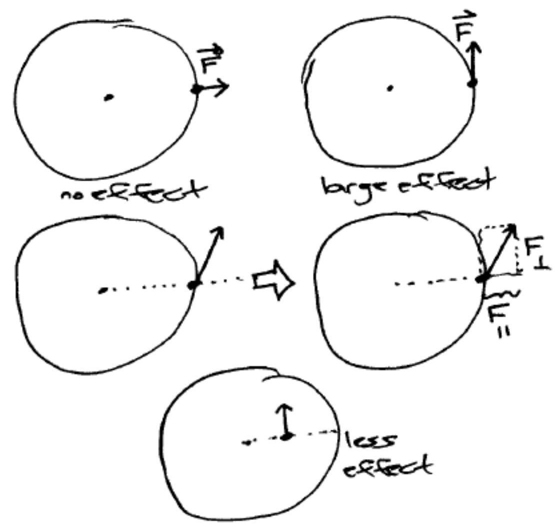
PH2213 Fox : Lecture 27
Chapter 11 : General Rotation (cross product)

Rotational Forces : Torque

Observation : Suppose we have bicycle wheel suspended on a stand so that the axle is fixed in place but the wheel can rotate about that point. We then apply the same magnitude of **force** at different **locations** on the wheel, and at different **angles**, and observe the resulting **angular acceleration** of the wheel.

- outer edge, radially (no effect)
- outer edge, tangent to wheel (considerable effect)
- outer edge but at an angle now: radial component has no effect; only the tangential component
- change where we apply the force now: move in close to the axis
- same force but closer in: smaller effect



The angular acceleration is proportional to r (how far out from the axis the force is applied), and is also proportional to the component of the force **tangent** (perpendicular) to $\vec{r}$.

Define the **torque** to be:

$\tau = rF_{tan} = rF \sin\phi$

where ϕ is the angle between the direction of $\vec{r}$ and the direction of $\vec{F}$.

Units: $N \cdot m$: $(newtons) \times (meters)$

English: $lb \cdot ft$ (sometimes $ft \cdot lb$)

Rotational analog to Newton's Laws:

$$\sum \tau = I\alpha$$

> NOTE: $(force) \times (distance)$ is also units of work (energy, joules) so torque and energy *technically* represent the same fundamental units, but by convention, torque is **never** written as having units of Joules.

That force, acting where it is, will then create rotation about an axis that is perpendicular to the plane that $\vec{r}$ and $\vec{F}$ are in.

Is there a vector mathematical operation that takes two vectors and produces a third vector with these desired properties (the given magnitude and direction we need)?

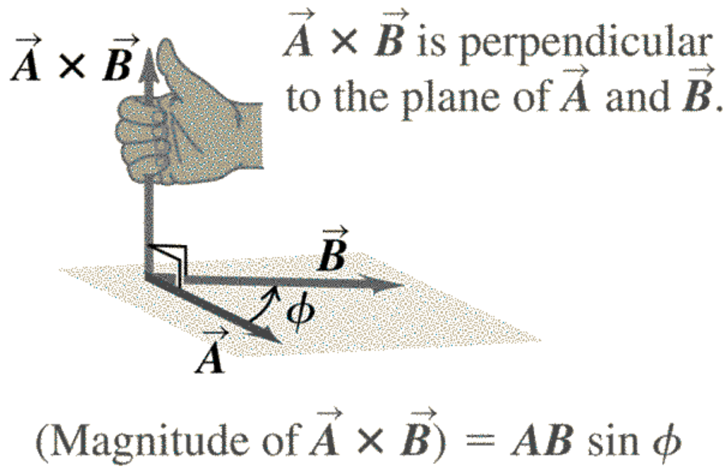
There is, and it allows us to write the torque equation compactly as: $\boxed{\vec{\tau} = \vec{r} \times \vec{F}}$.

Note the $\boxed{\times}$ symbol is NOT the same type of multiplication that we saw earlier with the DOT product, which used the $\boxed{\cdot}$ symbol (and created a SCALAR from two vectors).

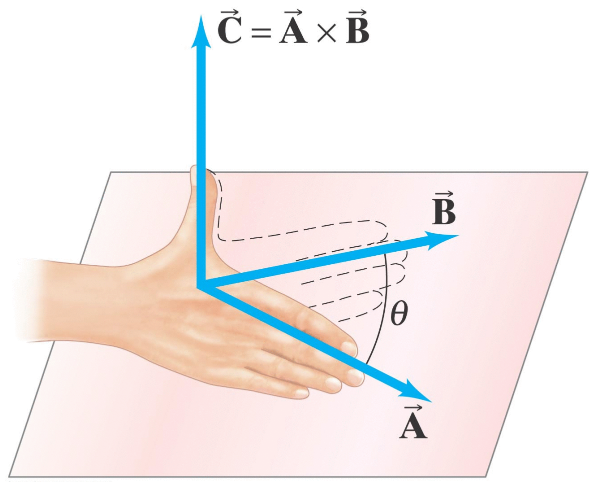
Cross Product

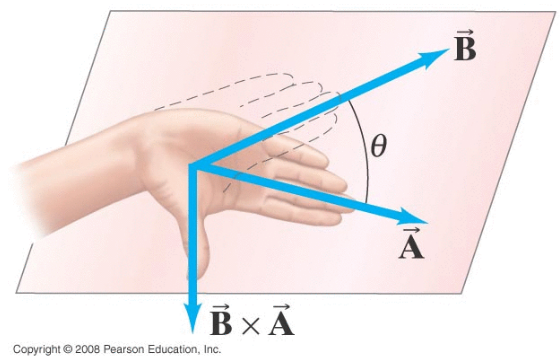
The cross product (or **vector product** of two vectors $\vec{A}$ and $\vec{B}$ produces a new **vector** with these properties:

- magnitude: $AB\sin\phi$
- direction: perpendicular to the plane defined by $\vec{A}$ and $\vec{B}$, with the direction found using the **right-hand rule** (starting at the first vector)

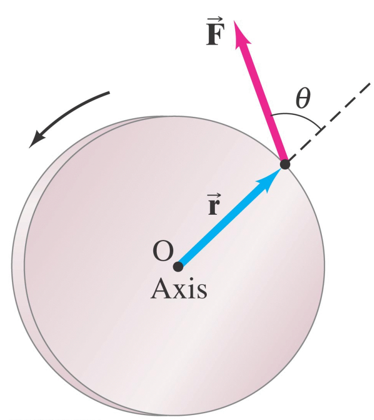


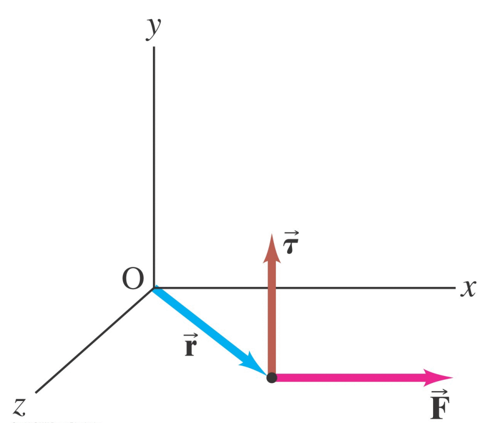
WARNING: this means that: $\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$, so we have to be careful to keep things in order (unlike 'normal' multiplication where the order doesn't matter).





Provides us with a more compact way of representing (and sometimes calculating) torque: $\vec{\tau} = \vec{r}\times\vec{F}$

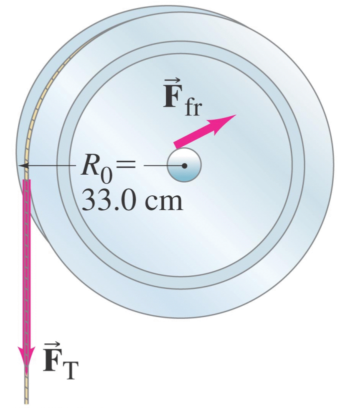




Torque RHR Sign and Angle Practice

Pulley (from earlier example) : a force of F_T is applied at the outer radius $R_o = 33\ cm$, and a frictional force of F_{fr} (I'd use the symbol f_k here...) is present at an inner radius of $R_i = 3\ cm$.

- Torque due to F_T : this force is tangent to the circle so it's entire magnitude will be F_{tan}. If that were the only force present, it would create a counter-clockwise rotation, so this will be a positive torque: $\tau_{F_T} = +(0.33\ m)(F_T)$.
- Torque due to the frictional force : this force is also tangent to the circle, so it's entire magnitude will be F_{tan}. If that were the only force present, it would create a **clockwise** rotation, so this will be a **negative** torque: $\tau_{F_{fr}} = -(0.03\ m)(F_{fr})$.



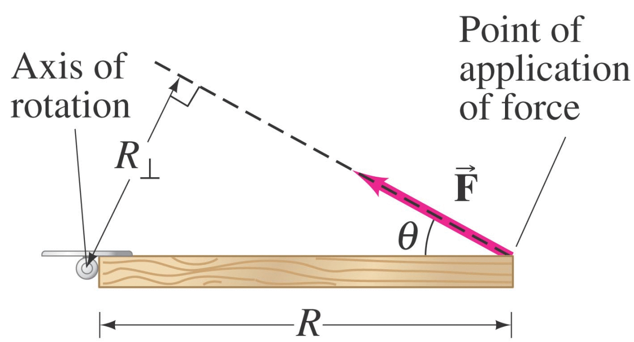
Shelf : a shelf is supported by a wire with some force $\vec{F}$ and the shelf is connected at the other end with a frictionless hinge, about which the shelf can rotate. Find the torque due to $\vec{F}$ about that axis.

The magnitude of the torque will be $\tau = rF\sin\phi$ where ϕ is the **angle between the directions of the two vectors**. (Figures like this are **why** I don't use θ in that equation since here θ isn't the angle we would directly use.)

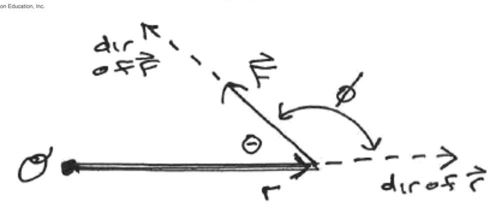
Extending each vector to make clear what the **directions** of each vector are, we see that $\phi = 180^o - \theta$.

Using the RHR, starting with our fingers pointed in the direction of $\vec{r}$, and curling towards the direction of $\vec{F}$, we find that we need to hold our hand so that our thumb is pointing up out of the page towards us. That's the $+Z$ (or $+\hat{k}$) direction, so this force will create a positive (counter-clockwise) torque.

$\tau = +rF\sin\phi$ (or if we want to use the original θ given in the figure, $\tau = +rF\sin(180 - \theta)$.

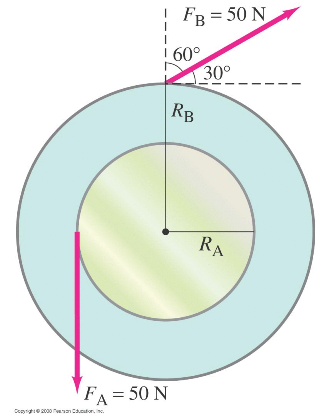


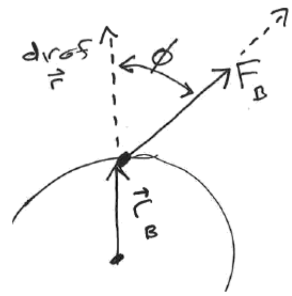
(a)



Another Pulley : axis of rotation at center of pulley

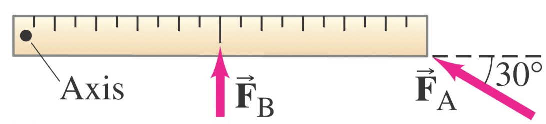
- Torque due to $F_A = 50\ N$: this force is being applied tangentially, so it's entire magnitude will be F_{tan}, and it's being applied R_A from the axis of rotation. That force alone would cause a CCW rotation, making it positive: $\tau_{F_A} = +(R_A)(50)$.
- Torque due to $F_B = 50\ N$ at an angle. Using the $|\tau| = rF\sin\phi$ approach this time, the magnitude of the torque will be $\tau = (R_B)(50\ N)\sin(60^o)$ and using the RHR (starting in the $\vec{r}$ direction and curling our fingers towards the $\vec{F}$ direction), our thumb is pointing down into the page, making this a negative torque: $\tau_{F_B} = -(R_B)(50)\sin(60^o)$.

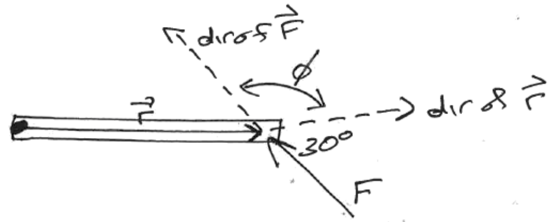




Ruler with Two Forces : axis of rotation at left end of ruler

- Torque due to F_B : this force is being applied perpendicular to the $\vec{r}$ vector from the axis to that point, so it's entirely tangential. Also, that force alone would create a CCW rotation, making this a positive torque. $\tau_{F_B} = +(r_b)(F_B)\sin(90^o) = +R_BF_B$.
- Torque due to F_A at an angle. Creating the 'direction of F' and 'direction of r' dotted lines, we see that the angle between those two directions will be $\phi = 180 - 30 = 150^o$. Using the RHR, our thumb ends up pointing down into the page, so this is a negative torque: $\tau_{F_A} = -(r_A)(F_A)\sin(150^o)$

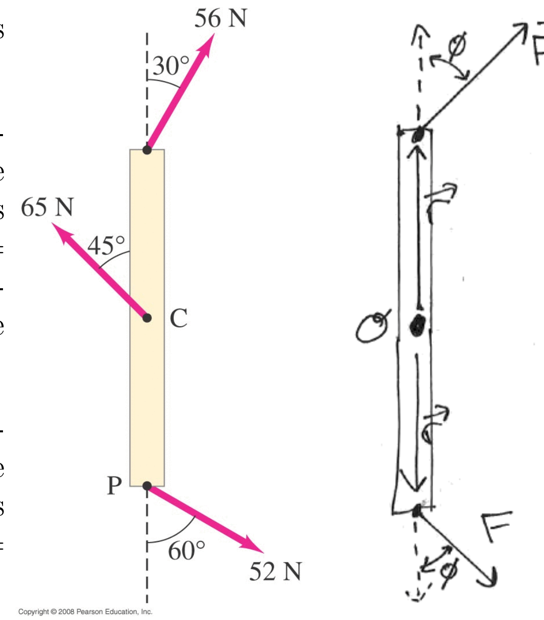




Pole with Three Forces : rotating about point C (the center of mass)

The pole is 2 m long, so the CM is 1 m in from each end.

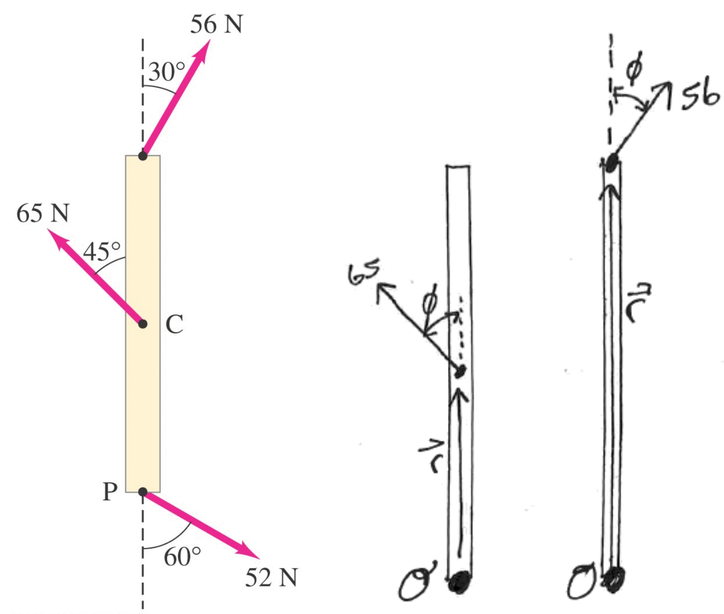
- Torque due to the 65 N force : ZERO since it's being applied right at the axis of rotation.
- Torque due to the 56 N force : this force is being applied 1 m from the axis. Looking at the drawing, ϕ here would be 30^o and the RHR says this will be a CW (negative) torque, so: $\tau_{56} = -(1\ m)(56\ N)\sin(30^o)$. (Note this time the angle they gave us turns out to be exactly the angle we need for ϕ.)
- Torque due to the 52 N force : this force is being applied 1 m from the axis. Looking at the drawing, ϕ here would be 60^o and the RHR says this will be a CCW (positive) torque, so: $\tau_{52} = +(1\ m)(52\ N)\sin(60^o)$.



Pole with Three Forces : rotating about point P (bottom of pole), this time.

The pole is 2 m long, so the CM is 1 m in from each end.

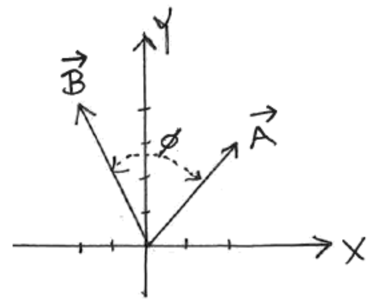
- Torque due to the 52 N force : ZERO since it's being applied right at the axis of rotation.
- Torque due to the 65 N force : this force is being applied 1 m from the axis at point P. Looking at the drawing, ϕ here would be 45^o and the RHR says this will be a CCW (positive) torque, so: $\tau_{65} = +(1\ m)(65\ N)\sin(45^o)$.
- Torque due to the 56 N force : this force is being applied 2 m from the axis. Looking at the drawing, ϕ here would be 30^o and the RHR says this will be a CW (negative) torque, so: $\tau_{56} = -(2\ m)(56\ N)\sin(30^o)$.



Example : Determine $\vec{C} = \vec{A} \times \vec{B}$ given:

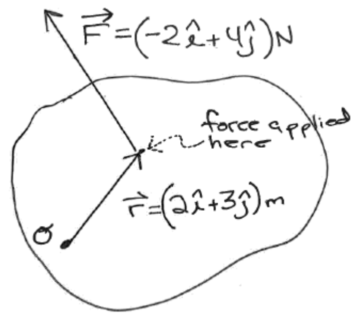
- $\vec{A} = 2\hat{\imath} + 3\hat{\jmath}$
- $\vec{B} = -2\hat{\imath} + 4\hat{\jmath}$

What is the angle between those two vectors?



This isn't just an artificial math example - it would be the exact process we would go through to determine the torque present on some object:

Suppose we have some object that is mounted so that it can rotate about an axis (labelled O in the figure). A force of $\vec{F} = (-2\hat{\imath} + 4\hat{\jmath})\ N$ is applied at a point whose vector location relative to the axis of rotation is $\vec{r} = (2\hat{\imath} + 3\hat{\jmath})\ m$
What torque does this represent?



The 'input' vectors here are in the (X,Y) plane and the cross-product creates a vector perpendicular to the plane defined by $\vec{A}$ and $\vec{B}$ so their cross product here **must** be entirely in the Z direction, and using the RHR (starting at A and curling towards B) we see if needs to be in the +Z direction, so that gives us a way to at least partly check our result:

$\vec{C} = \vec{A} \times \vec{B} = (2\hat{\imath} + 3\hat{\jmath}) \times (-2\hat{\imath} + 4\hat{\jmath})$.

We can use the usual FOIL method to expand out the product, but have to be **very** careful to keep everything in original order:

$\vec{C} = [(2\hat{\imath} \times (-2\hat{\imath})] + [(2\hat{\imath}) \times (4\hat{\jmath})] + [(3\hat{\jmath}) \times (-2\hat{\imath})] + [(3\hat{\jmath}) \times (4\hat{\jmath})]$

Looking at each term in order:

- $(2\hat{\imath} \times (-2\hat{\imath}) = (2)(-2)\hat{\imath} \times \hat{\imath} = 0$ since any vector 'crossed' with itself will be zero (remember the magnitude of the resulting vector involves the 'sine of the angle between the vectors', so if they're the same vector that angle will be zero and $sin(0) = 0$.)
- $(2\hat{\imath}) \times (4\hat{\jmath}) = (2)(4)\hat{\imath} \times \hat{\jmath} = 8\hat{k}$
  (RHR: Starting with fingers pointing towards the X axis, curl towards the Y axis; your thumb will be pointing in the $+Z$ direction, which is $\hat{k}$.)
- $(3\hat{\jmath}) \times (-2\hat{\imath}) = (3)(-2)\hat{\jmath} \times \hat{\imath} = (-6)(-\hat{k}) = +6\hat{k}$
  (RHR: Starting with fingers pointing towards the Y axis, curl towards the X axis; your thumb will be pointing in the $-Z$ direction, which is $-\hat{k}$.)
- $(3\hat{\jmath}) \times (4\hat{\jmath}) = (3)(4)\hat{\jmath} \times \hat{\jmath} = 0$ (same reason the first term was zero)

Finally: $\vec{C} = \vec{A} \times \vec{B} = 8\hat{k} + 6\hat{k} = 14\hat{k}$. (Entirely in the Z direction, as argued at the start.)

Let's use that to determine the **angle between the vectors** in the figure.

$C = |\vec{C}| = 14$ but it's also equal to $C = AB\sin\phi$ so:

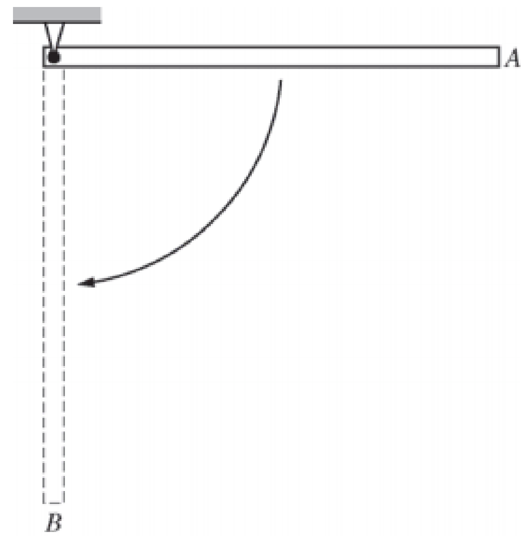
$A = |\vec{A}| = \sqrt{(2)^2 + (3)^2} = \sqrt{13}$

$B = |\vec{B}| = \sqrt{(-2)^2 + (4)^2} = \sqrt{20}$

Finally then: $C = AB\sin\phi$ so $14 = \sqrt{13}\sqrt{20}\sin\phi$ which leads to $\sin\phi = 0.8682...$ and finally $\phi = 60.3^o$ (which seems at least plausible, looking at the figure).

WARNING: the 'angle between the vectors' is always in the range from 0^o to $+180^o$, but the inverse sine function on your calculator always returns an angle between -90^o and $+90^o$. If you're interested in finding the angle between two (potentially 3-D) vectors, it's best to stick with the dot product. That involves the cosine function, and the inverse cosine will yield an angle in the 0 to 180 deg range, as needed.

A meter-stick is suspended from one end and can rotate freely about that point. If it's released at rest in position A, determine the (initial) angular acceleration of the stick.
(Also: show why α is NOT CONSTANT here, so we can't use angular equations of motion to determine the angular speed when the stick swings down to the vertical position. We were able to use CoE last time to determine that, at least.)



Note: model the 'ruler' as a long thin rod of mass M and length L, rotating about one of it's ends. Consulting the table of moments of inertia, we see $I = \frac{1}{3}ML^2$ for this geometry.

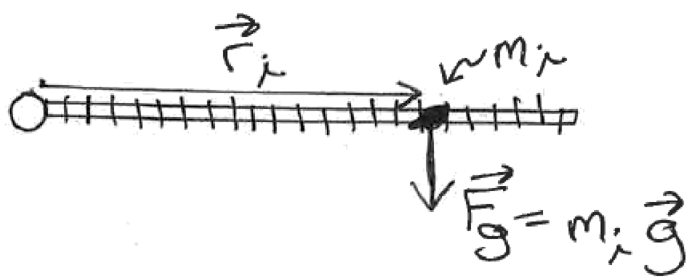
Our object is rotating about an axis coming up out of the page at the pivot point (the left end of the stick). Our angular acceleration and torque are technically vectors, but remember for rotation the vectorness is represented by the axis about which the rotation is occurring. So $\sum \vec{\tau} = I\vec{\alpha}$ means that the sum of all the torques about the Z axis is creating an angular acceleration about the Z axis, with the proportionality constant being the moment of inertia (about the Z axis also, of course).

Technically every force present may also be introducing a torque, since $\vec{\tau} = \vec{r} \times \vec{F}$. We have the force of gravity acting downward on the object, but we also have forces acting at the pivot the stick is rotating about. Fortunately, those forces are acting right at the axis of rotation, so when we try to calculate the torques they are creating, $\tau = rF\sin\phi$ and since these forces are right on top of the axis of rotation, $r = 0$ for them, meaning $\tau = 0$ also.

The only force affecting the rotation here is $\vec{F_g} = M\vec{g}$.

We don't have a point mass here: the mass of the stick is uniformly distributed all along it's length, and each of those mass elements is introducing it's own amount of torque.

If we break the object into a large number of tiny pieces of mass m_i located at some distance r_i from the axis of rotation, then each mass fragment is creating a torque of $\vec{\tau_i} = \vec{r_i} \times (m_i\vec{g})$.
The total torque that all of these mass fragments are creating then is:
$\vec{\tau} = \sum \vec{\tau_i} = \sum \vec{r_i} \times (m_i\vec{g})$.



We can shuffle the terms around here a bit:

$\vec{\tau} = \sum(m_i\vec{r_i}) \times \vec{g}$ and since $\vec{g}$ is constant, we can write that as $\vec{\tau} = (\sum m_i\vec{r_i}) \times \vec{g}$. That remaining sum though is just $M\vec{r}_{cm}$ so $\vec{\tau} = M\vec{r}_{cm} \times \vec{g}$ or finally:

Torque due to gravity : $\boxed{\vec{\tau} = \vec{r}_{cm} \times (M\vec{g}).}$

That means that the torque being created by that distributed mass is exactly the same torque that a **point mass located at it's CM** would create. (That's an EXTREMELY useful generic result that greatly simplifies a lot of statics and dynamics problems now and later on...)

Here, the CM of the meter stick is right at it's geometric center, $L/2$ out from the axis of rotation. $\vec{r}$ points directly to the right from the axis to that point. $\vec{F_g}$ points directly downward at that point. The angle between those two vectors is 90^o so the **magnitude** of the gravitational torque here would be $rF\sin\phi = (L/2)(Mg)\sin(90^o)$ or just $|\tau_{F_g}| = 0.5LMg$.

We can use the RHR to determine the sign. Starting with our fingers pointing from the axis out in the direction of $\vec{r}$, we curl our fingers down towards $\vec{F_g}$, and our thumb is now pointing **into** the page: this is a negative torque: $\tau_{F_g} = -0.5LMg$.

Finally then: $\sum\tau = I\alpha$ becomes $-0.5LMg = (\frac{1}{3}ML^2)\alpha$

We can cancel out M from both sides (and one factor of L) leaving us with: $\alpha = -\frac{3}{2}\frac{g}{L}$.

For an $L = 1\ m$ meter stick, $\alpha = -14.7\ rad/s^2$ (at least right at the beginning).

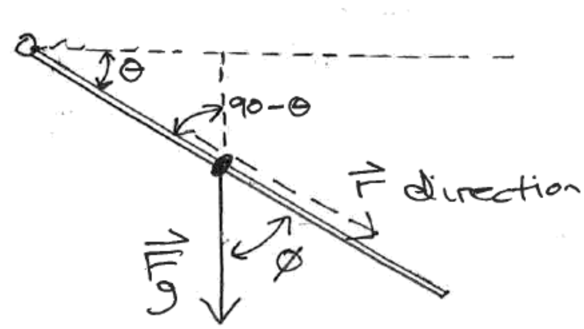
What would the linear tangential acceleration of a point on the far right end of the meter stick be?

$a_{tan} = r\alpha = (L)(-\frac{3}{2}\frac{g}{L})$ so here $\boxed{a_{tan} = -\frac{3}{2}g}$. That means that point on the free end starts off accelerating downward with an acceleration that's actually higher than g. If a stone or other small object were placed on that free end, the meter stick would accelerate downward faster than the stone would, leaving it behind. (Note that this result didn't depend on L, so any long thin object freely rotating about one end just due to the torque created by its weight has this feature.)

OK, so we have the angular acceleration at the instant the meter stick is released and it starts swinging around the pivot point. **Why can't we use angular equations of motion** to find, say, how fast it's moving after rotating through 90^o (which was the version of this scenario we analyzed via conservation of energy last time)?

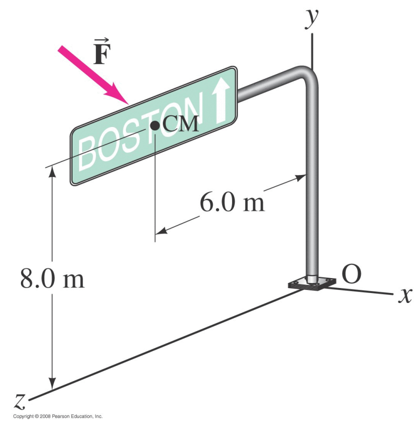
The problem is that the angular acceleration isn't constant here.

Let's look at the situation a little while later, when the meter stick has rotated through some angle θ. What is the gravitational torque now? $\tau_{F_g} = rF_g\sin\phi$ and from the figure we see that $\phi = 90^o - \theta$, so $\tau_{F_g} = -\frac{L}{2}Mg\sin(90-\theta)$. That means the amount of gravitational torque present depends on what angle the meter stick has rotated through. Since $\tau = I\alpha$, a non-constant torque means we have a **non-constant angular acceleration** also and thus **we can't use any of our rotational equations of motion**.



The only mechanism we have that can deal with non-constant accelerations is work-energy or conservation of energy (like we did with springs earlier). We can use those methods to determine how fast the meter stick is moving when it passes through the bottom (i.e. after rotating through 90 degrees) but we have no mechanism to determine how much **time** it took for the meter stick to get to that point...

Additional Example : An engineer estimates that under the most adverse expected weather conditions, the total force on the highway sign (see figure) will be $\vec{F} = (2400\hat{i} - 4100\hat{j})\ N$, acting at the CM. What torque does this force exert about the base O?
NOTE: this example uses a 'trick' for computing cross products that is useful for people who have had **matrices** and are familiar with a matrix operation called the **determinant**.



Before we start, let's think about that force for a moment. It has a (downward) vertical component that's probably the force of gravity pulling down on the sign. Wind blowing on the sign will push it too, with the maximum force occurring when the strongest wind happens to be blowing directly in the +X.

$\vec{\tau} = \vec{r} \times \vec{F}$ where $\vec{r}$ goes from the point about which we want to compute the torque to the point where the force is being applied.

In this case then, the $\vec{r}$ vector will go from the point O to the point labelled CM in the figure. That represents a 'displacement' of 6 in the +Z direction and 8 in the +Y direction, so we would write that vector as $\boxed{\vec{r} = 0\hat{i} + 8\hat{j} + 6\hat{k}}$ (with units of meters).

We have both $\vec{r}$ and $\vec{F}$ in unit-vector notation, and we can put these components into a matrix as shown below. Conveniently, if we take the determinant of this matrix (expanding it along the top row), the operations involved are identical to the various multiplications that appear in the cross product:

$$\vec{\tau} = \vec{r} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 8 & 6 \\ 2400 & -4100 & 0 \end{vmatrix}$$

or:

$$\vec{\tau} = \hat{i}[(8)(0) - (-4100)(6)] - \hat{j}[(0)(0) - (2400)(6)] + \hat{k}[(0)(-4100) - (2400)(8)]$$

$$\vec{\tau} = 24600\hat{i} + 14400\hat{j} - 19200\hat{k} \text{ (in units of } N\ m\text{).}$$

(Note : If we don't want the sign and pole to rotate at all, we'll need $\sum \vec{\tau} = 0$ (along with $\sum \vec{F} = 0$), which means that the various bolts and welds and whatever that are connecting the pole to the ground will need to exert a torque exactly the opposite of what we just found. That's a topic for Chapter 12 though!)