

Chapter 30 Examples : Inductance

NOTE: this is being revised and expanded to include several examples related to **impedance** in AC circuits, which are missing entirely from this version. I'll let you know when the updated pdf is available.

Key concepts: (See chapter 29 also.)

$\xi_2 = -M dI_1/dt$: A changing current in a coil of wire (1) will induce an EMF in a second coil (2) placed nearby. The **mutual inductance** (M) is defined as the proportionality constant that relates these directly.

Mutual inductance of two solenoids of the same length and (essentially) the same cross sectional area, one inside the other: $M = \mu_o N_1 N_2 A/l$ where N_1, N_2 are the number of loops in each solenoid, A is their cross-sectional area, and l is their length.

$\xi = -L dI/dt$: within a **single coil**, a changing current induces an opposing EMF, so the coil has a 'self-inductance' which uses the symbol L instead of M.

Self-inductance of a solenoid of length l and number of turns N: $L = \mu_o N^2 A/l$

Energy stored in an inductor when a current I is flowing through it: $U = \frac{1}{2} LI^2$

L-R circuit : when an inductor L, a resistor R and a constant voltage source V_o are connected in series, the current will rise according to: $I = \frac{V_o}{R} (1 - e^{-t/\tau})$ where $\tau = L/R$, with the current eventually levelling out at $I = V_o/R$.

If the battery is suddenly switched out (shorted across, say), the energy stored in the inductor is released, producing a current $I(t) = I_o e^{-t/\tau}$

L-C circuit : if a charged-up capacitor is connected in series with an inductor, the current will oscillate with $I(t) = I_o \sin \omega t$ where $I_o = \omega Q_o$ and $\omega = 1/\sqrt{LC}$

L-R-C circuit : if a charged-up capacitor is connected in series with an inductor and a resistor, the resistor will gradually remove energy from the system, resulting in a damped motion. If R is low enough, the system will still oscillate, but at a different frequency than the L-C case. If R is high enough, the charge will just exponentially decline. (This section was not really covered, so there are no related examples.)

Simple Example 1

We want to create a 200 mH inductor by wrapping wire around the cardboard tube from the center of a toilet-paper roll. The tube is 10 cm long and has a diameter of 3 cm . How many turns of wire do we need to wrap around the tube?

We're basically creating a solenoid geometry here, which has a self-inductance of $L = \mu_o N^2 A / l$.

We're looking for N , so rearranging: $N = \sqrt{\frac{lL}{\mu_o A}}$.

Here, $l = 10 \text{ cm} = 0.10 \text{ m}$. The cross sectional area will be $A = \pi r^2$ and here $d = 3 \text{ cm}$ so $r = 1.5 \text{ cm} = 0.015 \text{ m}$ so $A = \pi(0.015)^2 = 7.07 \times 10^{-4} \text{ m}^2$, and we were given that $L = 200 \text{ mH} = 200 \times 10^{-3} \text{ H} = 0.2 \text{ H}$.

$$N = \sqrt{\frac{lL}{\mu_o A}} = \sqrt{\frac{(0.1)(0.2)}{(4\pi \times 10^{-7})(7.07 \times 10^{-4})}} = 4745.$$

Each loop of wire represents one diameter around the paper tube, so each loop will be $C = 2\pi r = (2)(\pi)(0.015 \text{ m}) = 0.0942 \text{ m}$ long. The total length of wire we would need then would be: $(4745)(0.0942 \text{ m}) = 446 \text{ m}$. (That's a lot of wire, so it will need to be pretty thin; we'll deal with that in the next problem.)

Simple Example 2

The thinner the wire, the higher the resistance, so assume we have a coil made from very thin wire that has an inductance of 440 mH , and a resistance of 3.25 Ω .

Suppose that at some point in time, the current is 3 A but is also increasing at 3.60 A/s . What will be the potential (voltage) difference across the coil at this moment?

We have two effects going on here. The 3 A current flowing through this 3.25 Ω resistance will result in a voltage drop of $V = IR = (3)(3.25) = 9.75 \text{ volts}$.

The increasing current will cause the flux in the coil to increase though, creating an EMF in the opposite direction of $\xi = -LdI/dt = -(440 \times 10^{-3})(3.60 \text{ A/s}) = -1.584 \text{ V}$.

Following the direction the current is flowing in, the voltage drops across the resistor. Since the current is **increasing** in that direction (causing the flux to increase), the EMF induced will be in the opposite direction: it's dropping in the direction the current is flowing, meaning it's also a decrease in voltage. Overall then, the total decrease in voltage across the inductor will be $-9.75 - 1.584 = -11.33 \text{ V}$.

Energy Stored in Inductors (1) : Suppose we have a solenoid that is 38 *cm* long and 2.1 *cm* in diameter, and consists of 10000 turns of wire. When a (constant) current of $I = 10 \text{ A}$ is flowing through it, how strong is the magnetic field inside the solenoid and how much energy is it storing?

(a) The magnetic field in a solenoid is $B = \mu_o NI/l$ so here, taking care to convert everything to standard units: $B = (4\pi \times 10^{-7})(10000)(10)/(0.38) = 0.33 \text{ T}$. (Remember, 1 *T* = 10,000 *gauss* so this is 3300 *gauss* or about 6000 times stronger than the Earth's magnetic field. This would be a very strong field.)

(b) The energy stored in an inductor is $U = \frac{1}{2}LI^2$, so we'll need to determine L for this geometry.

For a solenoid, $L = \mu_o N^2 A/l$. The cross sectional area here is $A = \pi r^2$ and we have $d = 2.1 \text{ cm}$ so $r = 1.05 \text{ cm} = 0.0105 \text{ m}$ so $A = (\pi)(0.0105 \text{ m})^2 = 3.46 \times 10^{-4} \text{ m}^2$.

Putting things together then: $L = \mu_o N^2 A/l = (4\pi \times 10^{-7})(10000)^2(3.46 \times 10^{-4})/(0.38) = 0.1144 \text{ H}$ or 114.4 *mH*, so finally $U = \frac{1}{2}LI^2 = (0.5)(0.1144)(10)^2 = 5.72 \text{ J}$.

Energy Stored in Inductors (2) : The magnetic field inside an air-filled solenoid is measured to be 0.6 *T*. The solenoid is 38 *cm* long and has a diameter of 2.1 *cm* but here we don't know the number of turns or the current. How much energy is this inductor holding?

We're missing some of the bits we used in the previous example, but we do have the magnetic field this time, so maybe that will be enough.

For this geometry, we have: $B = \mu_o NI/l$ and $L = \mu_o N^2 A/l$ and we want to find $U = \frac{1}{2}LI^2$.

We don't know what the current is, so let's try to eliminate I from that equation. We can rearrange the B equation to solve symbolically for I : $I = \frac{Bl}{\mu_o N}$. In our equation for U , if we replace L with $L = \mu_o N^2 A/l$, and replace I with what we just found:

$$U = \frac{1}{2}LI^2 = \frac{1}{2}\left(\frac{\mu_o N^2 A}{l}\right)\left(\frac{Bl}{\mu_o N}\right)^2$$

After expanding and cancelling common terms, we end up with: $U = \frac{1}{2} \frac{AB^2 l}{\mu_o}$.

$(A)(l)$ is just the **volume** of the solenoid, so this is: $U = \left(\frac{1}{2} \frac{B^2}{\mu_o}\right) \times (\text{volume})$.

If we divide both sides by the volume, the left hand side becomes $(U)/(\text{volume})$ which is usually written with a lower-case u and represents the **energy density** (J/m^3) so we've arrived at $u = \frac{1}{2} \frac{B^2}{\mu_o}$.

This gives us a short-cut now. If we know the strength of the magnetic field, we know the energy density present, so all we need to do is multiply by the volume involved.

Here, $B = 0.6 \text{ T}$ so $u = \frac{1}{2} \frac{(0.6)^2}{4\pi \times 10^{-7}} = 143,240 \text{ J/m}^3$.

The volume contained in the solenoid is $(\text{volume}) = (A)(l) = \pi r^2 l = \pi(0.0105 \text{ m})^2(0.38 \text{ m}) = 1.316 \times 10^{-4} \text{ m}^3$ so the actual energy would be $U = (u)(V) = (143240 \text{ J/m}^3)(1.316 \times 10^{-4} \text{ m}^3) = 18.9 \text{ J}$.

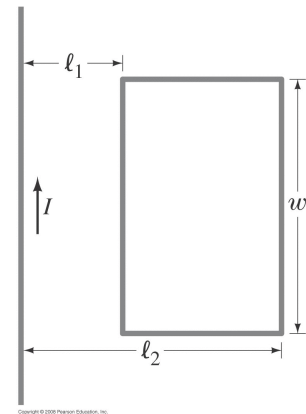
(As a check, we can redo part of the previous problem. In that case, where we had the same physical dimensions but also knew the current and the number of turns of the wire, we found that $B = 0.33 \text{ T}$, which leads to an energy density $u = \frac{1}{2} \frac{B^2}{\mu_o} = \frac{1}{2} \frac{(0.33)^2}{4\pi \times 10^{-7}} = 43,433 \text{ J/m}^3$. Multiplying by the volume we just found yields $U = 5.72 \text{ J}$ (same as we found before).

Inductive Charging : Loop of wire near a current

(a) A long straight wire and a small rectangular wire loop lie in the same plane. Determine the mutual inductance in terms of l_1 , l_2 , and w . (Assume the wire with the main current I is very long compared to the other dimensions given.) **Warning** : bad choice of symbols here; don't confuse w with ω that will appear later.

The text in the 'assume...' part means that we can say that the magnetic field created by the long current-carrying wire will be $B = \frac{\mu_o I}{2\pi r}$.

The magnetic flux through the loop then will be: $\Phi_B = \int \vec{B} \cdot d\vec{A}$. The current is flowing in the direction shown, so the magnetic field vectors in the loop will be going into the page. Let's choose the \vec{A} direction to also be into the page.



Choosing a coordinate system with X horizontally to the right and Y parallel to the current and pointing towards the top of the figure, $dA = dx dy$ so the flux integral becomes:

$$\Phi_B = \frac{\mu_o I}{2\pi} \int \frac{1}{x} dx dy.$$

Nothing changes in the Y direction, so the integral in that direction just produces w . The X integral produces $\ln(x)$ evaluated between the two end-points, so ultimately:

$$\Phi_B = \frac{\mu_o I w}{2\pi} \ln(l_2/l_1), \text{ so: } \xi = -d\Phi_B/dt = -(\frac{\mu_o w}{2\pi} \ln(l_2/l_1)) dI/dt$$

But the mutual inductance is defined via $\xi = -MdI/dt$ so we can pick off the result directly:

$M = \frac{\mu_o w}{2\pi} \ln(l_2/l_1)$ (and if everything is measured in standard metric units (meters, kg, sec, etc) then M will have units of 'Henries')

(b) Inductive chargers can use geometries like this to transfer power into devices like phones. Suppose the l and w parameters are 'phone-sized' (a few centimeters) and the wire is carrying a current of 10 A. What frequency does the current need, to induce 10 volt of EMF in the loop?

Suppose $w = 4 \text{ cm} = 0.04 \text{ m}$, $l_1 = 0.2 \text{ cm} = 0.002 \text{ m}$ and $l_2 = 4 \text{ cm} = 0.04 \text{ m}$. Then $M = \frac{\mu_o w}{2\pi} \ln(l_2/l_1) = \frac{(4\pi \times 10^{-7})(0.04)}{2\pi} \ln(0.04/0.002) = 2.4 \times 10^{-8} \text{ Henries}$

The induced EMF then will be $\xi = -MdI/dt$. If the current is varying with some (angular) frequency ω , then we can write $I = I_o \cos(\omega t)$ and $dI/dt = -I_o \omega \sin(\omega t)$ resulting in an induced voltage of $\xi = MI_o \omega \sin(\omega t)$.

We want the amplitude of this EMF to be 10 volt and the supply current is $I_o = 10 \text{ A}$ so: $10 = (2.4 \times 10^{-8})(10)(\omega)$ leading to $\omega = 4.2 \times 10^7 \text{ s}^{-1}$.

The 'regular' frequency is related to the angular frequency: $\omega = 2\pi f$ so $f = \omega/(2\pi)$ which here would be $f = 6.6 \times 10^6 \text{ Hz} = 6.6 \text{ MHz}$.

Real inductive chargers can have more than one loop (remember, each loop will result in the given EMF, so if we had 10 loops we'd create 10 times as much EMF in the daughter circuit), or could allow the same EMF to be created at 10X lower frequency in the supply circuit. (The 'Qi' wireless charger uses a frequency between 110 kHz and 205 kHz, apparently, which is about 30X lower than the one we derived, but the same EMF could be generated by using a loop with 30 turns, for example.)

L-R circuit : lighting effect

Suppose we want to create an effect where when we turn on a light switch, the light gradually goes from dark to full brightness over a short time period, instead of coming on abruptly. Let's say we want the light to reach 90% of its full power after 0.5 sec. Other details we'll need: suppose the bulb itself has a resistance of 2Ω and is connected to a 12 V battery. What inductor (L) do we need to stick in series with the lightbulb to achieve this effect?

In an L-R circuit, the current flowing will be $I(t) = \frac{V_o}{R}(1 - e^{-t/\tau})$ where $\tau = L/R$, with the current eventually levelling out at $I = V_o/R$.

Here, the final current flowing would be $I_{final} = (12 V)/(2 \Omega) = 6 A$ at which point the light bulb is putting out a power of $P = I^2 R = (6)^2(2) = 72 W$ (or $P = V^2/R = (12)^2/(2) = 72 W$).

We want the power to be 90% of this value at $t = 0.5 \text{ sec}$ so we want $P = (0.9)(72 W) = 64.8 W$ at that time.

That means we'll need the current to be: $P = I^2 R$ so $(64.8) = (I)^2(2)$ from which $I = 5.692 A$ at $t = 0.5 s$.

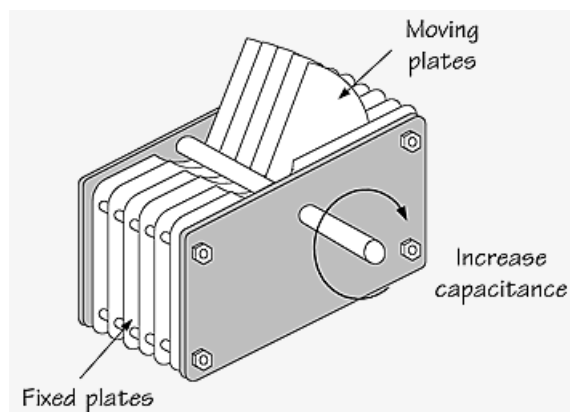
$I(t) = \frac{V_o}{R}(1 - e^{-t/\tau})$ so here we have:
 $5.692 = 6.0(1 - e^{-t/\tau}) = 6 - 6e^{-t/\tau}$ where $t = 0.5$.

Rearranging a bit, we have $e^{-t/\tau} = (6 - 5.692)/6 = 0.05132$ and taking the natural log of both sides:
 $-t/\tau = -2.97$ or $\tau = t/2.97 = (0.5)/(2.97) = 0.168 s$.

The time constant $\tau = L/R$ so here $(0.168) = (L)/(2)$ or $L = 0.337 H$.

L-C circuits : radio tuning

We found that L-C circuits have a natural oscillation frequency, an effect that is exploited in radio frequency tuners. Here, the capacitor is usually a series of metal plates that are interleaved in such a way that you can manually alter the effective area of the plates (see figure).



Suppose that a capacitance C of 1350 pF is needed to tune in an AM radio station operating at $f = 550 \text{ kHz}$.

- (a) What must the inductance be in this circuit?

The natural frequency of an L-C circuit is $\omega = 1/\sqrt{LC}$ where $\omega = 2\pi f$ and we have C so let's rearrange to solve for L :

$L = \frac{1}{\omega^2 C}$ and we know $C = 1350 \times 10^{-12} \text{ F}$ and $\omega = (2)(\pi)(550,000 \text{ s}^{-1})$ which leads to $L = 6.21 \times 10^{-5} \text{ H}$ or $L = 62.1 \mu\text{H}$.

- (b) What capacitance C is needed if we want to tune in to a radio station operating at $f = 1600 \text{ kHz}$?

We can solve this independently without even knowing L .

The natural frequency of an L-C circuit is $\omega = 1/\sqrt{LC}$ so rearranging to solve for C :

$C = \frac{1}{\omega^2 L}$. That means that C is inversely proportional to ω^2 or that $(C)(\omega^2)$ will be constant (and equal to $1/L$). Thus:

$$(C_{new})(\omega_{new})^2 = (C_{old})(\omega_{old})^2$$

$$\text{or } C_{new} = C_{old} \left(\frac{\omega_{old}}{\omega_{new}} \right)^2$$

$$\text{Here then: } C_{new} = (1350 \text{ pF}) \left(\frac{550 \text{ kHz}}{1600 \text{ kHz}} \right)^2 = 159.5 \text{ pF}.$$

The frequency we wanted to listen to went up by a factor of $(1600/550) = 2.91$ which required the capacitance to go **down** by a factor of $(2.91)^2 = 8.46$.

Note that to tune to a higher radio frequency, we needed to (significantly) reduce the capacitance. For a parallel-plate capacitor, recall we had $C = \epsilon_o \frac{A}{d}$. The plate-separation distance (d) is usually fixed, so to lower the capacitance, we need to (significantly) reduce the area of the plates. In the context of the figure, we'd turn the knob counter-clockwise (opposite to the arrow direction shown in the figure) which would swing the movable plates out from the fixed plates until just a small fraction of their area overlaps with the fixed plates.

Note also that the frequency varies with $1/C^2$ but C varies linearly with the overlapping area (which in turn varies linearly with the angle we turn the knob) so the frequencies on the AM dial are pretty significantly non-linear. (This is true for FM also, but the range of frequencies represented by the FM band is much narrower, so the effect on the 'dial' is less noticeable.)