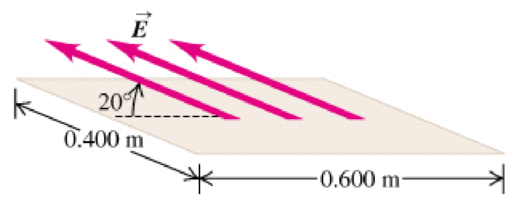
Chapter 22 Examples : Gauss's Law

Key concepts:

- Flux is a measurement of how much of something passes through a given area in a given time. This often appears in the context of fluid motion but can be extended to other fields, including electricity and magnetism.
- Electric Flux is defined as $\Phi_E = \vec{E} \cdot \vec{A}$, giving the flux in units of $(N/C) \cdot m^2$ or $N \cdot m^2/C$.
- Gauss's Law states that if we sum (or integrate) the flux over an entire **closed** surface, then this total flux **through** the closed surface is related to the charge contained **within** that surface: $\oint \vec{E} \cdot d\vec{A} = Q_{encl}/\epsilon_o$. (Note the little circle in the middle of the integration symbol: that represents integration over a complete closed surface.)
- By choosing the closed surface cleverly, we can use Gauss's Law to determine the electric field for various simple geometries.

A Simple Flux Calculation : A flat sheet is in the shape of a rectangle with sides of length 0.400 m and 0.600 m. The sheet is immersed in a uniform electric field of magnitude 75.0 N/C that is directed at 20^o from the plane of the sheet. Find the **magnitude** of the electric flux through the sheet. (Why can we not determine the sign of the flux?)



Here, we have a constant electric field cutting through a flat surface, so the flux is just: $\Phi = \vec{E} \cdot \vec{A}$ where $\vec{A}$ is the vector version of the area. $\vec{A}$ has a magnitude equal to the area of the rectangle, and a direction that is perpendicular to the 'sheet'. One of our forms for the dot product is $\Phi = \vec{E} \cdot \vec{A} = |E|\ |A|\ \cos\theta$ where θ is the angle between the two vectors involved. That is NOT the angle they gave us, of course. They gave us the angle between the electric field and the plane of the sheet, but the angle between the field and the vector version of the area will be 90 degrees minus this angle, or 70^o.

So finally: $\Phi = (75\ N/C) * (0.4\ m) * (0.6\ m) \cos 70^o = 6.156 Nm^2/C$.

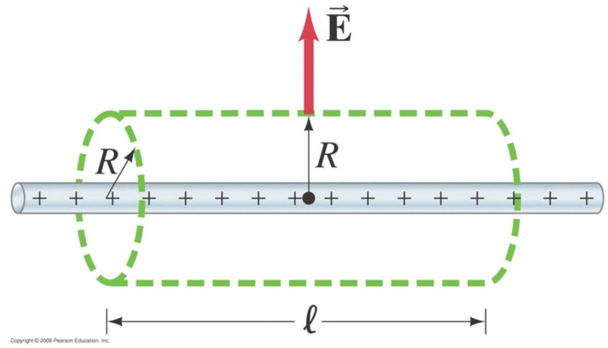
NOTE: why did they only ask us for the magnitude of the flux, and not the actual signed value?

$\Phi = \vec{E} \cdot \vec{A}$ which means we're treating both the electric field **and** the area as vectors. $\vec{A}$ is a vector whose magnitude is the area, and whose **direction** is a unit vector perpendicular to the plane formed by the area element. We have two choices though. If we think of the area as a sheet of paper, which side of the paper is the unit vector pointing out from? With an isolated area element like this, there's no right answer.

The only time (in this context anyway) where the direction is uniquely determined is in the case of applying Gauss's Law, which means we have a **closed surface** surrounding some charges. In that case, the direction of each little $d\vec{A}$ area element is **defined** to be **from the inside** of the surface **towards the outside**. The area elements in that case are all defined to be pointing 'outward'.

Applying Gauss's Law : Flux from a Line of Charge

It was shown in Example 22-6 (Section 22-3) in the textbook that the electric field due to an infinite line of charge is perpendicular to the line and has magnitude $E = \frac{\lambda}{2\pi\epsilon_0 r}$. Consider an imaginary cylinder with a radius of $r = 0.250\ m$ and a length of $l = 0.400\ m$ that has an infinite line of positive charge running along its axis. The charge per unit length on the line is $\lambda = 6.00 \mu C/m$.



(a) What is the electric flux through the cylinder due to this infinite line of charge?

(b) What is the flux through the cylinder if its radius is increased to $r = 0.500\ m$?

(c) What is the flux through the cylinder if its length is increased to $l = 0.800\ m$?

LONG WAY : From symmetry, the direction of the electric field will be perpendicular to the line (so radially outward or inward - outward in this case, since the charge is positive). Since the cylinder has the line of charge going right along its center axis, the area elements $d\vec{A}$ are also radial. Thus in the flux integral: $\Phi_E = \oint \vec{E} \cdot d\vec{A}$, we have field and area elements pointing in the same direction. The angle between them is zero, so the dot product simplifies to: $\Phi_E = \int E dA$, What about the part of the integral that involves the flat 'end caps' of the cylinder? The vector electric field is pointing radially outward but the $d\vec{A}$ area elements of these flat ends are perpendicular to that direction: they're pointing horizontally in the figure. Thus the dot product $\vec{E} \cdot d\vec{A}$ is **zero** for all the little area elements on those flat ends.

The electric field varies only with the distance from the charged line, so if our cylinder has a constant radius of r, the electric field magnitude is constant and can come out of the integral, leaving: $\Phi_E = E \int dA$, but the integral is just the definition of the area of the cylinder, which is $2\pi r l$, giving us an expression for the flux through the cylinder: $\Phi_E = 2\pi r l E$. Now, the electric field from an infinite line of charge is $E = \frac{\lambda}{2\pi\epsilon_o r}$ so substituting in this expression for E gives us our final symbolic $\Phi_E = 2\pi r l \frac{\lambda}{2\pi\epsilon_o r}$ or cancelling some common terms: $\boxed{\Phi_E = \lambda l/\epsilon_o}$.

SHORT WAY : According to Gauss's law, $\Phi_E = \oint \vec{E} \cdot d\vec{A} = Q_{encl}/\epsilon_o$. That is, the flux through the closed surface is just equal to the total charge enclosed. If the cylinder is of length l and has a charge/length of λ, then $Q_{encl} = \lambda l$ so $\Phi_E = \lambda l/\epsilon_o$ and we're done.

Our final symbolic derivation showed that, interestingly enough, the flux didn't depend on r at all: only on the charge density and the length of the cylinder. Parts (a) and (b) then have the same answer: $\Phi_E = \lambda l/\epsilon_o = (6.00 \times 10^{-6} C/m)(0.400\ m)/(8.854 \times 10^{-12} Nm^2/C)$ or $\Phi_E = 2.71 \times 10^5 Nm^2/C$.

For part (c), we have changed the length of the cylinder from $0.400\ m$ to $0.800\ m$, so we could just plug in this new value for l and recompute Φ_E, but we can short-cut the solution here. We doubled the length, thus exactly twice as much charge is enclosed, so the flux is exactly twice the earlier value: $\Phi_E = (2)(2.71 \times 10^5 Nm^2/C) = 5.42 \times 10^5 Nm^2/C$.

Applying Gauss's Law : Flux from Some Point Charges

A point charge $q_1 = 4.00nC$ is located on the x-axis at $x = 2.00\ m$, and a second point charge $q_2 = -6.00nC$ is on the y-axis at $y = 1.00m$. What is the total electric flux due to these two point charges through a spherical surface centered at the origin and with radius (a) 0.500 m, (b) 1.50 m, (c) 2.50 m?

Discussion: The total flux through the Gaussian surface is equal to the enclosed charge divided by ϵ_o. **It doesn't matter how the charge is distributed**: as long as it is **inside**, that's what counts. (The electric field lines can be quite complex, hitting the surface at odd angles, making the direct evaluation of the integral $\Phi_E = \int \vec{E} \cdot d\vec{A}$ **very** complex, but if the integral is over a closed surface (as they are in this example), then Gauss's Law says that the integral is just equal to $\Phi_E = Q_{encl}/\epsilon_o$.

HINT: draw the locations of these two charges on an xy graph and then add the various gaussian surfaces. The first one is so small that neither of the two charges are inside it. The second one is large enough that q_2 is inside but q_1 is not. The third one is large enough that both charges are inside.

(a) In this case, neither charge is located inside the sphere, so the total flux through this sphere is zero.

(b) In this case, the second charge is inside the sphere, but the first charge is outside, so $Q_{encl} = -6.00nC$ and therefore $\Phi_E = Q_{encl}/\epsilon_o = (-6.00\times10^{-9}C)/(8.854\times10^{-12}C^2/Nm^2) = -678Nm^2/C$.

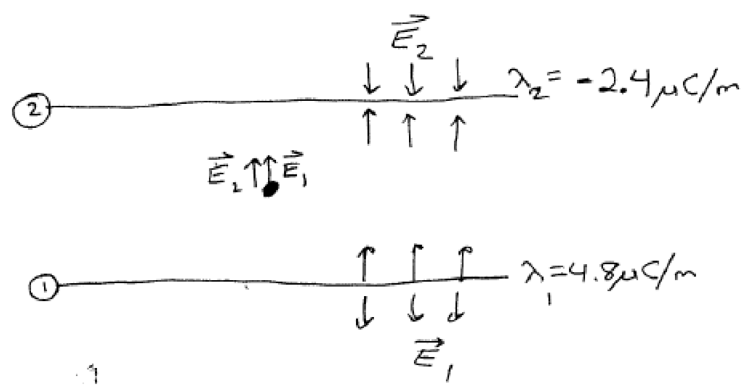
(c) In this case, both charges are inside the sphere, so $Q_{encl} = +4nC - 6nC = -2nC$, and the flux through the sphere will be: $\Phi_E = Q_{encl}/\epsilon_o = (-2.00 \times 10^{-9}C)/(8.854 \times 10^{-12}C^2/Nm^2) = -226Nm^2/C$.

Electric Field Between Two Lines of Charge

A very long uniform line of charge has charge per unit length $4.80\mu C/m$ and lies along the x-axis. A second long uniform line of charge has charge per unit length $-2.40\mu C/m$ and is parallel to the x-axis at $y = 0.400m$. What is the net electric field (magnitude and direction) at the following points on the y axis: (a) $y = 0.200\ m$ and (b) $y = 0.600\ m$?

Discussion: The electric field from an infinite line of charge is $\vec{E} = \frac{\lambda}{2\pi\epsilon_o r}\hat{r}$. For positive λ, this will give us something positive times $\hat{r}$, meaning that the field will be pointing radially outward from the line. For negative λ, this will give us something negative times $\hat{r}$, meaning that the field will be pointing radially inward towards the line.

(a) This point is located between the two lines. I've annotated the figure to show the directions of the electric fields created by each line: radially outward from line 1, and radially inward towards line 2. At the point of interest, then, **both** electric fields are pointing in the $+Y$ direction.

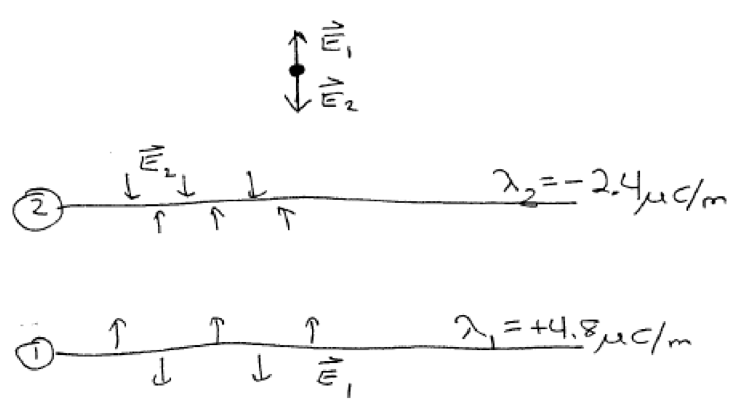


The magnitude of E_1 will be $|E_1| = \frac{|\lambda_1|}{2\pi\epsilon_o r}$ where r is the (positive) distance from the line to the point of interest. Here, the line is at $y = 0$ and the point is at $y = 0.200\ m$ so $r = 0.2$, giving us: $|E_1| = \frac{|4.8\times10^{-6}|}{2\pi 8.854\times10^{-12}(0.200)} = 431,412\ N/C$.

The magnitude of E_2 will be $|E_2| = \frac{|\lambda_2|}{2\pi\epsilon_o r}$ where r is the (positive) distance from the line to the point of interest. Here, the line is at $y = 0.400$ and the point is at $y = 0.200\ m$ so $r = (0.4 - 0.2) = 0.2$, giving us: $|E_2| = \frac{|2.4\times10^{-6}|}{2\pi 8.854\times10^{-12}(0.200)} = 215,706\ N/C$.

At the point of interest, these fields are both pointing upward, so the total electric field will be $(431,412\ N/C) + (215,706\ N/C) = 647,118\ N/C$ (in the $+Y$ direction).

(b) The point of interest is now located above the second line. I've annotated the figure to show the directions of the electric fields created by each line: radially outward from line 1, and radially inward towards line 2. At the point of interest, $\vec{E}_1$ will be pointing in the $+Y$ direction, but $\vec{E}_2$ will be pointing in the $-Y$ direction.



The magnitude of E_1 will be $|E_1| = \frac{|\lambda_1|}{2\pi\epsilon_o r}$ where r is the (positive) distance from the line to the point of interest. Here, the line is at $y = 0$ and the point is at $y = 0.600\ m$ so $r = 0.6$, giving us:

$|E_1| = \frac{|4.8\times10^{-6}|}{2\pi 8.854\times10^{-12}(0.600)} = 143,804\ N/C$.

The magnitude of E_2 will be $|E_2| = \frac{|\lambda_2|}{2\pi\epsilon_o r}$ where r is the (positive) distance from the line to the point of interest. Here, the line is at $y = 0.400$ and the point is at $y = 0.600\ m$ so $r = (0.6 - 0.4) = 0.2$, giving us: $|E_2| = \frac{|2.4\times10^{-6}|}{2\pi 8.854\times10^{-12}(0.200)} = 215,706\ N/C$.

At the point of interest, we can put the signs back into the situation, with E_1 having a positive y component and E_2 being negative, so the overall field here will be $(+143,804\ N/C)+(-215,706\ N/C) = -71,902\ N/C$, giving us a magnitude of $71,902\ N/C$ in the $-Y$ direction.

Electric Field Calculations for Various Geometries

(a) At a distance of 0.200 cm from the center of a **charged conducting sphere** with radius 0.100 cm, the electric field is 480 N/C. What is the electric field 0.600 cm from the center of the sphere?

We showed, using Gauss's Law, that a charged conducting sphere produces an electric field identical to that of a point charge located at the center of the sphere (as long as we are outside the sphere). So $E = kQ/r^2$. We're given the field at a given radius, so could use this to find the charge Q, then use that charge to find the field at the new location, but it's much easier to just do this as a ratio problem. Let r be the distance from the center of the sphere. Then the electric field at r_2 can be compared to the electric field at some other distance r_1 by: $E_2/E_1 = (kQ/r_2^2)/(kQ/r_1^2) = (r_1/r_2)^2$ or finally: $E_2 = E_1(\frac{r_1}{r_2})^2$. In this situation, we know $E_1 = 480\ N/C$ at $r_1 = 0.200\ cm$ and we want to find the field at $r_2 = 0.600\ cm$, so: $E_2 = (480\ N/C)(\frac{0.2}{0.6})^2 = 53.33\ N/C$. (Note that since we're dealing with the **ratio** of r_1 to r_2, we just left each value in units of cm, since the units cancel out.)

(b) At a distance of 0.200 cm from the axis of a **very long charged conducting cylinder** with radius 0.100 cm, the electric field is 480 N/C. What is the electric field 0.600 cm from the axis of the cylinder?

Following how we solved part (a), the electric field from a very long conducting cylinder acts just like a line charge located along the axis of the cylinder, for which the electric field is $E = \frac{\lambda}{2\pi\epsilon_o r}$ where r is the distance from this line charge to the point of interest. Again setting this up as a ratio, $E_2 = E_1\frac{r_1}{r_2}$ for this case, so $E_2 = (480\ N/C)\frac{0.2}{0.6} = 160\ N/C$.

(c) At a distance of 0.200 cm from a **large uniform sheet of charge**, the electric field is 480 N/C. What is the electric field 1.20 cm from the sheet?

Treating this as an infinitely large sheet, the electric field is $E = \frac{\sigma}{2\epsilon_o}$ which is **constant** and does not depend on how far we are away from the sheet. Thus the electric field at the new distance is the same as it was at the original distance: $E = 480\ N/C$ still.

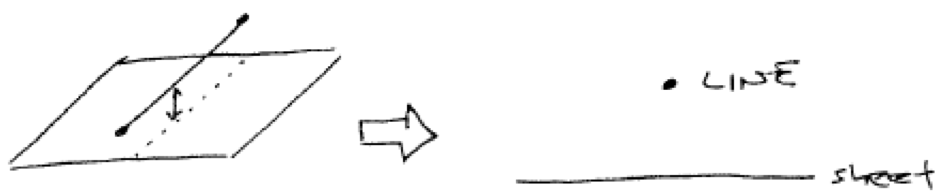
Equilibrium Point(s) for a line and sheet of charge

A long line carrying a uniform linear charge density $+50.0\mu C/m$ runs parallel to and 10.0 cm from the surface of a large, flat plastic sheet that has a uniform surface charge density of $-100\mu C/m^2$ on one side.

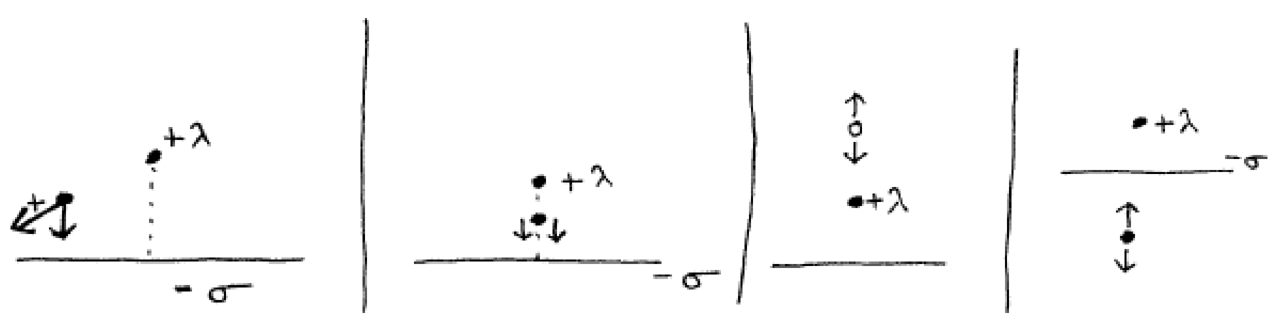
Find the location of all points where an α particle would feel no force due to this arrangement of charged objects.

An α particle is basically a bare helium nucleus, which has a positive charge. The force on any charged particle in an electric field is given by $\vec{F} = q\vec{E}$. We are looking for some magic location where the force is zero, so that is equivalent to finding a point where the total electric field is zero: i.e. where the (vector) electric field from the line charge and the (vector) electric field from the sheet of charge exactly cancel one another out.

Let's look at the situation from a perspective in line with the line charge:



There are several possibilities for where the field might be zero. In the left most figure below, we consider a point off to the side somewhere. Here, the electric field from the line charge will be radially outward from it (shown as the vector drawn at the angle). The electric field from the negatively charged sheet will be straight downward. There is **no possible location** where these two vectors can exactly cancel each other out, since the radial vector from the line charge will always have an x component that the electric field from the sheet charge does not have. Thus if any point exists, it must be along a line drawn through the line charge perpendicular to the sheet. That leaves us with three possibilities:



In the second picture, we are between the line and the sheet, but here both electric fields are pointing down, so cannot cancel each other. In the third picture, we are somewhere above the line. The electric field from the line is UP and the electric field from the sheet is DOWN, so here we might find a location where they cancel. In the fourth picture, we are somewhere below the sheet and here the electric field from the line charge is DOWN and the electric field from the sheet is UP, so again we could get cancellation.

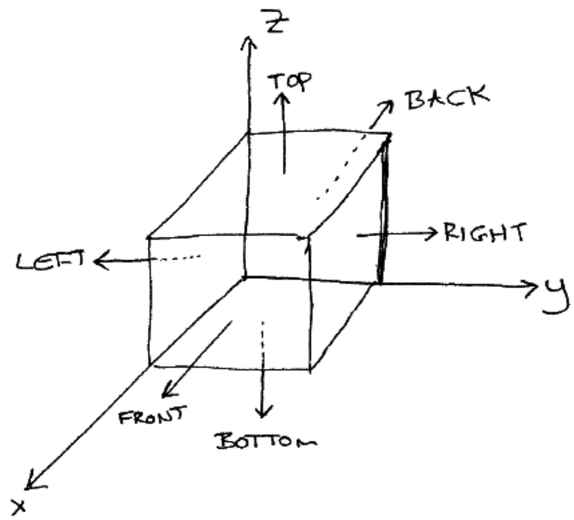
Now the electric field magnitude from a sheet of charge is constant while the electric field from a line depends on the distance r we are from the line, so we can solve both cases by setting the two field magnitudes equal: $|E_{line}| = |E_{sheet}|$ or: $\frac{|\lambda|}{2\pi\epsilon_o r} = \frac{|\sigma|}{2\epsilon_o}$ from which: $r = |\lambda|/(\pi|\sigma|)$ or $r = \frac{50\mu C/m}{\pi 100\mu C/m^2} = 0.159\ m$ or $15.9\ cm$.

At this distance directly above or below the line of charge, the two fields will cancel.

Flux due to a given (varying) electric field

Suppose we know that in some region, the electric field is given by $\vec{E} = (10x)\hat{\imath} + 5\hat{\jmath} + 20\hat{k}$ (with the constants having whatever units we need so that E comes out in the standard metric units of N/C). (Note that E_x varies with x, so this electric field is **not** constant.)

Compute the flux out through each side of a cube that is **3 meter** along each side, with one corner at the origin and the other sides aligned along the coordinate axes, as shown in the figure. The various sides are named in the figure.



This looks more complicated than it is. Recall in the earlier examples that when $\vec{E}$ is constant, $\Phi_E = \vec{E} \cdot \vec{A}$, and the dot product is essentially picking out just the component of $\vec{E}$ that is in the same direction as the area vector $\vec{A}$.

Here, we have a varying electric field, so will need to use the integral version of the flux: the flux through some surface can be computed as $\Phi_E = \int \vec{E} \cdot d\vec{A}$ but let's do whatever we can do avoid having to actually **do** any integrals...

Top surface : On this surface, the little $d\vec{A}$ area elements will be entirely in the Z direction, so will only have a $\hat{k}$ component. When we do the dot product inside the integral then, **only** the $\hat{k}$ component of $\vec{E}$ will survive. It's X and Y components will be gone so we can write this integral as just $\int E_z dA_z$. The Z component of E is constant though ($E_z = 20$), so it comes out of the integral: $\Phi_E = E_z \int dA_z$. $E_z = 20$ and the integral now just represents the area of the top. It has a magnitude of 9 square meters and it pointing in the +Z direction, so $A_z = +9$ and the integral just reduces to $\Phi_E = (20\ N/C)(+9\ m^2) = +180\ N\ m^2/C$.

Bottom Surface : In this case, when we break this surface into little $d\vec{A}$ elements, they're all pointing exactly in the negative Z direction, so again when we do the dot product with the electric field vector, the only term that will survive will be the Z component of $\vec{E}$: $\int \vec{E} \cdot d\vec{A} = \int E_z dA_z$. $E_z = 20$ which is constant, so comes out of the integral, leaving $\Phi_E = (20) \int dA_z$ but the integral is just the area of the bottom, which has a magnitude of 9 m^2 and it pointing in the negative Z direction, so $\int dA_z = -9\ m^2$.

The flux through the bottom surface then is just $\Phi_E = (20)(-9) = -180\ N\ m^2/C$.

Front Surface : In this case, the $d\vec{A}$ elements are all pointing in the positive X direction, so will only have components in the (positive) $\hat{\imath}$ direction and thus will only pick up the X component of the electric field along that surface. Any component of $\vec{E}$ in the Y or Z directions will be eliminated by the dot product. If we look at how $\vec{E}$ varies though, at $x = 3$, $E_x = 10x = (10)(3) = 30$ which is constant, so will come out of the integral again. $\Phi_E = \int \vec{E} \cdot d\vec{E} = \int E_x dA_x = E_x \int dA_x$. The integral represents the area of the front surface. As a vector, it's pointing in the +X direction, so $\int dA_x = +9\ m^2$ and finally the flux integral just reduces to $\Phi_E = (30\ N/C)(+9\ m^2) = +270\ N\ m^2/C$.

Back Surface : In this case, the $d\vec{A}$ elements are all pointing in the negative X direction, so will only have components in the (negative) $\hat{\imath}$ direction and thus will only pick up the X component of the electric field along that surface. Any component of $\vec{E}$ in the Y or Z directions will be eliminated by the dot product. If we look at how $\vec{E}$ varies though, at $x = 0$, $E_x = 10x = (10)(0) = 0$ and we can stop there since the integrand is zero, making the overall integral zero as well. The flux through the back surface is just zero.

Left surface : In this case, the $d\vec{A}$ elements are all pointing in the negative Y direction, so will only have components in the (negative) $\hat{\jmath}$ direction and thus will only pick up the Y component of the electric field along that surface. Any component of $\vec{E}$ in the X or Z directions will be eliminated by the dot product. Fortunately, $E_y = 5\ N/C$ which is **constant** so will come out of the integral, making this one trivial also: $\int \vec{E} \cdot d\vec{A} = \int E_y dA_y = E_y \int dA_y$ and the integral is just the (signed) area of that side, which will be $-9\ m^2$ so the integral reduces to $(5\ N/C)(-9\ m^2) = -45\ N\ m^2/C$.

Right surface : In this case, the $d\vec{A}$ elements are all pointing in the positive Y direction, so will only have components in the (positive) $\hat{\jmath}$ direction and thus will only pick up the Y component of the electric field along that surface. Any component of $\vec{E}$ in the X or Z directions will be eliminated by the dot product. Again, $E_y = 5\ N/C$ which is **constant** so will come out of the integral, making this one trivial also: $\int \vec{E} \cdot d\vec{A} = \int E_y dA_y = E_y \int dA_y$ and the integral is just the (signed) area of that side, which will be $+9\ m^2$ so the integral reduces to $(5\ N/C)(+9\ m^2) = +45\ N\ m^2/C$.

Summary

1. Flux through the top surface $\boxed{+180\ N \cdot m^2/C}$
2. Flux through the bottom surface $\boxed{-180\ N \cdot m^2/C}$
3. Flux through the back surface $\boxed{0\ N \cdot m^2/C}$
4. Flux through the front surface $\boxed{+270\ N \cdot m^2/C}$
5. Flux through the left surface $\boxed{-45\ N \cdot m^2/C}$
6. Flux through the right surface $\boxed{+45\ N \cdot m^2/C}$

Charge Enclosed

If we add up the flux through all six sides, we've created a **closed surface** so can use Gauss's Law to compute how much charge there must be inside the cube. $\oint \vec{E} \cdot d\vec{A} = Q_{enc}/\epsilon_o$ or $Q_{enc} = \epsilon_o \oint \vec{E} \cdot d\vec{A}$.

Adding up all the fluxes we just computed, the total flux is $180 - 180 + 0 + 270 - 45 + 45 = +270\ N\ m^2/C$ so or $Q = (8.854 \times 10^{-12})(270) = +2.4 \times 10^{-9}\ C$ or $+2.4\ nC$.

We can't say any more than that (like exactly **how** the charge is spread out), but at least we know that the total charge inside that cube is the given value.