PH2233 Fox : Lecture 01
Chapter 14 : Oscillations

In this chapter we look at 'oscillations' - a type periodic motion that appears frequently, even when not desired (think: Tacoma Narrows bridge collapse).

The next chapter involves the machinery involved when these oscillations also propagate through materials (water waves, sound waves, etc).

The topics we'll cover include:

- 14.1 : oscillations of a spring
- 14.2 : simple harmonic motion
- 14.3 : energy in the simple harmonic oscillator
- 14.4 : SHM related to circular motion
- 14.5 : the simple pendulum
- 14.6* : the physical pendulum
- 14.7 : damped harmomic motion
- 14.8 : forced oscillations : resonance

Introduction

Tacoma Narrows bridge example:

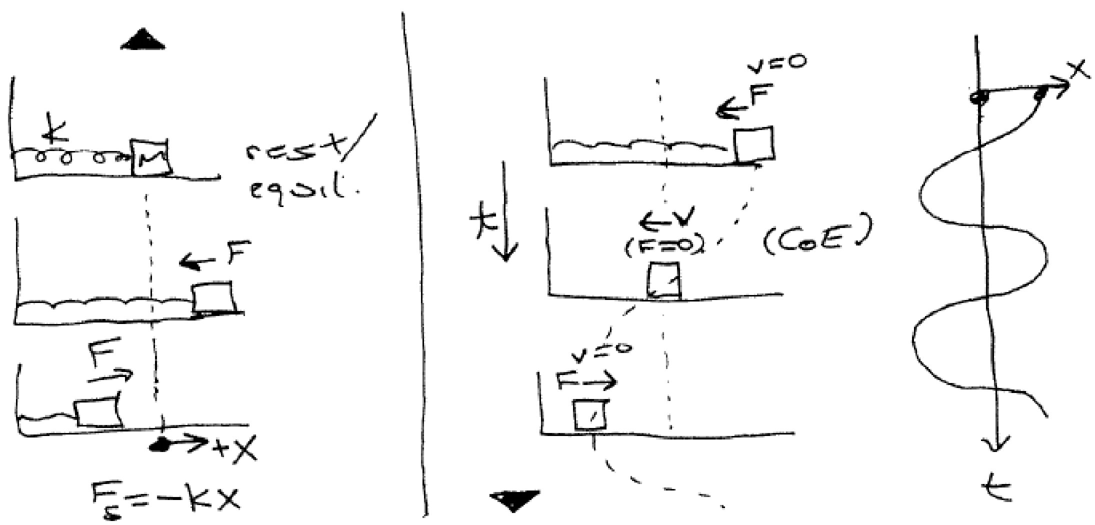
- short video: `https://www.youtube.com/watch?v=y0xohjV7Avo`
- long video: `https://www.youtube.com/watch?v=j-zczJXSxnw`
- Practical Engineering explanation: `https://www.youtube.com/watch?v=mXTSnZgrfxM`

14.1 : oscillations of a spring

Let's review the 'mass on a spring' scenario from PH2213.

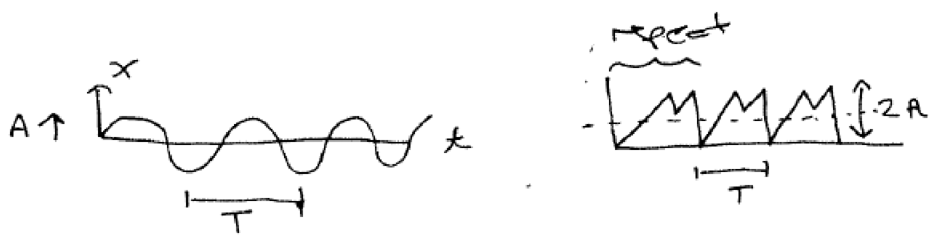
On the left, we see a mass attached to an unstretched spring and let's assume we have no friction here. If we pull the mass to the right, the spring will exert a force to the left so when we let the mass go, it will slide to the left. The spring force and displacement are both to the left, so the spring ends up doing positive WORK on the mass, increasing its kinetic energy. By the time the mass returns to its original position, it's picked up quite a bit of K so it doesn't stop there - it continues moving left. It's now COMPRESSING the spring though, and the spring will be exerting a force to the right as the object continues moving left. The spring is now doing NEGATIVE work on the object, eventually bringing it to a stop. It won't sit there though, thanks to the large spring force to the right. This back and forth motion continues forever unless some other force (friction most likely) is present that can continuously remove energy from the motion.

On the right, we see a graph of the position of the mass as a function of time. It's periodic and looks cosinusoidal and we'll soon see that it is in fact a cosine.



DEFINITIONS : Whether it is a perfect cosine (or sine) or not, we can still define some **generic terms** describing this class of motion: **amplitude (A)**, **cycle**, **period (T)**, which is related to: **frequency** $f = 1/T$ (def) and **angular frequency** $\omega = 2\pi f$.

By convention A, T, f, ω are generally taken to be **positive** values so we get a unique solution. $y(t) = -2\sin(-3t) = 2\sin(3t)$ identical equations actually, even though we'd have different values for A and ω without using the usual convention.



Examples of PERIODIC MOTION

14.2 : Simple Harmonic Motion

Suppose we're in a pretty common scenario where we're dealing with a **linear restoring force**. A spring is certainly a good example of that (as long as we don't stretch or compress the spring too badly), so we'll stick with the horizontal spring example from the previous page.

$F = -kx$ but $F = ma = m\frac{d^2x}{dt^2}$ so in one step applying Newton's Laws to the mass on a spring creates a differential equation: $\boxed{\frac{d^2x}{dt^2} = -\frac{k}{m}x}$, sometimes written as $\boxed{x'' = -(k/m)x}$ where the primes indicate differentiation WRT time and sometimes as $\boxed{\ddot{x} = -(k/m)x}$.

If you've taken a DFQ class, you know that a 2nd order linear DFQ like this will have **two** unique/independent (aka ORTHOGONAL) solutions. They are usually taken to be: $\boxed{x_1(t) = \cos(\omega t)}$ and $\boxed{x_2(t) = \sin(\omega t)}$ where $\omega = +\sqrt{k/m}$ (the positive root taken by convention).

(Plug each of these into the original DFQ and show that in fact they are solutions.)

Our DFQ is <u>linear</u>, so once we find two unique orthogonal solutions, we're done. Any linear combination of these two is also a solution, so we can write this as: $\boxed{x(t) = C_1\cos(\omega t) + C_2\sin(\omega t)}$

CONVENTION : for most of what we'll do with this type of motion, a more convenient form is: $\boxed{x(t) = A\cos(\omega t + \phi)}$

where $\omega^2 = k/m$, i.e. $\omega = +\sqrt{k/m}$ (taken positive by convention). ϕ can be either positive or negative.

(Substitute that expression into the original DFQ and verify that it's another way of writing the solution.)

Let's use this trig identify to show these are the same thing: $\boxed{\cos(a+b) = \cos a\cos b - \sin a\sin b}$

- Use that trig identity to expand out the form $x(t) = A\cos(\omega t + \phi)$:
- $x(t) = A\cos(\omega t)\cos\phi - A\sin(\omega t)\sin\phi$.
- Collecting terms: $x(t) = (A\cos\phi)\cos(\omega t) - (A\sin\phi)\sin(\omega t)$
- But that's just the general solution we found if we set $c_1 = A\cos\phi$ and $c_2 = -A\sin\phi$.

Moving forward, we'll usually use the $\boxed{x(t) = A\cos(\omega t + \phi)}$ form when we're talking about this type of periodic motion.

FYI : Any solution to that DFQ can be written in that form. Suppose the actual motion looks like a SINE wave. SHOW that this sine wave can be written as a COSINE wave that has been shifted to the right (forward in time) by a phase shift of 90 degrees ($\pi/2$ radians). So $x(t) = A\sin(\omega t)$ becomes $x(t) = A\cos(\omega t - \pi/2)$.

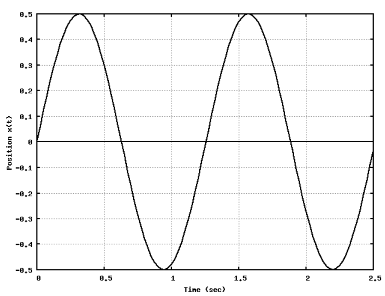
Shifted to the **right** means a **negative** phase shift; shifting the function left means a positive phase shift.

SIMPLE EXAMPLE

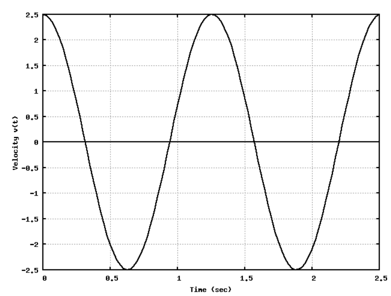
Suppose we let a $\boxed{2\ kg}$ mass oscillate on a spring of unknown spring constant k and find that it's position can be written as: $\boxed{x(t) = 0.5\cos(5t - \pi/2)}$ (with all units in standard metric: meters, rad/s, etc).

Let's extract everything conceivable about this motion (period, amplitude, frequency, spring constant, initial position and velocity, acceleration, <u>maximum</u> velocity, <u>maximum</u> acceleration, ...)

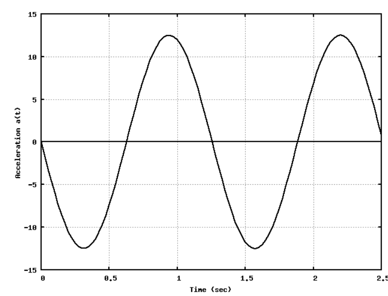
Differentiating the expression we have for $x(t)$ yields $v(t) = dx/dt = -2.5\sin(5t - \pi/2)$ and $a(t) = dv/dt = -12.5\cos(5t - \pi/2)$:



Position



Velocity



Acceleration

Generic equation: $x(t) = A\cos(\omega t + \phi)$ so we can pick off some of these answers right away:

$\boxed{A = 0.5\ m}$ and $\boxed{\omega = 5\ rad/s}$

$\omega = \sqrt{k/m}$ so $k/m = \omega^2 = (5)^2 = 25$ so $(k/2) = 25$ or $\boxed{k = 50\ N/m}$.

Max v and a : let's stick with our generic equation for now: $x(t) = A\cos(\omega t + \phi)$ and differentiate:

$v(t) = dx/dt = -A\omega\sin(\omega t + \phi)$ which means we have something oscillating back and forth (the sine term) varying between plus and minus $A\omega$. That's the maximum velocity then: $v_{max} = A\omega$ or here $(0.5\ m)(5\ rad/s) = 2.5\ m/s$.

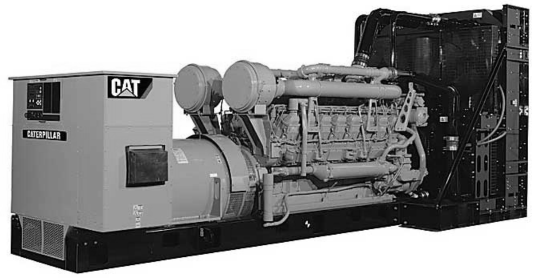
$a(t) = dv/dt = -A\omega^2\cos(\omega t + \phi)$ so similarly we see $a_{max} = A\omega^2 = (0.5)(5)^2 = 12.5\ m/s^2$ in this case.

So from the position equation, we can immediately write down the velocity and acceleration equations of motion.

Note that we can also go backwards here. An **accelerometer** (you probably have one in your phone) is a simple device since some crystals generate a (tiny) voltage when squeezed (which will happen under acceleration). Suppose we find that $a(t) = -20\cos(10t)$. Then $\omega = 10$ and $A\omega^2 = 20$ so $A = 20/10^2 = 0.2\ m$.

EXAMPLE 14-5 (modified)

Suppose we have a large diesel generator designed to create 60 Hz AC voltage. Recall from PH2223 that we can create an AC voltage by changing the magnetic flux, either by rotating coils of wire through a magnetic field, or rotating the magnet itself. Either way, something very heavy is rotating and unless everything is perfectly balanced, the machine is likely to vibrate.



We don't want the generator to migrate across the floor like a washing machine or dryer, so we bolt the machine to the floor.

The machine is still unbalanced but now it will cause the floor itself to vibrate slightly.

So: suppose the floor near the machine is oscillating up and down at $f = 60\ Hz$ with an amplitude of just 1 millimeter.

What side effects will this vibration have on other items sitting on the floor (nearby machines, people, etc)?

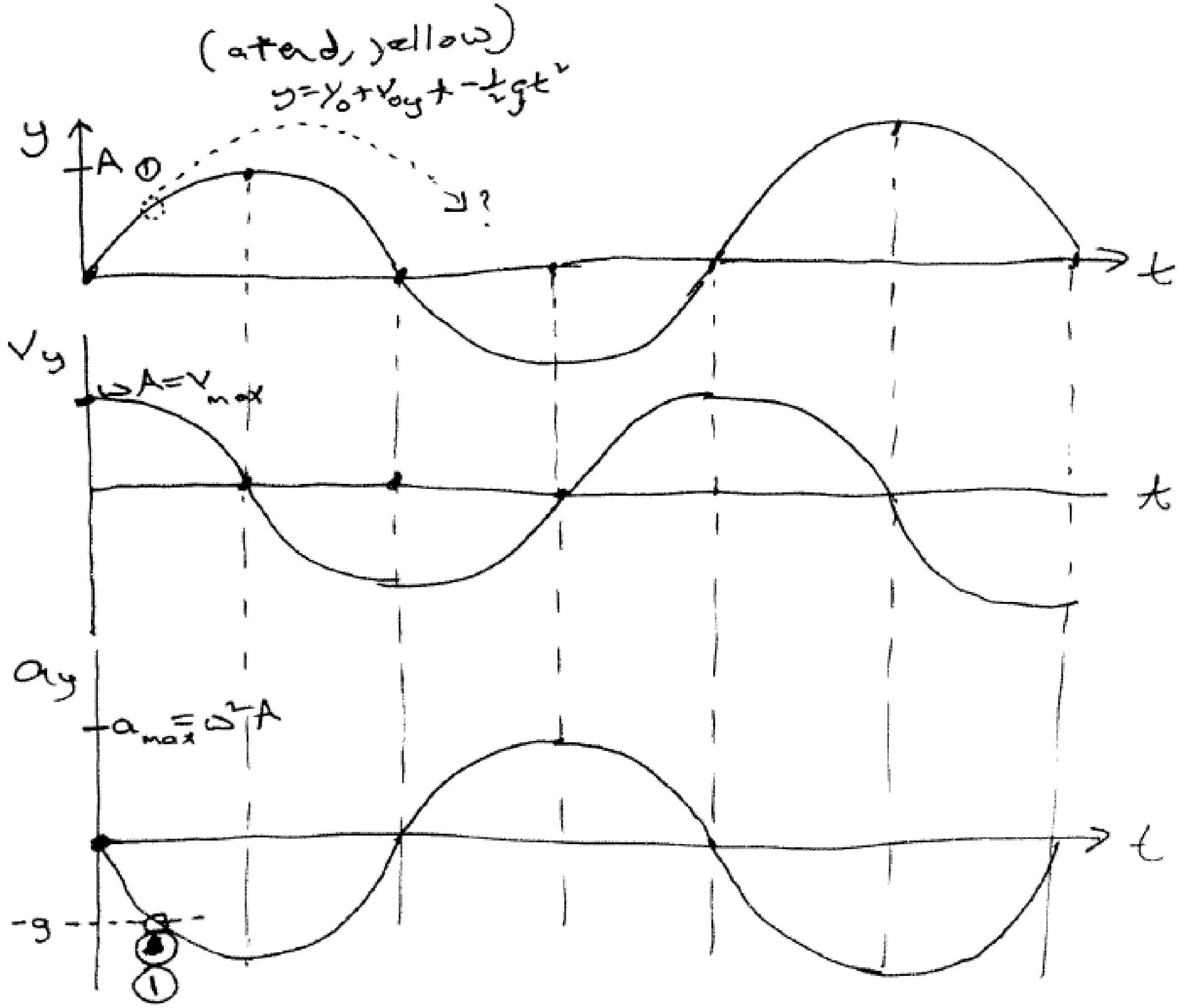
The floor level oscillating up and down with $f = 60\ Hz$, so $\omega = 2\pi f = 376.8\ rad/s$ and an amplitude of $A = 1\ mm = 0.001\ m$.

The figures on the previous page are still relevant. The velocity of the floor will vary between $\pm\omega A = 0.3768\ m/s$ and more importantly the **acceleration** of the floor will vary between $\pm\omega^2 A = 142\ m/s^2$.

For nearly half of the motion, the floor is accelerating **downward** with an acceleration larger than g.

In class, I held an object in my hand and then accelerated my hand away faster than g, resulting in the object (momentarily) undergoing freefall.

Quickly, the floor will rise up and meet this object falling downward. That's a **collision** which means the object will feel a 'collision force' of $F = \Delta p/\Delta t$. If the floor and object are pretty solid, Δt will be **very small** resulting in a large force.



And this scenario will occur 60 times each second. Essentially we're pounding on the object with a jackhammer... (Hopefully there's nothing delicate like electronics, or screws that this 'impact driver'-like force will work loose.)

We'll see later how we can protect nearby objects (basically adding a 'damping' force like a mat or rug between the vibrating floor and the loose objects resting on the floor.