# PH2213 : Examples from Chapter 11 : Cross Product

The only topic we will use from this chapter is the definition of the vector cross product, since we need that to compute torques. The symbol $\boxed{\times}$ is used to indicate the vector cross product, to differentiate it from the scalar (or dot) product, which used a $\boxed{\cdot}$ symbol. These symbols may mean the same thing for regular scalar numbers but they are entirely different when we're talking about vectors.

The full vector definition of the torque created by some force $\vec{F}$ being applied to an object is $\vec{\tau} = \vec{r} \times \vec{F}$ where $\vec{r}$ is the position vector, starting from the axis of rotation and pointing to the spot where the force is being applied.

Suppose we have a bicycle wheel mounted vertically so it can spin freely around its axle, and let's call the plane of the wheel the XY plane. If we apply a tangential force to the edge of the tire, the wheel will start to spin but the axis of rotation will be in the Z direction, even though the vector force and the vector position where the force was applied are in the XY plane. The induced angular acceleration also depends on the angle between the position and force vectors. The cross product gives us a way to do this calculation automatically (needed if we're doing this in a spreadsheet or computer program).

Cross Product : The cross product basically takes two vectors and produces a third vector that is perpendicular to both the 'inputs' and has a magnitude that depends on the angle between them: If $\vec{C} = \vec{A} \times \vec{B}$, the figure at right illustrates how we can determine $\vec{C}$:
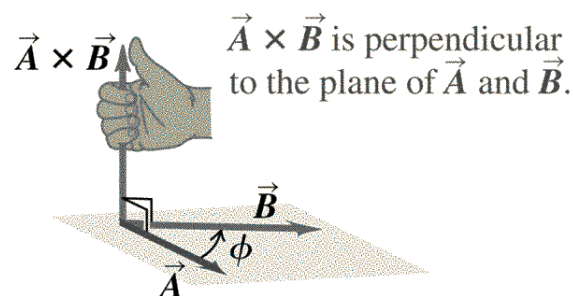
This basically illustrates the manual steps we go through, using the **right-hand-rule** (RHR) to determine the direction, and the magnitude is just $|C| = |A|\,|B|\,\sin\phi$. Just like the case with the dot product, that angle ϕ is the angle **between** the two vectors, **not** any other angle that might be in the figure.

$\vec{A} \times \vec{B}$ is perpendicular to the plane of $\vec{A}$ and $\vec{B}$.

(Magnitude of $\vec{A} \times \vec{B}$) = $AB \sin \phi$

How can we turn this into a direct numerical calculation? We know we can write vectors using the unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ so let's see what the cross products of those basic units vectors are.

$\hat{i} \times \hat{i}$: what happens if we take the cross product of any unit vector with itself? According to the definition above, the magnitude of the result will be the magnitude of the first vector (which will be just 1.0) times the magnitude of the second vector (also 1.0) times the sine of the angle **between** the two vectors. Well these are the same vector, so the angle between them is 0° and $\sin 0 = 0$ so that means **the cross product of any vector with itself is zero**. (And this applies to any vector, not just unit vectors.)

$\hat{i} \times \hat{j}$: the first vector is in the +X direction and the second vector is in the +Y direction. Their magnitudes are both 1 and the angle between them is 90° so the magnitude of this vector will be $(1)(1)\sin 90 = 1.0$ and how about the direction? Using the RHR, curling our fingers from the X axis towards the Y axis, we see our thumb is pointing in the direction of the +Z axis, so $\hat{i} \times \hat{j} = \hat{k}$.

$\hat{j} \times \hat{i}$: we've flipped the order now. The overall magnitude of the result will remain the same but the direction is reversed now. In order to be able to curl our fingers from the first vector to the second vector, we have to flip our hand over so that our thumb ends up pointing in the -Z direction.

We can go through all the unit vectors the same way:

$$\hat{i} \times \hat{i} = 0 \qquad \hat{i} \times \hat{j} = \hat{k} \qquad \hat{i} \times \hat{k} = -\hat{j}$$

$$\hat{j} \times \hat{i} = -\hat{k} \qquad \hat{j} \times \hat{j} = 0 \qquad \hat{j} \times \hat{k} = \hat{i}$$

$$\hat{k} \times \hat{i} = \hat{j} \qquad \hat{k} \times \hat{j} = -\hat{i} \qquad \hat{k} \times \hat{k} = 0$$

Example: Let $\vec{A} = 2\hat{i} + 3\hat{j}$ and $\vec{B} = 4\hat{j} + 5\hat{k}$. Find $\vec{C} = \vec{A} \times \vec{B}$.

$\vec{C} = (2\hat{i} + 3\hat{j}) \times (4\hat{j} + 5\hat{k})$

We can do the usual FOIL expansion here but we have to be very careful since the cross product is **not** commutative: we have to keep everything in order:

$\vec{C} = (2\hat{i}) \times (4\hat{j}) + (2\hat{i}) \times (5\hat{k}) + (3\hat{j}) \times (4\hat{j}) + (3\hat{j}) \times (5\hat{k})$

Now we can do all the individual unit vector cross products (using the table at the top of the page), producing:

$\vec{C} = (2)(4)\hat{k} + (2)(5)(-\hat{j}) + (3)(4)(0) + (3)(5)\hat{i}$

or: $\vec{C} = 8\hat{k} - 10\hat{j} + 15\hat{i}$

These are usually written in x-y-z order, so finally $\vec{C} = 15\hat{i} - 10\hat{j} + 8\hat{k}$.

Pseudo-Determinant : The operations involved in doing the cross product happen to correspond to the same operations that are used in finding the determinant of a matrix. If $\vec{A} = A_x\hat{i} + A_y\hat{i} + B_z\hat{k}$ and $\vec{B} = B_x\hat{i} + B_y\hat{i} + B_z\hat{k}$ then we can write $\vec{C} = \vec{A} \times \vec{B}$ as:

$$\vec{C} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

If we 'do' this determinant by **expanding across the first row**:

$\vec{C} = (A_yB_z - A_zB_y)\hat{i} - (A_xB_z - A_zB_x)\hat{j} + (A_xB_y - A_yB_x)\hat{k}$

or (applying the negative sign on the middle term):

$\vec{C} = (A_yB_z - A_zB_y)\hat{i} + (A_zB_x - A_xB_z)\hat{j} + (A_xB_y - A_yB_x)\hat{k}$

Using the same input vectors as before:

$$\vec{C} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 0 \\ 0 & 4 & 5 \end{vmatrix}$$

or: $\vec{C} = \hat{i}(3*5 - 4*0) - \hat{j}(2*5 - 0*0) + \hat{k}(2*4 - 0*3) = 15\hat{i} - 10\hat{j} + 8\hat{k}$