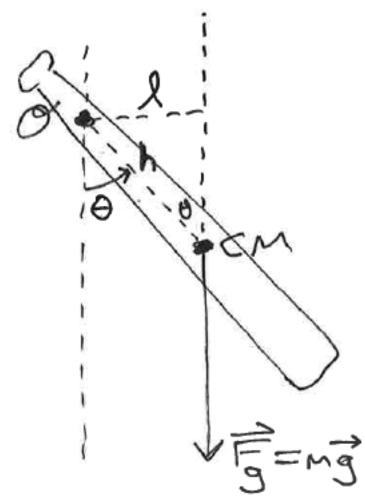
PH2233 Fox : Lecture 03
Chapter 14 : Oscillations

Pendulums : another type of periodic motion

Suppose we have some object (like the baseball bat in this figure) that is constrained to rotate about some point (labelled O in the figure).
The object will ultimately be rotating about point O, so let's attack this via rotational equations and concepts like torque.
$\sum \vec{\tau} = I\vec{\alpha}$
(Technically a vector equation, but suppose all the motion is constrained to occur in the plane of the paper, so basically we're rotating about an axis coming up out of the page.)
where each (external) force acting on the object will produce a torque of $\vec{\tau} = \vec{r} \times \vec{F}$.
Recall that $\vec{r}$ is the vector that point from the axis of rotation to the point where the force is being applied.



There's some normal force acting on the object right at the rotation axis, but $|r|$ is essentially ZERO there, so it won't introduce any torque.

In the real world there's almost certainly some friction between the object and the nail or whatever is being used to cause the object to rotate about O but for now we'll ignore friction.

The only other external force acting is **gravity**. Every tiny dm element making up the object is exerting it's own force at it's own location, but we showed in PH2213 that mathematically we can treat that force as if it were all being applied at the center of mass of the object.

Let h represent the distance from the axis of rotation to the CM of the object.

We have various ways of computing torque. Let's use the **lever-arm** approach here. $\tau = Fl$ where l is the lever arm. We draw the **line of action** (basically the dotted line extending the $\vec{F}_g$ vector and the l is the perpendicular distance from that line to the axis of rotation.

Minimal propagation of angles shows us that $l = h \sin\theta$.

Note here we're measuring θ in the usual 'CCW is positive' convention but our X axis starts off pointing straight down.

Then (with everything representing rotations about $+Z$ coming up out of the board, our torque equation becomes: $\boxed{\tau = Fl = (-1)(mg)(h\sin\theta) = I\alpha}$ (Note the (-1) out front since this force will induce a **clockwise** torque and CW is negative. And I will be the moment of inertia about the O axis.)

Well, α is the angular acceleration, which is the derivative of the angular velocity ω which itself is the derivative of the angular position θ so this equation becomes:

$I\frac{d^2\theta}{dt^2} = -mgh\sin\theta$ or: $\boxed{\frac{d^2\theta}{dt^2} = -\frac{mgh}{I}\sin\theta}$

This is **NOT** the same differential equation we had with the mass on a spring (linear restoring force), so the solution isn't going to be a simple cosine or sine function this time.

What if the angles remain small though? In a grandfather clock, the mass at the end of the rod only moves a few centimeters back and forth and the maximum displacement angle is probably only 10 degrees or less.

If that's the case, $\sin\theta$ is almost equal to θ (in radians anyway). The series expansion for $\sin(x)$ is:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \text{ (with } x \text{ in radians)}$$

At $\theta = 10^o = 0.1745..\ rad$ the terms in this sum are:

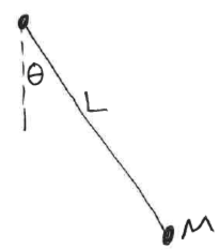
$$\sin(0.1745..) = 0.1745329 - 0.000881 + 0.000001349...$$

The exact value of the sine here is 0.173648... so just keeping that first term, we're making an error of only about a half of one percent.

If we approximate $\sin\theta$ as being very near θ, our differential equation turns into: $\boxed{\frac{d^2\theta}{dt^2} = -\frac{mgh}{I}\theta}$ which **IS** exactly the same form we had for the mass on a spring and the solution would be of the same type: $\boxed{\theta(t) = \theta_{max}\cos(\omega t + \phi)}$ where $\omega = \sqrt{mgh/I}$.

(Be very careful here: the ω in this equation is **not** the angular speed but the angular frequency of the motion.)

Special Case : Simple Pendulum



Suppose our 'pendulum' is a tiny mass (basically a point-mass M) on the end of a (massless) string of length L.
Then $I = \sum mr^2 = ML^2$, $h = L$ in our equation and the angular frequency reduces to just $\omega = \sqrt{\frac{MgL}{ML^2}}$ or just $\boxed{\omega = \sqrt{g/L}}$.

The angular frequency is connected to the frequency and period though: $\omega = 2\pi f$ with $f = 1/T$ so ultimately the period of a pendulum can be written as:

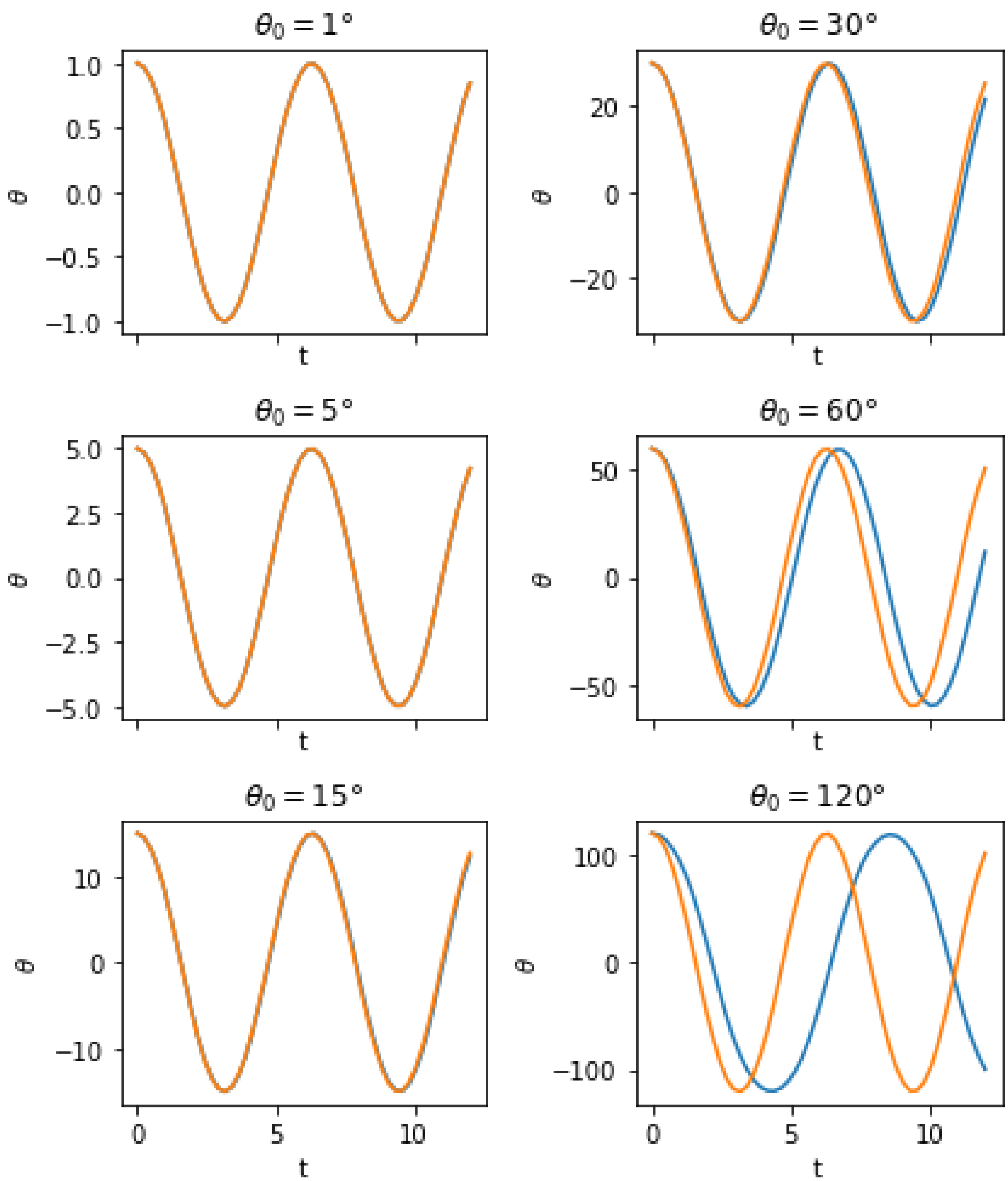
- $T = 2\pi/\omega = 2\pi\sqrt{L/g}$: point-mass ('simple') pendulum
- $T = 2\pi/\omega = 2\pi\sqrt{\frac{I}{mgh}}$: real ('physical') pendulum

Both of those are **approximations** since we simplified the actual DFQ. Before we go too far we can actually compare the approximate and exact solutions for the point-mass case.

The exact solution to the differential equation where we **don't** approximate $\sin\theta$ as θ does exist.

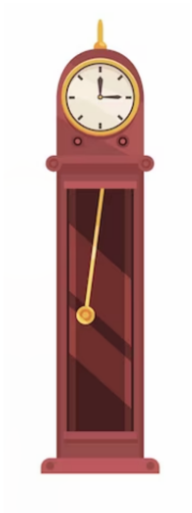
Unfortunately it involves something called an elliptic function which is expressed as an integral we can't do exactly and I've never seen a calculator that has that function built in, so it's not particularly practical but we can compare the approximate and exact solutions for various starting angles as seen in the figure here.

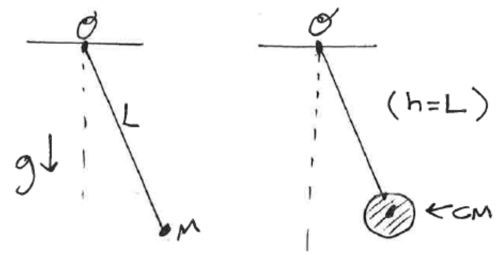
For small angles, the exact solution (blue line) looks very much like the approximate solution, but as the starting angle gets larger, the differences get more obvious and the actual period for the motion is longer than our 'simple pendulum' formula yields.



Simple vs Physical Pendulum Comparison

Let's consider the sort of pendulum in a grandfather clock, consisting of a rod with some mass connected to a heavier flat metal disk. Suppose the distance from the axis of rotation to the CM of the disk is exactly 1 meter, the disk has a mass of 4 kg and a radius of 5 cm, and the rod has a mass of 0.2 kg and a length of 1.05 m (so that it extends behind the disk all the way to the outer edge of the disk; we'll see later that in real clocks like this, there's an adjustment screw so that the disk can be slid slightly up or down on the rod).





Point Mass Solution : Here we have a point mass at the end of a massless rod of length $L = 1\ m$, so $T = 2\pi\sqrt{L/g} = \mathbf{2.0071}\ sec.$

A step in the right direction : Let's account for the extended nature of the disk, but ignore the mass of the connecting rod.

The moment of inertia for a disk rotating about an axis that is perpendicular to the disk and passes through its CM is $I_{cm} = \frac{1}{2}MR^2 = (0.5)(4\ kg)(0.05\ m)^2 = 5 \times 10^{-3}\ kg\ m^2$.

The disk is actually rotating about an axis coming out of the page at the point marked **O** in the figure, which is 1 meter away from the center (CM) of the disk, so we'll need to use the **parallel axis theorem** to find its true moment of inertia: $I = I_{cm} + Md^2$ where d is the distance from the axis of rotation to the CM of the object, so here:

$I = (5 \times 10^{-3}\ kg\ m^2) + (4\ kg)(1\ m)^2 = 4.005\ kg\ m^2$.

This produces a period of $T = 2\pi\sqrt{I/(mgh)}$ where h is the distance from the axis of rotation to the CM of the object (1 meter here), so: $T = (2)(\pi)\sqrt{\frac{4.005}{(4)(9.8)(1)}} = 2.00834..\ sec.$ (Very slightly longer than the point-mass equation gave.)

Better estimate : This time we'll include the rod.

We have to deal with a 'composite object' now: the object being the disk plus the rod. Fortunately we can just add the moments of inertia to find it.

The moment of inertia of the disk part (with the given axis of rotation at point O) we already found to be 4.005 $kg\ m^2$.

For the rod, we have a long, thin rod of mass 0.2 kg and length $L = 1.05\ m$ rotating about it's end and for that geometry $I = \frac{1}{3}ML^2 = (0.2\ kg)(1.05\ m)^2/3 = 0.0735\ kg\ m^2$.

The entire object then has $I = 4.0785\ kg\ m^2$.

We're not quite done yet though since in $T = 2\pi\sqrt{I/(mgh)}$ we need to know h which is the distance from the axis point to the CM of the (now composite) object.

That's also easy to calculate for composite objects since we can treat each 'part' as a point mass located at it's own CM and then use the point-mass center-of-mass formula.

Here we replace the disk with a point mass of 4 kg located at $r = 1\ m$ from the axis.

We replace the rod with a 0.2 kg point mass located at the CM of the rod, which will be at its midpoint, which is $1.05/2 = 0.525\ m$ from the axis.

$r_{cm} = \frac{\sum m_i r_i}{\sum m_i} = \frac{(4\ kg)(1.0\ m)+(0.2\ kg)(0.525\ m)}{(4\ kg)+(0.2\ kg)} = 4.105/4.2 = 0.977381\ m.$

So $h = 0.977381\ m$ (distance from axis to CM of the overall object).

We have all the parts we need now: $T = 2\pi\sqrt{I/(mgh)} = (2)(\pi)\sqrt{\frac{4.0785}{(4.2)(9.8)(0.977381)}} = 2.0006\ sec.$

These are all **incredibly close**, (only 2 ms apart) but using the more accurate formula gives us a period that is **slightly** less than the point-mass 'simple' pendulum formula yielded.

We'd like the period to be **exactly** 2 seconds so that a simple mechanical part could advance the clock by 1 second every time the pendulum passes through the lowest point (moving either left to right, or right to left).

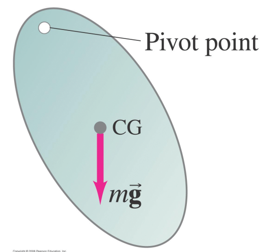
Hence the adjustment screw seen in this figure. It basically lets the user adjust the position of the disk up and down as needed, which **slightly** changes both the moment of inertia and the location of the composite CM and thus will change the period of the pendulum.
(An adjustment device like this would be needed anyway since g itself isn't the same everywhere, and even changes slightly with elevation, so even centuries ago this was the solution.)

Application : Determining I for complicated objects

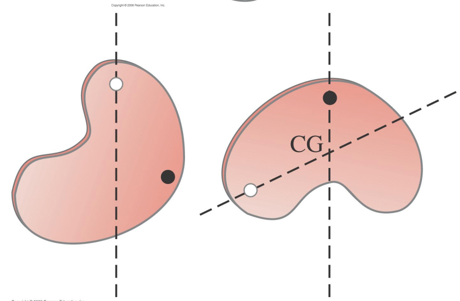
For small oscillations, we found the period to be related to the moment of inertia (about the rotation axis) and the distance to the center of mass of the object: $T_{physical} = 2\pi\sqrt{I/(mgh)}$.

This actually gives us a mechanism to **calculate** the moment of inertia of some object by suspending it at some point and allowing it to oscillate back and forth.

We'll need to find where the CM is also, but we saw one trick to doing that back in chapter 9.



If we hang and object from some point (the white dot in the top figure), the torque due to gravity will cause it to rotate unless the center of mass is directly below the pivot point.

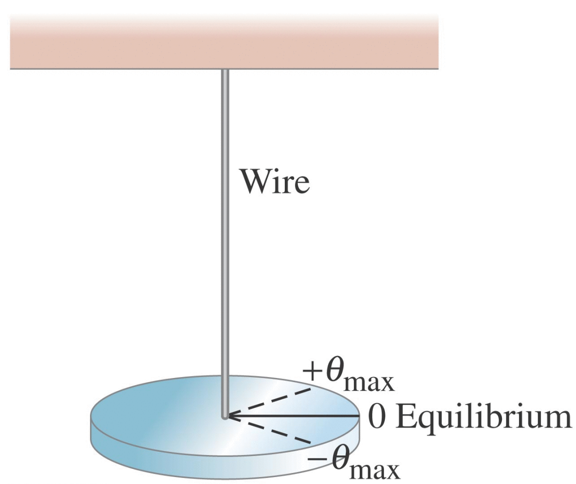


In the lower figure, with the object hanging at rest, we know the CM must be somewhere along the vertical dotted line from the axis. Hanging the object from a different point gives us another line the CM must be located along, and where these lines intersect provides where the CM must be.

Letting the object oscillate back and forth now, we know h and can measure T, so that's enough information to give us I, which would be the moment of inertia about the chosen pivot point. And if we wanted to report the moment of inertia for rotations about the center of mass, we can use the parallel axis theorem to get that: $I = I_{cm} + Md^2$ where d would be the distance between the pivot point and the CM.

Torsion Pendulum

Another type of oscillatory motion is a **torsion pendulum** in which some object (like the disk shown in the figure) is suspended by a wire. Twisting the object about that vertical axis will twist the wire, resulting in a 'linear restoring force' that's expressed as an angular force (i.e. a torque): $\boxed{\tau = -\kappa\theta}$ (linear restoring torque).
$\sum \tau = I\alpha$ yields the same differential equation we had with the mass on a spring and we can directly pick off the angular frequency of the motion to be $\omega = \sqrt{\kappa/I}$ so the period would be $T = 2\pi\sqrt{I/\kappa}$.
Doing this with a known object gives us a way to determine that torsion constant κ for the wire too. (NOTE: mechanical/civil engineers define a very different quantity that they call the 'torsional constant', so don't confuse that with the definition used here.)

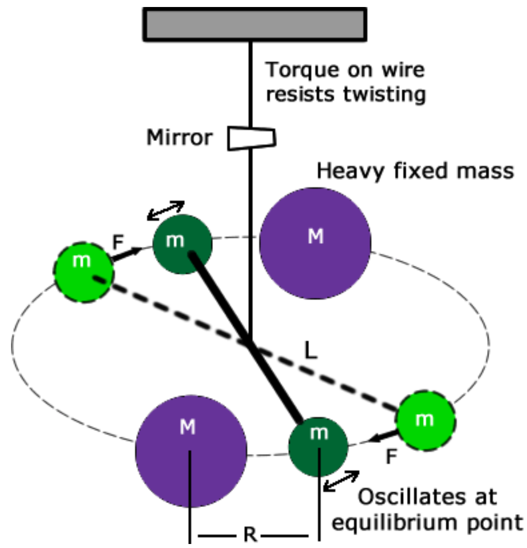


NOTE: if you watch the Tacoma Narrows bridge collapse videos, especially the longer ones, you'll see that the bridge is actually undergoing this 'torsional' motion leading up to the collapse.

`https://www.youtube.com/watch?v=j-zczJXSxnw&t=243s`

Application : Early Estimate for G

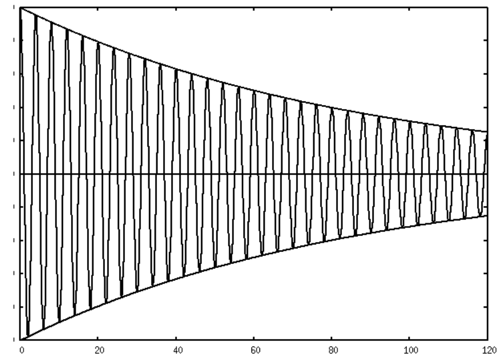
Around 1798, Henry Cavendish did an experiment that yielded the first good estimate for the universal gravitational constant G (the constant in the $F_G = G\frac{Mm}{r^2}$ gravitational force equation). (Actually he was more interested in finding the average density of the Earth and didn't bother to include the G value he found, but others later repeated the experiment focusing on estimating G.)



In that experiment, the rod connecting the lighter metal balls (m) in the figure was suspended by a thin wire and then brought near the heavy fixed masses (M). The (incredibly tiny) gravitational force between the objects caused the lighter ones to move slightly, twisting the wire (a light shining on the mirror attached to the wire would make a spot that moved slightly on the wall so that the tiny angles could be measured).

At the equilibrium point, the restoring torque $\tau = \kappa\theta$ created by the wire would just match the torque created by the gravitational forces between the objects. They'd need the value of κ to complete this though and the figure above shows how: the period of the disk in the upper figure oscillating rotationally would tell them what κ needed to be. (Alternately, and the way Cavendish approached it, if the masses start off slightly out of equilibrium, the gravitational torque would cause them to overshoot in their rotation, resulting in angular oscillations, the period of which could be used to determine what κ was, and the amplitude could be used to determine how far the equilibrium point shifted due to the additional (tiny) gravitational force between the large and small balls.

14.7 : damped harmomic motion



When we have a linear restoring force, we found that the object oscillates with the generic solution of $x(t) = A\cos(\omega t + \phi)$ but that cosine would go on forever, and in the real world these oscillations usually (always?) diminish over time.

In class, I held a meter-stick over the edge of the table and pulled up on the free end. When I let it go, that end vibrated up and down but and amplitude quickly died away.

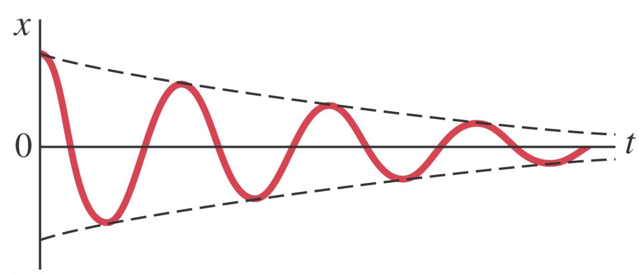
A major reason for this is air resistance. We (briefly) looked at **resistive forces** in PH2213 and a common model for them is a force that depends on how fast the object is trying to move through some medium. For relatively slow motion, a good first approximation is $F_R = -bv$ and that's what we'll use here. (At higher speeds, like those involving vehicles, the resistive force is more likely to be proportional to the square of the velocity but that's much more difficult to handle mathematically so we'll stick with the linear version.)

NOTE: a resistive force is always doing negative work since it's always in the direction opposite the direction of motion (which v gives us).

As a result, the object involved is **continuously slowing down**. For periodic motion means that the period will be longer, which in turn means that f or ω will be smaller than what we had before.

Sticking with our linear restoring force, $\sum F = ma$ becomes $-kx - bv = ma$ and noting that $a = d^2x/dt^2$ and $v = dx/dt$ we end up with a new differential equation:

$$\boxed{\frac{d^2x}{dt^2} + (b/m)\frac{dx}{dt} + (k/m)x = 0}$$



Based on how real objects behave, we'll 'guess' that the solution will be a cosine with an exponentially decaying amplitude factor:

$$\boxed{x(t) = Ae^{-\gamma t}\cos(\omega' t)}$$

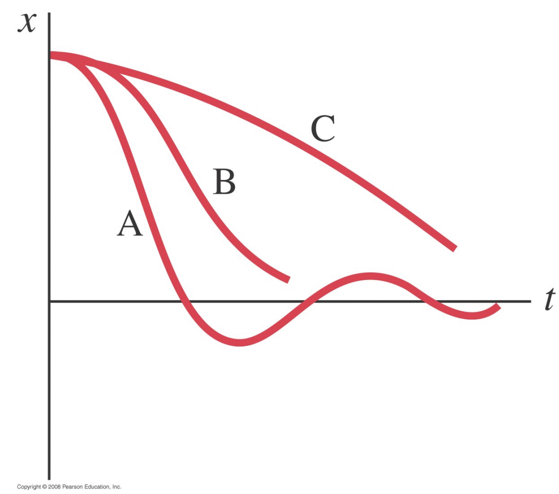
Plugging that guess into the differential equation, it works and we find that the exponential decay factor is $\gamma = b/(2m)$ and the new frequency of the oscillations will be: $\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$

Before we go too far though, let's look at the equation for the new angular frequency. If there's too much damping (i.e. if b is too large) the second term under the square root can get large enough that we end up with either zero or even a negative number there and we can't take the square root of a negative number (yet). Mathematically, we end up with what's called an imaginary number.

Can we still get a solution for this case?

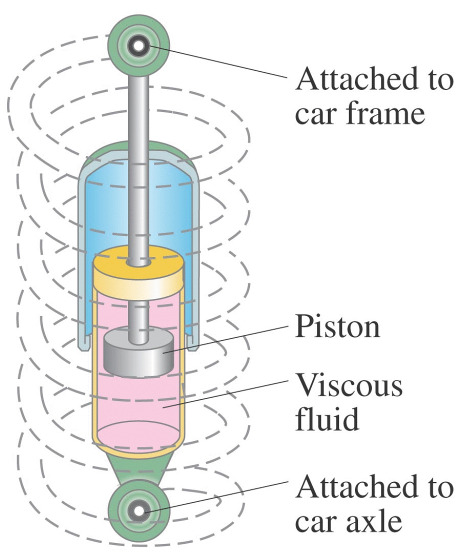
First, suppose b is such that the expression under the square root is exactly zero. This occurs if $\frac{k}{m} = \frac{b^2}{4m^2}$ which implied that $\boxed{b = 2\sqrt{km}}$. If the resistive force constant (in $F_R = -bv$) is exactly that value, then $\omega' = 0$ which just means there isn't any oscillating going on at all: the period goes to infinity for the oscillating part and we're really just left with a decaying exponential. This is called **critical damping** : the object starts off at some displacement and just gradually returns to equilibrium, never 'overshooting' or oscillating at all. (This is usually the desired result when damping mechanisms are being designed.)

If the expression under the square root is negative, we get an imaginary number for ω' and the result of an imaginary number being used as an argument to the cosine term mathematically ends up just producing another exponential decay factor. This situation is called **overdamping** and physically it means that the damping dominates the scenario, resulting in the object taking a longer time to return to equilibrium.

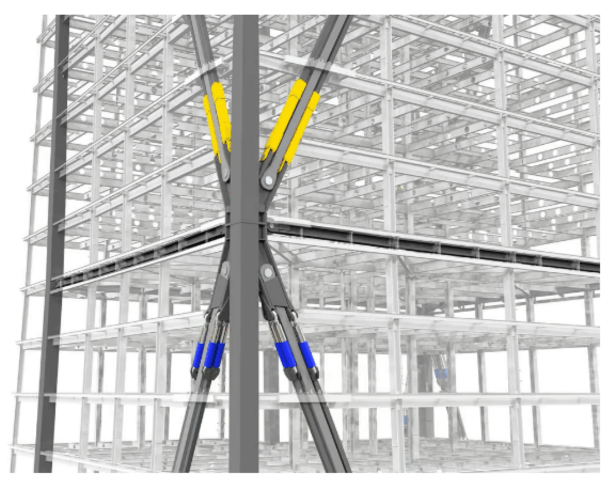


A : underdamped
B : critically damped
C : overdamped

To complete the definitions, if we just have a little damping (not enough to trigger the purely exponential behaviour), that situation is referred to as **underdamped** motion: the object oscillates, but the oscillations taper off.



Shock Absorbers in cars.

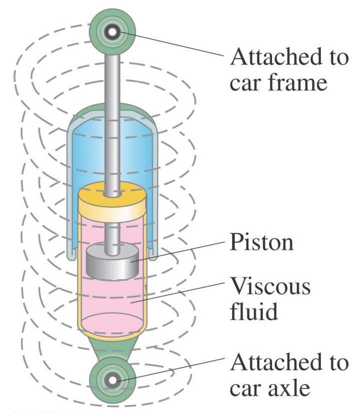


Shock Absorbers in **buildings**.

Damped Motion Example : Car Springs and Shock Absorbers

I looked up some numbers for actual cars and found one to use as an example.

The car has a mass of $M = 1200\ kg$ and each of the four springs has a spring constant of $k = 5000\ N/m$ and a damping factor of $b = 545.5\ N\ s/m$. In combination then, the effective spring constant for this situation is $k = 20,000\ N/m$ and $b = 2182\ N\ s/m$.

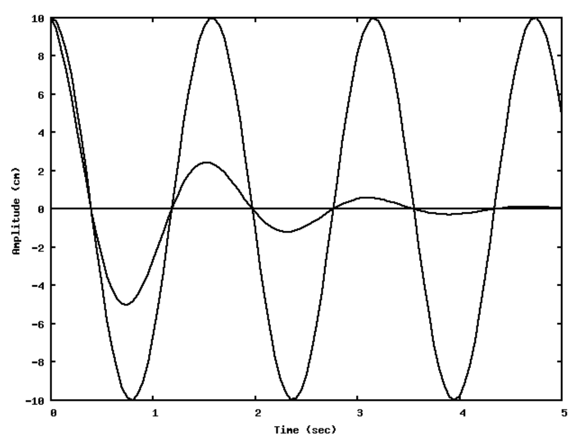


The **undamped angular frequency** is $\omega = \sqrt{k/m} = 4.08248..\ s^{-1}$ representing an oscillation with a period of $T = 2\pi/\omega = 1.5391\ sec$.

The new angular frequency is:

$$\begin{aligned}\omega' &= \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \\ &= \sqrt{\frac{20000}{1200} - \frac{(2182)^2}{4(1200)^2}} \\ &= \sqrt{16.666.. - 0.8266..} \\ &= 3.97996..\ s^{-1}\end{aligned}$$

representing a period of $T = 1.5787\ sec$.
Note that the period didn't change too much: from about 1.54 s to about 1.58 sec.



Including the damping, we now have an exponential decay factor of $\gamma = b/(2m) = 0.909166\ s^{-1}$. As a result, the behavior of the oscillations changed drastically. Instead of endless cosinusoidal motion, the oscillations are pretty quickly damped away after a few small bounces.

ADDENDUM: What damping factor b would be need if we don't want the car to bounce at all? I.e., what b will yield the critically damped situation?

This occurs when $\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} = 0$, so we need $\frac{k}{m} = \frac{b^2}{4m^2}$ or rearranging to solve for b:

$$b = 2\sqrt{mk} \text{ for critically damped}$$

In our case, $m = 1200\ kg$ and $k = 20,000\ N/m$ yields $b = 9800\ N\ s/m$ or about FOUR TIMES the damping we have now.

(NOTE: apparently a few car models over the years have used 'active suspension' systems to completely (ideally) remove any oscillations, allowing the car to remain travelling horizontally as the wheels bounce up and down over rough surfaces. The wikipedia article on 'active suspension' is vague but hints at hydraulics or putting bits of metal in the fluid so that an electromagnet can somehow adjust the viscosity dynamically. There are at least quite a few references listed if you're curious!)

(NOTE: we didn't get to this in class today, so will do it at the start next time...)

14.8 : forced oscillations : resonance

Suppose we have an object undergoing simple-harmonic motion (like a mass on a spring, pendulum, etc) and we apply an external force that's at the same frequency.

Imagine a person on a swing and we give them an extra push during each cycle, timed so that we're pushing the person in the same direction they're moving. That means that on every cycle, the external force is doing positive work, adding to the energy of the motion. We found that $E = \frac{1}{2}kA^2$ so that means the amplitude of the motion will keep growing endlessly.

Fortunately in the real world there's always **some** amount of damping or friction, so energy is being removed too. We never reach infinite amplitudes for the motion, but they can get quite large.

Adding a time-varying external force to our system: $\sum F = ma$ so $\boxed{ma = -kx - bv + F_o \cos(\omega t)}$ (where ω represents the angular frequency involved in the external force, not necessarily the same as the natural frequency of the object in the presence of the restoring force).

Our differential equation becomes: $\boxed{\frac{d^2x}{dt^2} + (b/m)\frac{dx}{dt} + (k/m)x = \frac{F_o}{m}\cos(\omega t)}$

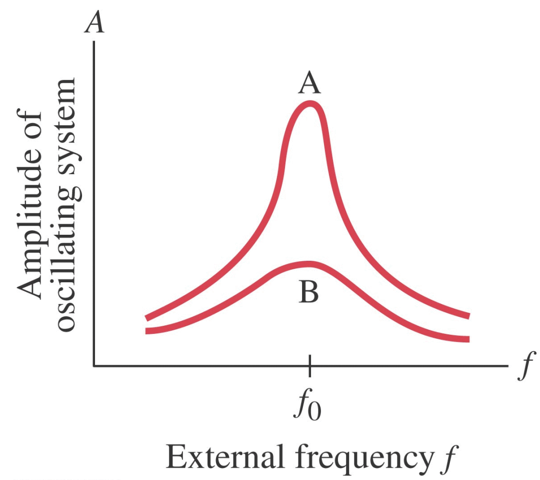
Technically this is solvable if we assume a solution of the form $x(t) = Ae^{i\omega t}$ and allow A and ω to be complex numbers.

The book takes the approach of 'guessing' a form $x(t) = A_o \sin(\omega t + \phi)$ and leaves it to the reader to try that and see what happens, but the actual time varying behavior of the solution is more complicated than that.

Ultimately, 'it can be shown' that the amplitude of the motion will be:

$$A_o = \frac{F_o}{m\sqrt{(\omega^2 - \omega_o^2)^2 + b^2\omega^2/m^2}}$$

where $\omega_o = \sqrt{k/m}$ (the 'natural' undamped, unforced angular frequency of the motion).
The maximum amplitude occurs when the frequency of the forcing function is almost the same as the natural frequency of the motion and can result in very large amplitude fluctuations, a phenomenon called **resonance**.



Real world examples: opera singers breaking glass, Tacoma Narrows bridge collapse, buildings in earthquakes, etc.

Good videos of sound breaking a wine glass in slow motion, showing the oscillations building up in the glass itself:

`https://www.youtube.com/watch?v=BE827gwnnk4`
`https://www.youtube.com/watch?v=sxRkOQmzLgo`