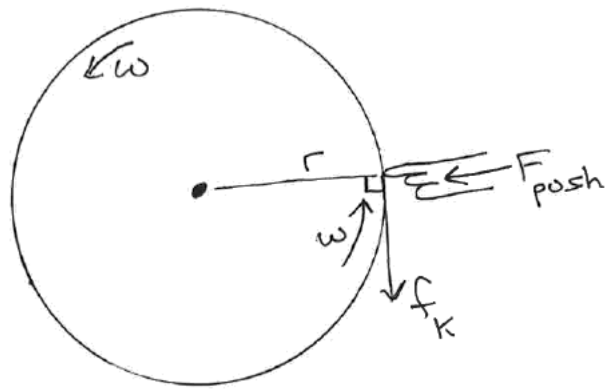
PH2213 Fox : Lecture 26
Chapter 10 : Rotational Motion, Angular Equations of Motion

The bike wheel in the figure has a mass of 4 kg and a radius of 0.4 m. (Treat the wheel as a ring where all the mass is at that distance from the axis.)
What is the moment of inertia of the wheel for rotations about it's (fixed) axle?
The wheel is rotating at 4 rev/sec and we want to bring it to a stop by pushing on it with the palm of our hand. If we push with $F = 20\ N$ and the coefficient of kinetic friction between our hand and the rubber tire is $\mu_k = 0.5$, how long will it take to bring the wheel to a stop?



Moment of inertia : all the mass is at $r = R$ so the 'integral' is just $I = MR^2$. We can also use the table on the previous page since this geometry is the same as the hollow cylinder rotating about an axis down it's center (where the 'height' of the cylinder has shrunk down to nothing). So here: $I = MR^2 = (4\ kg)(0.4\ m)^2 = 0.64\ kg\ m^2$.

Torque : our pushing force is essentially the normal force between our hand and the tire, so here $f_k = \mu_k F_N = (0.5)(20\ N) = 10\ N$. This force is tangent to the circle representing the wheel, so:

Torque on the wheel: $\tau = F_{tan} r = (10\ N)(0.4\ m) = 4\ N\ m$. BUT: we need to think about the sign here too. If this were the only force acting on the tire, it would cause it to rotate clockwise, and clockwise is the negative direction, so really $\tau = -4\ N\ m$ here. (If multiple forces are creating torques on an object, we have to go through a similar hand-waving argument to decide what sign to attach to that torque. We'll have a more rigorous way of doing this once we cover the 'cross product' operation in chapter 11.)

$\sum \tau = I\alpha$ so $-4.0 = (0.64)(\alpha)$ or $\boxed{\alpha = -6.25\ rad/s^2}$.

As the wheel turns, it is slowing down with that angular acceleration.

How long will it take to bring the wheel to a stop then?

What was the wheel's initial angular speed? The 4 $revolutions/sec$ value provided is actually the <u>frequency</u> f, but we can convert that to <u>angular speed</u>: $\omega = 2\pi f = 8\pi\ s^{-1}$.

$\boxed{\omega = \omega_o + \alpha t}$ so: $0 = (8\pi) + (-6.25)(t)$ from which $t = 4.02\ s$. It would take just over 4 seconds to bring the wheel to a stop.

How many revolutions will the tire have made while coming to a stop?

We have multiple options now since we know the starting and ending (angular) speeds, the acceleration, and the time:

- $\omega^2 = \omega_o^2 + 2\alpha\Delta\theta$ so $(0)^2 = (8\pi)^2 + (2)(-6.25)(\Delta\theta)$ from which $\Delta\theta = 50.53\ radians$ which is $(50.53\ rad) \times \frac{1\ rev}{2\pi\ rad} = 8.04\ revolutions$.
- $\theta = \theta_o + \omega_o t + \frac{1}{2}\alpha t^2$: $\theta = 0 + (8\pi)(4.02) + (0.5)(-6.25)(4.02)^2 = 50.53\ rad$ (same result)

- $\omega_{avg} = \Delta\theta/\Delta t$ so $\Delta\theta = (\omega_{avg})(\Delta t) = (\frac{8\pi+0}{2})(4.02) = 50.52\ rad$ (ok, finally a tiny bit of round-of there in the 4th significant figure but I rounded t to $4.02\ s$ so that's not unexpected...).

Work done by friction

The wheel made 8.04 revolutions as friction slowed it down to a stop. How much tire rubber passed by our finger? Each rotation represents a full circumference of the circle, which is a length of $C = 2\pi r = (2)(\pi)(0.4\ m) = 2.513\ m$ so our finger slid along $(2.513\ m/revolution) \times (8.04\ revolutions) = 20.21\ m$.

The **work** that friction did was $W_{f_k} = -f_k d = -(10\ N)(20.21\ m) = -202.1\ J$.

Friction did that much negative work (which would have turned into that much heat).

The fact that friction is removing that much energy bringing the wheel to a stop means that the spinning wheel must have started out with that much kinetic energy. Even though the wheel's center of mass didn't move at all here, just the fact that it was rotating means kinetic energy is present.

Rotational Kinetic Energy

At the end of chapter 9, we started breaking the motion of object up into two parts. First, there's its center of mass motion (due to any external forces like gravity acting on the object), where the (CM of the) object follows the nice equations of motion we've used since chapter 2. In addition to that motion, the object may also be rotating about an axis through it's center of mass, and that is the focus of chapters 10 through 12, introducing rotational equations of motions, rotational forces (torques). Today we'll focus on 'rotational energy' and how we can use that to analyze the motion of real, extended objects instead of just pretending everything is a point mass (finally).

Consider an object rotating about some axis:

We can break the object into a vast number of tiny mass elements m_i, and if the entire object is rotating about some axis with an angular velocity ω, then every little m_i element of the object is doing the same thing.
Each of those elements is located some distance r_i **from** the axis of rotation though, so this angular motion means that each element is tracing out a circular path.
The linear speed of that element (i.e. the usual m/s type speed) will be $v_i = r_i\omega$, so that little element represents a kinetic energy of $K_i = \frac{1}{2}m_i v_i^2$ and replacing v_i with $r_i\omega$ we can write it's kinetic energy as: $K_i = \frac{1}{2}m_i(r_i\omega)^2 = \frac{1}{2}m_i r_i^2\omega^2$.

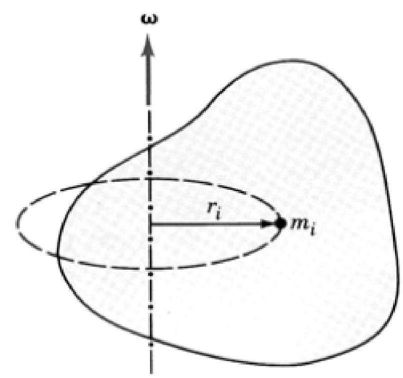


Figure 1

Summing up all these kinetic energies over the entire object, we have:

$K = \frac{1}{2}(\sum m_i r_i^2)\omega^2$ which is $\boxed{K = \frac{1}{2}I\omega^2}$.

The kinetic energy introduced by the object rotating is called the **rotational** kinetic energy: $\boxed{K_{rot} = \frac{1}{2}I\omega^2}$

to differentiate it from the kinetic energy due to the object as a whole moving (i.e. how is it's center of mass moving): $\boxed{K_{cm} = \frac{1}{2}mv_{cm}^2}$

(NOTE: K_{trans} (for 'translational') is another common way to denote K_{cm}.)

$$\boxed{K = K_{cm} + K_{rot} = \frac{1}{2}mv_{cm}^2 + \frac{1}{2}I\omega^2}$$

We're almost ready to produce the complete Work-K and CoE equations but need to talk about work done by torques first:

Rotational Work

If we start with one of the angular equations of motion: $\omega^2 = \omega_o^2 + 2\alpha\Delta\theta$.

and then multiply that equation by $\frac{1}{2}I$, we arrive at:

$\frac{1}{2}I\omega^2 = \frac{1}{2}I\omega_o^2 + I\alpha\Delta\theta$ or:

$K_{rot,final} = K_{rot,initial} + (\tau)(\Delta\theta)$

Compare that to our previous (linear) Work-K relation: $K_{final} = K_{initial} + (F)(d)$

In the linear case, the work done by a force in the same direction as the displacement was $W = Fd$.

In the rotational case, again our linear relationships map into their angular equivalents. The work done by a torque τ as it rotates something through an angle $\Delta\theta$ is $\boxed{W = \tau\Delta\theta}$.

Similarly, the **rate** at which a rotational force (torque) does work would be $\boxed{P = \tau\omega}$ (compare to $P = Fv$ in the translational case).

> NOTE: technically $P = \vec{F} \cdot \vec{v}$ and same with the rotational situation: $P = \vec{\tau} \cdot \vec{\omega}$ but since all our angles, angular speeds, torques, etc are all occuring about the same Z axis coming up out of the board, the dot product reduces to just $W = \tau\Delta\theta$ and $P = \tau\omega$.

Conservation of Energy

Work-K and CoE are still good ways to analyze situations but we have to account for the possibility of rotational work and rotational kinetic energy:

Work-K : $K_b = K_a + \sum_{all} W$

CoE : $(K + U_g + U_s)_b = (K + U_g + U_s)_a + \sum_{other} W$

where in each case $K = K_{cm} + K_{rot}$ now, and where we may have rotational work terms (torque)X(angle) in addition to the old (force)X(distance) type of work.

Example : Grinding Wheel
(HW 10.37, modified)

Problem 10.37 A grinding wheel is a uniform cylinder with a radius of 8.50 cm and a mass of 0.380 kg. Calculate:

(a) **the moment of inertia about its axis of rotation:**
From table: $I = \frac{1}{2}MR^2$ for this shape so $I = (0.5)(0.380\ kg)(0.085\ m)^2 = 1.373 \times 10^{-3}\ kg\ m^2$.

(b) **the applied torque needed to accelerate it from rest to** 1500 rpm **in** 0.50 sec
$\tau = I\alpha$ so what is α here? In 5 seconds, the disk goes from rest to 1500 rev/min which represents an angular speed of $(1500\frac{rev}{min}) \times \frac{2\pi\ radians}{1\ rev} \times \frac{1\ min}{60\ s} = 157.1\ rad/s$. $\omega = \omega_o + \alpha t$ so $157.1 = 0 + (\alpha)(0.5\ s)$ so $\alpha = 314.16\ rad/s^2$. ($100\pi\ rad/s^2$ exactly, as it turns out.)
Finally, $\tau = I\alpha = (0.001373\ kg\ m^2)(314.16\ rad/s^2) = 0.43\ N\ m$. (The motor in a real grinding wheel would probably put out much more torque than that...)

Bringing in our earlier angular topics from this chapter:

(c) **How fast (in m/s) is the outer edge of the grinding wheel moving?**
$v = r\omega$ so here $v_{edge} = (0.085\ m)(157.1\ rad/s) = 13.35\ m/s$ (about 30 miles/hr).

(d) **What is the radial acceleration of a point on the outer edge of the grinding wheel?**
$a_r = v^2/r = r\omega^2 = (0.085\ m)(157.1\ rad/s)^2 = 2097\ m/s^2$ (about 214 g's)

(e) **How many revolutions did the wheel rotate through during this spin-up?**
One solution: use the angular equations of motion to find the angle θ the wheel has turned through, then convert that into revolutions: $\theta = \theta_o + \omega_o t + \frac{1}{2}\alpha t^2$ so $\theta = 0+0+(0.5)(314.16\ rad/s^2)(0.5\ s)^2 = 39.27\ rad$ and each 2π radians represents one rotation, so dividing by 2π yields 6.25 $revs$.
Another option: $\omega_{avg} = \Delta\theta/\Delta t$ so $\Delta\theta = \omega_{avg}\Delta t$ and let's just leave everything in units of revolutions. $(1500\frac{rev}{min}) \times \frac{1\ min}{60\ s} = 25\ revs/second$. The <u>average</u> angular speed will be $w_{avg} = \frac{1}{2}(\omega_o + \omega)$ so $\omega_{avg} = 12.5\ rev/sec$. Then $\Delta\theta = \omega_{avg}\Delta t = (12.5\ rev/s)(0.5\ s) = 6.25\ revs$.

(f) **How much energy is 'stored' in the rotating grinding wheel?**
$K_{rot} = \frac{1}{2}I\omega^2 = (0.5)(0.001373\ kg\ m^2)(157.1\ rad/s)^2 = 16.9\ J$

(g) **What average power must the motor be putting out here?**
$P_{avg} = (work)/(time) = (16.9\ J)/(0.5\ s) = 33.8\ W$

In real grinding wheel tools (well, in nearly any scenario really) there is some internal friction, so let's account for that:

Suppose we find that if we disconnect the motor from the wheel, it is seen to slow down from 1500 rpm to rest in 5 s.

(h) **How much frictional torque must be present?**
We can do the same process as above. $\tau = I\alpha$ but what is the α for this 'spin-down' phase? The wheel starts at $\omega_o = 157.1\ rad/s$ and ends at $\omega = 0$ after 5 sec so $\omega = \omega_o + \alpha t$ yields $\alpha = -31.4\ rad/s^2$.
$\tau = I\alpha$ so the <u>frictional</u> torque present here must be $\tau_{friction} = I\alpha = (0.001371\ kg\ m^2)(-31.4\ rad/s^2) = -0.0431\ N\ m$.

(i) **How much heat was generated as the wheel spun down**?
(I.e., how much work did this frictional torque do?) (SHORTCUT HERE!)
Once the motor is out of the picture, all the 16.9 J of (rotational) kinetic energy stored in the wheel is being removed solely by the frictional torque, so apparently it did $-16.9\ J$ of work, creating that much heat.

(j) **How much torque must the motor have really been putting out?**
$\sum \tau = I\alpha$ so $\tau_{motor} + \tau_{friction} = I\alpha$ so:
$\tau_{motor} - 0.0431 = (0.43\ N\ m)$ so $\tau_{motor} = 0.473\ N\ m$ (a little higher than we found before when we ignored friction)

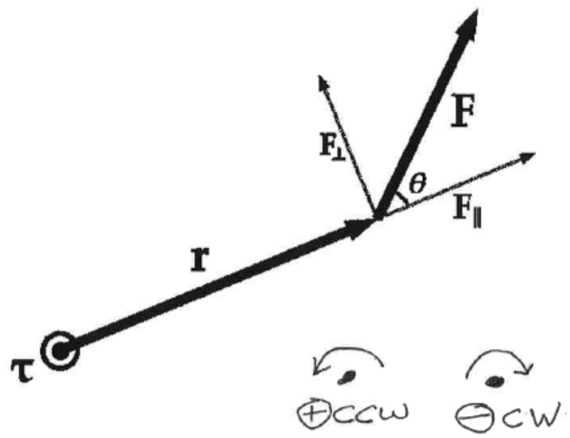
(k) **What average power was the motor putting out during the spinup phase?**
With friction included now, the work the motor did went into two buckets: most of it went into the kinetic energy of the rotating wheel, but it also had to do some additional work to account for the energy lost due to friction during the spinup. How much work did friction do during the spinup? $W = \tau\Delta\theta$ and the wheel turned through $\Delta\theta = 6.25\ revs = 39.27\ radians$ so the friction work during the spinup was $W_{friction} = \tau_{friction}\Delta\theta = (0.0431\ N\ m)(39.27\ rad) = -1.69\ J$. The motor had to do 16.9 J of work creating that much rotational kinetic energy in the wheel, plus another 1.69 J to account for the energy lost due to heat, so it did 18.59 J of work altogether. It did that in 0.5 sec, so the average power the motor was putting out was $P_{avg} = work/time = (18.59\ J)/(0.5\ s) = 37.2\ Watts$.

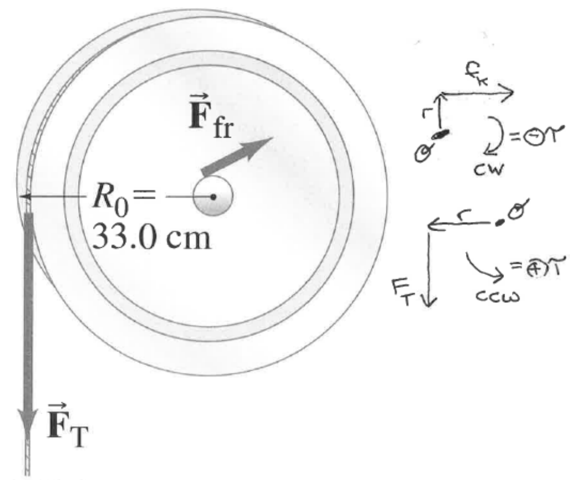
(l) **What instantaneous power did the motor have to be putting out right near the end of the spinup phase?**
$P = \tau\omega$ and the motor is putting out $(0.473\ N\ m)(157.1\ rad/s) = 74.3\ W$.
(If I'd kept enough signficant figures in all the intermediate result, this number should be exactly twice what it's average power was over the 5 second spinup...)

(m) **How about when the wheel is now turning at a constant** 1500 RPM**?**
Once the wheel is completely spun up, the motor only has to put out as much torque as friction is creating in the other direction since $\alpha = 0$ at this point. The motor torque is now just 0.0431 $N\ m$ (same magnitude as friction, just in the opposite direction) so the motor power now only needs to be $P = \tau\omega = (0.0431\ N\ m)(157.1\ rad/s) = 6.77\ Watts$ (and friction will be removing energy at the same rate, turning it into heat).

Review

- Rotational version of Newton's Laws: $\sum \tau = I\alpha$ where each $\tau = F_{tan} r$
  (all of those are found for a given axis of rotation)
- Signs: POSITIVE means counter-clockwise (CCW) about the axis of rotation
  NEGATIVE means clockwise (CW) about that axis





- Moment of Inertia: $I = \sum(m_i r_i^2)$ or $I = \int r^2 dm$
  (see table for common geometric shapes, rotating about particular axes)
- Rotational kinetic energy: $K_{rot} = \frac{1}{2} I\omega^2$ Work: $W = \tau\Delta\theta$
- Work-K and Conservation of Energy still viable, but potentially have **two** types of kinetic energy now: $K = K_{cm} + K_{rot} = \frac{1}{2} m v_{cm}^2 + \frac{1}{2} I\omega^2$

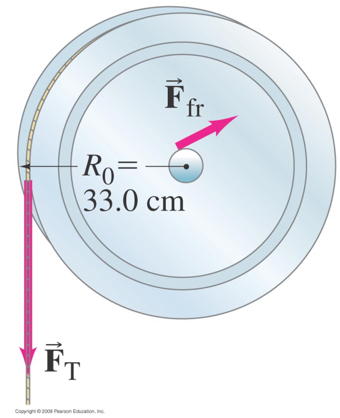
Angular Variables : rotation about a single axis

- $\omega_{avg} = \Delta\theta/\Delta t$ $\omega = d\theta/dt$ (angular velocity)
- $\alpha_{avg} = \Delta\omega/\Delta t$ $\alpha = d\omega/dt$ (angular velocity)

Angular Equations of Motion (requires constant (angular) acceleration):

- $\omega = \omega_o + \alpha t$
- $\theta = \theta_o + \omega_{avg}\Delta t$ $\theta = \theta_o + \omega_o t + \frac{1}{2}\alpha t^2$
- $\omega^2 = \omega_o^2 + 2\alpha\Delta\theta$

A 15 N force is applied to a cord wrapped around a pulley of mass $M = 4\ kg$ and radius $R_{outer} = 33\ cm$ and $R_{inner} = 3\ cm$. There is also a frictional torque at the axle of magnitude $|\tau_{friction}| = 1.1\ N\ m$.
If the system starts at rest, how fast will the pulley be turning after one complete rotation? ($\Delta\theta = 2\pi\ rad$)

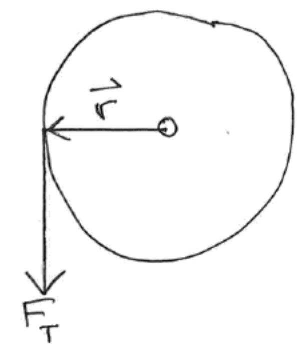


Moment of inertia for this geometry (from table) :

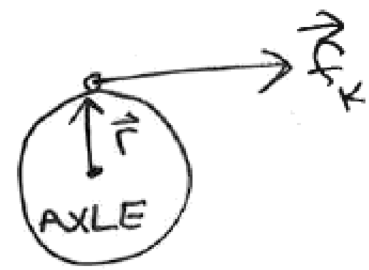
$$I = \frac{1}{2}M(R_{outer}^2 + R_{inner}^2) = (0.5)(4)(0.33^2 + 0.03^2) = 0.22345\ kg\ m^2$$

Let's first attack this using Work-K: $K_{final} = K_{initial} + \sum W$ where here the object is just rotating in place, so $K = \frac{1}{2}I\omega^2$ and we'll calculate all the works via $W = \tau\Delta\theta$.

Torque about the axis created by the tension in the cord. $\vec{r}$ points from the axis of rotation to the point where this force is being applied (right on the left edge of the pulley). $\vec{F}$ points in the direction of that force.
Magnitude: $|\tau| = rF_{tan} = (0.33\ m)(15\ N) = 4.95\ N\ m$
Direction: this force would cause a CCW (+) rotation, so $\tau = +4.95\ N\ m$.



Torque about the axis due to the frictional force. $\vec{r}$ points from the axis of rotation to the point where this force is being applied (the pulley is resting on the axle, so this would be the point shown in the figure). We found that the cord acting alone would cause the pulley to rotate counter-clockwise (due to being a positive torque), so the pulley material will be moving to the left scraping over the axle. That means $\vec{f_k}$ will be acting to the right on the pulley at that point.



They gave us the magnitude $|\tau| = 1.1\ N\ m$. Looking at the direction of f_k, a force in that direction would cause a **clockwise** angular acceleration, and CW is negative, so: the torque on pulley due to friction: $\tau_{f_k} = -1.10\ N\ m$.

(Note: Gravity $\vec{F_g}$ is also acting on the pulley but we'll see later why gravity isn't introducing any torque on the pulley here.)

Work done by pulley : $W = \tau\Delta\theta = (+4.95\ N\ m)(2\pi\ rad) = +31.102\ J$

Work done by friction : $W = \tau\Delta\theta = (-1.1\ N\ m)(2\pi\ rad) = -6.9115\ J$

$K_{final} = K_{initial} + \sum W = 0 + 31.102 - 6.9115 = +24.19\ J$.

$K = \frac{1}{2}I\omega^2$ and we found earlier that the moment of inertia of the pulley was $I = 0.22345\ kg\ m^2$ so $24.19 = \frac{1}{2}(0.22345)\omega^2$ from which $\omega = 14.7\ rad/s$.

Using Equations of Motion

We could also use $\sum \tau = I\alpha$ and equations of motion to solve this:

$\sum \tau = +4.95 - 1.10 = 3.85\ N\ m$ and $I = 0.22345\ kg\ m^2$ so $\alpha = (\sum \tau)/I = 17.23\ rad/s^2$.

Then: $\omega^2 = \omega_o^2 + 2\alpha\Delta\theta$ so $\omega^2 = (0)^2 + (2)(17.23)(2\pi) = 216.52$ from which $\omega = 14.7\ rad/s$.

How realistic is this?

The problem stated that the frictional force is producing a torque of magnitude $1.1\ N\ m$.

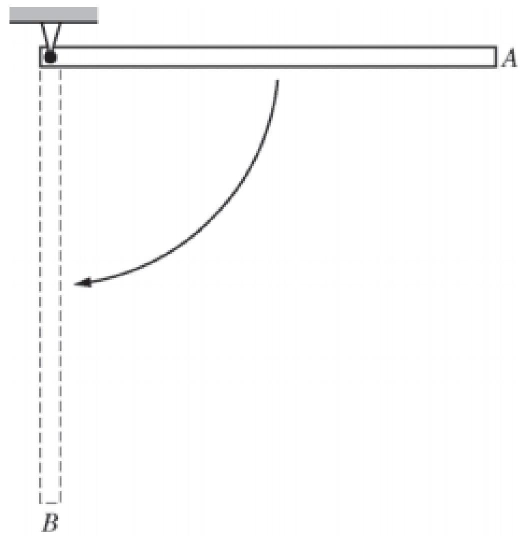
Well, $\tau = rF_{tan}$ and here the force is kinetic friction, so $\tau = rf_k = r(\mu_k F_N)$.

The disk is basically resting on the top of the axle, so looking at all the vertical forces, $\sum F_y = 0$ so $F_N - mg - F_{pull} =$ so $F_N = mg + F_{pull} = (4\ kg)(9.8\ m/s^2) + 15\ N = 54.2\ N$.

This force is being applied 3 cm from the axis of rotation, so $\tau = r\mu_k F_N$ means that $1.1 = (0.03)(\mu_k)(54.2)$ from which $\mu_k = 0.68$ which is a **lot** of friction.

A real-world pulley would likely have bearings or something to significantly reduce the friction. There will always be some friction, but the numbers given here are a stretch.

A meter-stick is suspended from one end and can rotate freely about that point. If it's released at rest in position A, what will be the angular speed of the ruler at position B? How fast will the far end of the ruler be moving (in m/s) at position B?



Note: model the 'ruler' as a long thin rod of mass M and length L, rotating about one of it's ends. Consulting the table of moments of inertia, we see $I = \frac{1}{3}ML^2$ for this geometry.
(See examples10-core.pdf for a similar 'falling pole' example.)

DETOUR : In a CoE sense here, the stick is changing elevation, but different parts are undergoing different changes. How do we deal with U_g when we have an extended object?

If we break the object up into an infinite number of infinitesimal pieces, with h being our vertical axis: $U_g = \sum m_i g h_i$ or $U_g = (\sum m_i h_i)g$ but $h_{cm} = \frac{\sum m_i h_i}{M}$ so apparently $\boxed{U_g = Mgh_{cm}}$. This is a **huge** help when it comes to dealing with extended (i.e. real) objects. It lets us essentially replace the object with a point mass located at its center-of-mass for some purposes (like determining U_g).

(We'll find this idea repeated a few times: often we can treat an extended object as if it were a point with the same mass but located at the CM of the object.)

The CM of the meter-stick will be exactly in the middle $L/2$. For purposes of U_g, let's measure all our heights relative to where the CM is located at position (B). Then:

$(K + U_g + U_s)_B = (K + U_g + U_s)_A + W_{other}$

We have no 'other' work here (just gravity), and no springs, and we're starting at rest, so:

$K_B + 0 + 0 = 0 + Mg(L/2) + 0 + 0.$

Now, what to do with the kinetic energy of the object... We could use $K_{cm} + K_{rot}$, as the CM of the object of certainly moving and then the stick is rotating about that point. It's simpler if we just compute K about the actual rotation axis instead of around the CM though for a problem like this. In that case, all of the kinetic energy of the object can be written as $K_{rot} = \frac{1}{2}I\omega^2$ **if** we let I be the moment of inertial about the sticks actual rotation axis (instead of about the CM). So here:

$\frac{1}{2}I\omega^2 = \frac{1}{2}MgL$

Consulting the table of moments of inertia, for this geometry we have $I = \frac{1}{3}ML^2$ so:

$\frac{1}{2}(\frac{1}{3}ML^2)\omega^2 = \frac{1}{2}MgL$ and after rearranging and cancelling some common terms, we arrive at $\omega = \sqrt{(3g/L)}$. Apparently the actual mass of the stick didn't matter - just it's length. (And for the $L = 1\ m$ case here, $\omega = 5.422\ rad/s$.

The **linear** speed of the point on the far end of the meter stick will be moving at $v = r\omega = (1\ m)(5.422\ rad/s) = 5.422\ m/s$.

Same Scenario, Different Solution (NOT recommended.)

As noted in the previous problem, we can look at the meter stick as either:

- every mass element rotating about the actual rotation axis
- CM is moving, and the meter stick is rotating about the CM

We used the first (simpler) approach on the previous page. Let's look at the second approach now (and see why it's usually avoided in situations like this).

In this approach we have two K terms: $K = K_{cm} + K_{rot}$

The CM of the object is right at it's midpoint, $L/2$ in from either end. The velocity of the CM then will be $v = r\omega$ so $v_{cm} = (\frac{L}{2})\omega$

Let's compute the kinetic energy terms in a way that uses ω as our variable, so we can compare to the previous solution.

Translational Kinetic Energy Term : We have $K_{cm} = \frac{1}{2}Mv_{cm}^2$. Replacing v_{cm} with $L\omega/2$, we end up with $K_{cm} = \frac{1}{8}ML^2\omega^2$. (That is the regular CM kinetic energy, but we're expressing it in terms of ω instead of v_{cm}.)

Rotational Kinetic Energy Term : we're now looking at the meter stick as rotating **about it's center of mass**, which means we need to use a **different moment of inertia** than we used before. For a long thin rod rotating about an axis through it's middle, $I = \frac{1}{12}ML^2$.

Thus: $K_{rot} = \frac{1}{2}I\omega^2 = \frac{1}{24}ML^2\omega^2$.

Finally combining both types of kinetic energy:

$$K = K_{cm} + K_{rot} = \frac{1}{8}ML^2\omega^2 + \frac{1}{24}ML^2\omega^2$$

which we can combine into: $K = \frac{1}{6}ML^2\omega^2$.

Following the previous page's approach, the initial U_g of $MgL/2$ is turning into this amount of K, so:

$\frac{1}{6}ML^2\omega^2 = MgL/2$ from which $\omega = \sqrt{(3g/L)}$, which is the same result we had before.

Summary

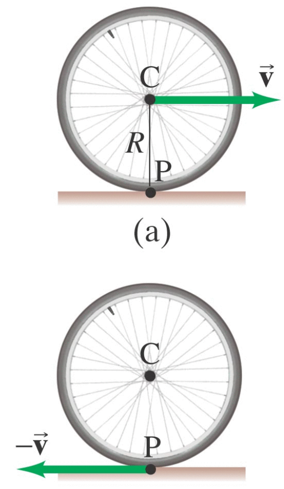
We can certainly treat rotation as the linear motion of the center of mass plus rotation **about** that center of mass, **but**:

When an object is rotating about a **fixed point**, it's almost always simpler to just treat the kinetic energy as entirely rotational, using the actual rotation axis that's present in the scenario. (That means using the proper moment of inertia: the one for the object rotating about the actual axis, not neccessarily the moment of inertial about it's CM. The parallel axis theorem $I = I_{cm} + Md^2$ may be needed to convert a moment of inertia from the table into one about the actual axis of rotation.)

Rolling without Slipping

When a ball or wheel **rolls** across a surface (without slipping), we can relate it's translational and rotational velocities:

It's center-of-mass speed and it's rotational speed are locked together:
$v = R\omega$



(a) (b)

Compute and compare the translational and rotational kinetic energies of a rolling bowling ball that has a translational (CM) speed of 8 m/s. ($M = 5\ kg$, $R = 0.21\ m$)

Translational : $K_{cm} = \frac{1}{2}mv^2 = (0.5)(5\ kg)(8\ m/s)^2 = 160\ J.$

Rotational : $K_{rot} = \frac{1}{2}I\omega^2$

Solid sphere: $I = \frac{2}{5}MR^2$ and $v = R\omega$ so $\omega = v/R$
Moment of inertia : $I = (0.4)(5\ kg)(0.21\ m)^2 = 0.0882\ kg\ m^2$
Angular speed : $\omega = v/R = (8\ m/s)/(0.21\ m) = 38.095\ rad/s$

$K_{rot} = (0.5)(0.0882\ kg\ m^2)(38.095\ rad/s)^2 = 64\ J$

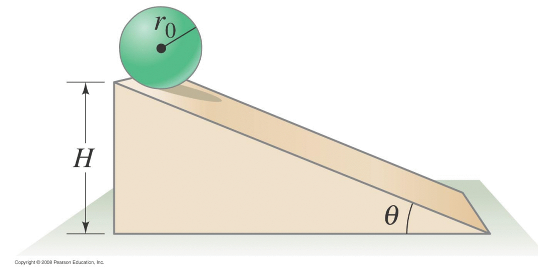
Symbolic Comparison :

$K_{rot} = \frac{1}{2}(\frac{2}{5}MR^2)(\frac{v^2}{R^2}) = \frac{1}{5}Mv^2 = \frac{2}{5} \times \frac{1}{2}Mv^2$ which we can write as: $\boxed{K_{rot} = \frac{2}{5}K_{cm}}$ so here $K_{rot} = (0.4)(160\ J) = 64\ J$ also.

In any event: Total $K = K_{cm} + K_{rot} = 224\ J.$

(Similar relationships between K_{rot} and K_{cm} can be derived for other geometric shapes.)

A basketball ($M = 0.65\ kg$, $R = 23\ cm$, hollow sphere) is released (at rest) at the top of a 30^o ramp. If it rolls (without slipping) down the ramp, how fast will it be moving after travelling $4\ m$ along the ramp? (Compare to a point mass sliding down a frictionless ramp.)



(see book and **examples10-core.pdf** for similar examples with a solid pool or bowling ball)

We'll use CoE again but this time the ball will have both translational and rotational kinetic energies.

The point where the ball touches the ramp is dropping vertically a distance H, and this is a solid ball that isn't deforming, so the CM (the geometric center of the ball) is also dropping vertically the same amount. Let's measure U_g relative to where the CM of the ball is when it reaches the bottom of the ramp. Then:

$(K + U_g + U_s)_{bottom} = (K + U_g + U_s)_{top} + W_{other}$

No other work here and no springs, and it's starting at rest at the top:

$K_{bottom} + 0 + 0 = 0 + MgH + 0 + 0$

BUT: $K = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$.

This is rolling without slipping, so $v = R\omega$ or $\omega = v/R$

Hollow sphere, so $I = \frac{2}{3}MR^2$. So:

$K_{bottom} = MgH$ becomes:

$\frac{1}{2}Mv^2 + \frac{1}{2}(\frac{2}{3}MR^2)(v/R)^2 = MgH$ or:

$\frac{1}{2}Mv^2 + \frac{1}{3}Mv^2 = MgH$. (The left term is the regular translational kinetic energy; the next term is the rotational kinetic energy converted into terms involving M and v instead of I and ω so we see that K_{rot} is 2/3 the value of K_{trans} : the rotational kinetic energy of a hollow sphere is a very significant contributor to the overall kinetic energy of the ball.

In any event, combining terms: $\frac{5}{6}Mv^2 = MgH$ and finally $v = \sqrt{\frac{6}{5}gH}$. (For our particular case, $H = (4\ m)\sin 30 = 2\ m$ so $v = 4.85\ m/s$: linear speed of the ball when it reaches the bottom.)

For comparison, suppose we had a point mass object of mass M sliding (without friction) down the same ramp? Then basically the $U_g = MgH$ at the top is all being converted into $K = \frac{1}{2}Mv^2$ at the bottom, so $MgH = \frac{1}{2}Mv^2$ or $v = \sqrt{2gH}$, which here would be $v = 6.26\ m/s$.

Notice that the **rolling** ball is moving slower and would take more time to get to the bottom, due to the fact that the original MgH of potential energy has to go into two buckets (both linear and rotational kinetic energy) this time.

(Also note that in the rolling case, neither the mass nor the actual size of the ball mattered! See examples10-core.pdf and the similar book example of a **solid** ball rolling down an incline. That gives a different result. The mass and size don't matter, but **how** that mass is distributed inside the sphere does (hollow vs solid).