Physics 2233 : Chapter 14 Examples : Oscillations

Conventions: A (amplitude), T (period), ω (angular speed), f (frequency), usually all taken to be **positive**

$f = 1/T \qquad \omega = 2\pi/T = 2\pi f$

Restoring force: $F = -kx$ produces SHM with $\omega = \sqrt{k/m}$ so $f = \omega/(2\pi) = \frac{1}{2\pi}\sqrt{k/m}$ or $T = 2\pi\sqrt{m/k}$

$x = A\cos(\omega t + \phi) \qquad v_x = \frac{dx}{dt} = -\omega A \sin(\omega t + \phi) \qquad a_x = \frac{dv_x}{dt} = -\omega^2 A\cos(\omega t + \phi) = -\omega^2 x$

Given initial conditions x_o, v_{ox} : $\phi = arctan(-\frac{v_{ox}}{\omega x_o})$ and $A = \sqrt{(x_o)^2 + \frac{v_{ox}^2}{\omega^2}}$

Energy in SHM: $E = \frac{1}{2}mv_x^2 + \frac{1}{2}kx^2 = const = \frac{1}{2}kA^2 \qquad v_x = \pm\sqrt{\frac{k}{m}}\sqrt{A^2 - x^2} \qquad v_{max} = \sqrt{\frac{k}{m}}A = \omega A$

Angular SHM: $\tau_z = -\kappa\theta$ leads to $\omega = \sqrt{\frac{\kappa}{I}}$ and $f = \frac{1}{2\pi}\sqrt{\frac{\kappa}{I}}$ where I is the moment of inertia about the point of oscillation.

Simple pendulum (approximation): $\omega = \sqrt{\frac{g}{L}} \qquad f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{g}{L}} \qquad T = \frac{2\pi}{\omega} = \frac{1}{f} = 2\pi\sqrt{\frac{L}{g}}$

Simple pendulum (more exact, given a maximum angular displacement Θ):
$T = 2\pi\sqrt{\frac{L}{g}}(1 + \frac{1^2}{2^2}\sin^2\frac{\Theta}{2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2}\sin^4\frac{\Theta}{2} + \cdots)$

Physical Pendulum (approximation): $\omega = \sqrt{\frac{mgd}{I}} \qquad T = 2\pi\sqrt{\frac{I}{mgd}}$ (d is pivot to CM). I is moment of inertia about the pivot. Parallel axis theorem: $I = I_{cm} + Md^2$

Damped Oscillations: $\Sigma F_x = -kx - bv_x$: $x(t) = Ae^{-(\frac{b}{2m})t}\cos(\omega' t + \phi)$ where $\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$.
Note ω' is less than ω.

- Underdamped (decaying oscillations): $b < 2\sqrt{km}$
- Critically damped (no oscillations): $b = 2\sqrt{km}$
- Overdamped (no osc.): $b > 2\sqrt{km}$.

Problem 2 : If an object on a horizontal frictionless surface is attached to a spring, displaced, and then released, it will oscillate. If it is displaced a distance 0.125 m from its equilibrium position and released with zero initial speed, then after a time 0.800 s its displacement is found to be a distance 0.125 m on the opposite side, and it has passed the equilibrium position once during this interval. Find the (a) amplitude, (b) period, and (c) frequency.

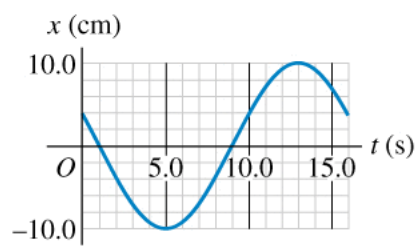
(a) The amplitude is the maximum displacement from equilibrium: in one period, the object goes from $+0.125\ m$ to $-0.125\ m$, so essentially by definition, the amplitude of this oscillation is $A = 0.125\ m$.

(b) From the description of the motion, the object has moved through only one half of a complete cycle, so $0.800\ s$ is equal to $T/2$: $(T/2) = (0.800\ s)$ or $T = 1.60\ s$.

(c) The frequency f and period T are related by $T = 1/f$ or $f = 1/T$ so here $f = 1/(1.60\ s) = 0.625\ s^{-1}$ or $0.625\ Hz$.

(For completeness, the angular frequency ω is related to the period by $\omega = 2\pi/T$ or since $T = 1/f$ we can write this as $\omega = 2\pi f$. So for this motion, $\omega = (2)(\pi)(0.625\ s^{-1}) = 3.93\ rad/sec$.)

Problem 4 : The displacement of an oscillating object as a function of time is shown in the figure. (a) What is the frequency? (b) The amplitude? (c) The period? (d) What is the angular frequency of this motion?



From the figure, we see that the object is moving between $+10\ cm$ and $-10\ cm$ so we can immediately identify the amplitude to be $A = 10\ cm$ or $A = 0.10\ m$. The figure seems to show one complete cycle, starting at $x = 2\ cm$ and ending at the same value, so the period T looks to be $16\ s$. So:

(a) $f = 1/T = 1/(16\ s) = 0.0625\ s^{-1} = 0.0625\ Hz$.

(b) $A = 10.0\ cm$ as we argued above.

(c) $T = 16\ s$, as described above.

(d) $\omega = 2\pi f = (2)(\pi)(0.0625\ s^{-1}) = 0.393\ rad/sec$

Problem 6 : In a physics lab, you attach a 0.200 kg air-track glider to the end of an ideal spring of negligible mass and start it oscillating. The elapsed time from when the glider first moves through the equilibrium point to the second time it moves through that point is 2.60 s. Find the spring's force constant.

The time between moving through the equilibrium positions (the points where $x = 0$) represents only one half of a complete period (sketch out one period of a sine or cosine wave - the zeroes of the function occur at intervals of one half the period of the wave). So from the information given, we find that $T/2 = 2.60\ s$ or $T = 5.20\ s$. That implies that $f = 1/T = 0.1923\ s^{-1}$ and $\omega = 2\pi f = 1.208\ s^{-1}$. But for a spring, $\omega = \sqrt{k/m}$ so $(1.208) = \sqrt{k/0.2}$ so $k = 0.292\ N/m$. (Alternatively, from $\omega = \sqrt{k/m}$ if we square both sides we have $\omega^2 = k/m$ then multiplying both sides by m we arrive at $k = m\omega^2$ so $k = (0.200\ kg)(1.208\ s^{-1})^2 = 0.292\ N$.)

Problem 7 : A body of unknown mass is attached to an ideal spring with force constant 120 N/m. It is found to vibrate with a frequency of 6.0 Hz. Find (a) the period, (b) the angular frequency, and (c) the mass of the body.

(a) The period and frequency are related simply by $T = 1/f$ so here $T = 1/(6.0\ Hz) = 1/(6.0\ s^{-1}) = 0.167\ s$.

(b) The angular frequency ω is related to the frequency by $\omega = 2\pi f$ so $\omega = (2)(\pi)(6.0\ s^{-1}) = 37.7\ rad/sec$.

(c) To find the mass, we know that $\omega = \sqrt{k/m}$ or squaring and rearranging terms, $m = k/\omega^2$ so $m = (120)/(37.7)^2 = 0.084\ kg$.

Problem 8 : When a 0.750 kg mass oscillates on an ideal spring, the frequency is 1.33 Hz. What will the frequency be if 0.220 kg are added to the original mass? Try to solve this problem without finding the force constant of the spring. What will the frequency be if 0.220 kg are subtracted from the original mass? Try to solve this problem without finding the force constant of the spring.

The angular frequency $\omega = \sqrt{k/m}$ but $\omega = 2\pi f$ so $f = \frac{1}{2\pi}\sqrt{k/m}$. The spring constant has some value here, so it is constant but the frequency will vary with the mass. Since k does not change, if we multiply this equation on both sides by $\sqrt{m}$ we end up with $f\sqrt{m} = \sqrt{k}/(2\pi)$ but the right hand side there is constant, so that means that f and m are related in a way that $f\sqrt{m}$ remains a constant.

If we let our initial unaltered mass and frequency be labeled as m_1 and f_1 then for some other mass m_2, we have a different frequency f_2 but whatever those are, they must be related such that $f_1\sqrt{m_1} = f_2\sqrt{m_2}$. Since we are going to be altering the mass and want to know how that alters the frequency, a more useful version of this would be $f_2 = f_1\sqrt{m_1/m_2}$

(a) When the mass is increased by the given value, $m_2 = 0.750kg + 0.220kg = 0.970kg$ so $f_2 =$

$(1.33\ Hz)\sqrt{0.750/0.970} = 1.17\ Hz$.

(b) When the mass is decreased by the given value, $m_2 = 0.750kg - 0.220kg = 0.530kg$ so $f_2 = (1.33\ Hz)\sqrt{0.750/0.530} = 1.58\ Hz$.

Think about what is happening here. Hanging a heavier mass on the spring results in motion with a longer period (and therefore a smaller frequency).

Problem 11 : A frictionless block of mass 2.00 kg is attached to an ideal spring with force constant 300 N/m. At t=0 the spring is neither stretched nor compressed and the block is moving in the negative direction at a speed of 12.0 m/s. Find the amplitude and phase angle and write an equation for the position as a function of time.

From the mass of the block and the spring constant, we can find the angular frequency $\omega = \sqrt{k/m} = \sqrt{300/2} = 12.25\ s^{-1}$.

We have information about the position and velocity of the block at $t = 0$ so we can use our 'initial conditions' equations to find the amplitude and initial phase angle:

$A = \sqrt{x_o^2 + v_{ox}^2/\omega^2} = \sqrt{0 + (12)^2/12.25^2} = 0.98\ m$.

Looking at our generic equation for the position as a function of time, we have $x(t) = A\cos(\omega t + \phi)$. At $t = 0$, the block is located at $x = 0$ which means that $\cos\phi$=0 which implies that $\phi = \pi/2$ or $\phi = -\pi/2$. To figure out which one we need here, if we look now at the velocity equation, we have $v_x(t) = -A\omega\sin(\omega t + \phi)$. At $t = 0$ this becomes $v_{ox} = -A\omega\sin\phi$. A is defined to always be positive and $v_{ox} = -12\ m/s$ so it looks like we need to pick the ϕ in a way that makes $\sin\phi$ be positive. That forces us to pick $\phi = +\pi/2$ and rule out the $-\pi/2$ option.

So finally $x(t) = (0.98\ m)\cos(12.25t + \pi/2)$. We can push this a little further though using some trig identities: $\cos(\theta + \pi/2) = -\sin\theta$ so we can write our equation of motion as $x(t) = -(0.98\ m)\sin(12.25t)$ which has a simpler form but doesn't match the 'standard form' the book uses.

We have the same uncertainty about the exact value of ϕ if we use the equation $\phi = -arctan(\frac{v_{ox}}{\omega x_o}$. With the numbers we have here, $x_o = 0$ so the denominator is zero and we then want angles such that the tangent function is infinite. That would occur at $\pi/2$ or $-\pi/2$ (and any angles equal to those plus or minus integer multiples of π for that matter). But let's look at what is occurring at the instant just before the block reaches $x = 0$. The block is moving to the left, so right before it passes this point, the value of x is some tiny positive number. The velocity at that point is negative, so we've basically taking the arc-tangent of negative infinity. The first solution to that is $-\pi/2$. ϕ is the negative of the arc-tangent result though so that implies $\phi = +\pi/2$ here.

Either way, we need to do some hand-waving to narrow down the possibilities on ϕ to get the 'right' one.

Problem 15 : This procedure has actually been used to "weigh" astronauts in space. A 42.5 kg chair is attached to a spring and allowed to oscillate. When it is empty the chair takes 1.30 s to make one complete vibration. But with a person sitting in it, with her feet off the floor, the chair now takes 2.54 s for one cycle. What is the mass of the person?

As with an earlier problem, we can do this without actually ever computing the spring constant for the chair. $\omega = \sqrt{k/m}$ but $\omega = 2\pi f$ and $f = 1/T$ so $T = 1/f = 2\pi/\omega = 2\pi\sqrt{m/k}$. If we square both sides, we have $T^2 = 4\pi^2 m/k$. k is some (as yet unknown) constant but we're looking for how the period changes with the mass. If we put the mass on the right hand side, we end up with $T^2/m = 4\pi^2/k$ but that right hand side is constant. So T and m are related in a way that T^2/m is constant.

We have want to compare two situations then, let's label the first case (the chair oscillating by itself for example) with a subscript of 1 and the other case (the chair with the astronaut) with a subscript of 2. Then $(T_2)^2/m_2 = (T_1)^2/m_1$ or solving for m_2: $m_2 = m_1(T_2)^2/(T_1)^2$.

Here, we have $T_1 = 1.30\ s$, $T_2 = 2.54\ s$ and $m_1 = 42.5\ kg$ so $m_2 = (42.5)(2.54/1.30)^2 = 162.2\ kg$.

The mass in this case is the combined mass of the astronaut and the chair, so the mass of the astronaut alone would be $162.2 - 42.5 = 119.7\ kg$. (On earth, a mass of 1 kg has a weight of about 2.2 pounds, so on earth this astronaut would weigh 263 pounds, which is a bit heavier than any astronaut is likely to weigh, so perhaps they were wearing a space suit when this measurement was done...)

Problem 18 : A 0.500 kg mass on a spring has velocity as a function of time given by
$v_x(t) = (3.60\ cm/s)\sin[(4.71\ s^{-1})t - \pi/2]$.
What are (a) the period, (b) the amplitude, (c) the maximum acceleration of the mass, and (d) the force constant of the spring?

The generic equation for the velocity of an object in SHM is $v_x(t) = -A\omega\sin(\omega t + \phi)$, which is not exactly the form we were given. But $\sin(\theta + \pi) = -\sin(\theta)$ or $\sin(\theta) = -\sin(\theta + \pi)$. So we can take our original equation, add π to the argument of the sine function and multiply the result by -1 and end up with exactly the same equation. The original equation for the velocity becomes: $v_x(t) = -(3.60\ cm/s)\sin[(4.71\ s^{-1})t + \pi/2]$. This puts the equation into our standard form, but we note that by doing so, we didn't change the numeric values we're interested in. Since A and ω are defined to be positive by convention, we can still pick the terms we need off of the original equation if we wanted to.

(a) Looking at the form of the equation, we see that $\omega = 4.71\ s^{-1}$. $T = 2\pi/\omega$ so here $T = (2)(\pi)/(4.71\ s^{-1}) = 1.33\ s$.

(b) In the case of velocity, the constant in front of the sin represents $-A\omega$ so here $-(A)(4.71\ s^{-1}) = -0.036\ m/s$ or $A = 0.00764\ m$ or $A = 7.64\ mm$.

(c) In the equation for the acceleration, the magnitude of the maximum value will be $a_{max} = \omega^2 A$ so here $a_{max} = (4.71\ s^{-1})^2 \times (0.00764\ m) = 0.169\ m/s^2$.

(d) To find the force constant, we know that $\omega = \sqrt{k/m}$ so $\omega^2 = k/m$ or $k = m\omega^2$. Here, $k = (0.500\ kg)(4.71\ s^{-1})^2 = 11.1\ kg/s^2$.

That doesn't look like the normal units for a spring constant, which is usually given as some number of Newtons per meter. A Newton is a unit of force, which has the same units as mass times acceleration (F=ma). So Newtons has units of $kg\ m/s^2$. A spring constant has units of N/m so dividing $kg\ m/s^2$ by m we do in fact end up with kg/s^2, so apparently N/m is the same unit as kg/s^2 when we reduce each to the most basic units (kg, m, s). For part (d), then, the more traditional way of writing the units there would be $k = 11.1\ N/m$.

Problem 19 : A 1.50 kg mass on a spring has displacement as a function of time given by the equation $x(t) = (7.40\ cm)\cos[(4.16\ s^{-1})t - 2.42]$. Find (a) the time for one complete vibration, (b) the force constant of the spring, (c) the maximum speed of the mass, (d) the maximum force on the mass, (e) the position, speed and acceleration of the mass at $t = 1\ s$, and (f) the force on the mass at that time.

First, we compare the equation we have to the generic equation for the position as a function of time in SHM: $x(t) = A\cos(\omega t + \phi)$. Immediately, we can pick off $A = 0.074\ m$ and $\omega = 4.16\ s^{-1}$.

(a) The period: $T = 2\pi/\omega = (2)(\pi)/(4.16 rad/s) = 1.51\ s$.

(b) The force constant: we know that $\omega = \sqrt{k/m}$ or squaring both sides and rearranging terms, we have $k = m\omega^2$. For the situation here, $k = (1.50\ kg)(4.16\ rad/s)^2 = 26.0\ N/m$. (See the last paragraph in the preceding problem on getting the units of k into the traditional form.)

(c) The maximum speed for an object in SHM is simply $v_{max} = \omega A$ so here $v_{max} = (4.16\ rad/s)(0.074\ m) = 0.308\ m/s$ (or $30.8\ cm/s$)

(d) $F = ma$ so we can find the maximum force on the object by determining its maximum acceleration. But for an object in SHM, we know that $a_{max} = \omega^2 A$ so here $a_{max} = (4.16\ rad/s)^2(0.074\ m) = 1.281\ m/s^2$. Thus the maximum force will be $F_{max} = ma_{max} = (1.50\ kg)(1.281\ m/s^2) = 1.92\ N$.

Alternatively, the maximum force on the object will occur when it is located the farthest distance from its equilibrium position. That is, when $x = A$ at which point the force will have a magnitude of kA or $(26.0\ N/m)(0.074\ m) = 1.92\ N$.

(e) Here we want to find the position, speed, and acceleration at $t = 1\ s$. We have an equation for the position x as a function of time, so simply plugging in $t = 1$, we find that $x(1) = -0.0125\ m$. Given the equation for the position, the equation for velocity is $v = -A\omega\sin(\omega t + \phi)$ so here $v(t) = -(0.308\ m/s)\sin(4.16t - 2.42)$ and evaluating this at $t = 1$ gives $v(1) = -0.303\ m/s$. The problem asked for the SPEED, not the velocity, so at $t = 1\ s$, the SPEED of the object is just $0.303\ m/s$. We could derive an equation for the acceleration and plug in the desired time to find a but here we have a shortcut. The acceleration and position are related through $a = -\omega^2 x$ (remember when we differentiated the position, we brought out a factor of ω and converted the cosine into a sine. Differentiating again to get the acceleration brings out a second factor of ω and turns the sine back into a cosine, which is right where we started. The net result is simply $a = -\omega^2 x$. So here, $a = -(4.16\ rad/s)^2 \times (-0.0125\ m) = +0.216\ m/s^2$.

Note that here we have another way of finding the velocity at a given position. We know that $v_x = \sqrt{k/m}\sqrt{A^2 - x^2}$ also. $\sqrt{k/m}$ is just ω which we knew from the beginning was 4.16 rad/s. Here $A = 0.074\ m$ and $x = -0.0125$ (we found that at the beginning of part (e). So $v_x = (4.16)\sqrt{(0.074)^2 - (-0.0125)^2} = 0.303\ m/s$ also. (This method only gives us the magnitude of v_x, not its sign, but in this case we only wanted the speed anyway.)

(f) The force on the mass at this time is just $F = ma = (1.50\ kg)(0.216\ m/s^2) = 0.324\ N$

Problem 23 : A 0.500 kg glider, attached to the end of an ideal spring with force constant $k = 450N/m$, undergoes simple harmonic motion with an amplitude 0.040 m. Compute (a) the maximum speed of the glider, (b) the speed of the glider when it is at $x = -0.015\ m$, (c) the magnitude of the maximum acceleration of the glider, (d) the acceleration of the glider at $x = -0.015\ m$, and (e) the total mechanical energy of the glider at any point in its motion.

Rather than setting up equations for the position, velocity, and acceleration, we can approach this problem more easily by looking at the energy. At any point in the motion, we have kinetic energy of $K = \frac{1}{2}mv^2$ in the moving glider, plus elastic potential energy stored in the spring of $U_{el} = \frac{1}{2}kx^2$. The sum of these gives the total mechanical energy in the system and this quantity is conserved unless there are some other external forces acting in a way to do positive or negative work on the system. We have no such forces here.

When the glider is out at its maximum amplitude, all the energy is stored in the elastic potential energy of the spring. At that point, $x = A$ and the glider is not moving, so at that point it has a kinetic energy of zero and a potential energy of $U_{el} = \frac{1}{2}kA^2 = (0.5)(450\ N/m)(0.040\ m)^2 = 0.360\ N\ m = 0.360\ J$. The total mechanical energy then will always be $0.360\ J$ as the object swings back and forth - switching energy between kinetic and potential.

(a) The maximum speed of the glider occurs when the spring is at its equilibrium length, at which point no energy at all is stored in the spring, and all the available energy is in the motion of the object. So at this point, $\frac{1}{2}mv^2 = 0.360\ J$ or $(0.5)(0.500\ kg)(v)^2 = 0.360\ J$ and $v = 1.20\ m/s$.

(b) When the glider is located at $x = -0.015\ m$, we're midway through the motion. The object is not at the equilibrium point but has not yet reached the maximum amplitude either. At this point, the spring is storing potential energy of $\frac{1}{2}kx^2 = (0.5)(450\ N/m)(-0.015\ m)^2 = 0.0506\ J$. That means that $0.360J - 0.0506J = 0.3094\ J$ of energy must be in the form of the kinetic energy of the moving object. So at this point, $\frac{1}{2}mv^2 = 0.3094\ J$ so $(0.5)(0.5\ kg)(v)^2 = 0.3094\ J$ or $|v| = 1.11\ m/s$.

Alternatively, the book takes the same conservation of energy principle we just used and derives a useful equation that relates the velocity directly to the position: $|v_x| = \sqrt{k/m}\sqrt{A^2 - x^2}$. So here, $|v_x| = \sqrt{450/0.5}\sqrt{(0.04)^2 - (0.015)^2} = (30)(0.037) = 1.11\ m/s$ as well.

(c) The maximum acceleration occurs when the object is out at the maximum distance from equilibrium. $F = ma$ so $a = F/m$ but $F = -kx$ so $a = -kx/m$. We're just interested in the magnitude here, so $|a_{max}| = kA/m = (450\ N/m)(0.040\ m)/(0.50\ kg) = 36\ m/s^2$.

(d) To find the acceleration at $x = -0.015\ m$, just as in the previous step, we have $F = ma$ so $a = F/m$. The force is coming from the extended or compressed spring and has a value of $F = -kx$ so combining these, $a = -kx/m$. At $x = -0.015\ m$ then we have $a = -(450\ N/m)(-0.015\ m)/(0.5\ kg) = +13.5\ m/s^2$.

(e) Here we are looking for the total mechanical energy, which we already computed in the discussion before answering part (a) above, and found to be $0.360\ J$.

Problem 26 : A toy of mass $0.150\ kg$ is undergoing SHM on the end of a horizontal spring with force constant $k = 300\ N/m$. When the object is a distance $0.012\ m$ from its equilibrium position, it is observed to have a speed of $0.300\ m/s$. What are: (a) the total energy of the object at any point in its motion, (b) the amplitude of the motion, (c) the maximum speed attained by the object during its motion?

We have the initial position and velocity, so one might be tempted to use the equations the book derived to calculate the amplitude, but we can approach this entirely from a standpoint of conservation of energy.

As the object moves back and force, energy is migrating between the kinetic energy of the moving mass and the potential energy stored in the spring. The sum of these is the total mechanical energy, which is conserved, so will always have the same value anywhere in the motion.

(a) At the initial conditions, we have the spring extended by $0.012\ m$ so at this point it is storing potential energy of $U_{el} = \frac{1}{2}kx^2 = (0.5)(300\ N/m)(0.012\ m)^2 = 0.02160\ J$. We also have the moving toy, which has a kinetic energy of $K = \frac{1}{2}mv^2 = (0.5)(0.150\ kg)(0.300\ m/s)^2 = 0.00675\ J$. At this point, then, the total mechanical energy is $E = 0.02160J + 0.00675J = 0.02835\ J$. This amount of energy will always be there, no matter where the object is located as it moves back and forth.

(b) When the toy reaches its maximum displacement from the equilibrium length of the spring, all of the available energy has been stored in the spring, leaving nothing for the kinetic energy of the toy (causing it to stop there). At this location, $K = 0$ and $U_{el} = \frac{1}{2}kA^2$. We have to have $0.02835\ J$ of energy at all times though, so we can use that fact to compute how far the spring has been displaced (i.e. the amplitude of the motion of the toy): $0.02835\ J = \frac{1}{2}kA^2 = (0.5)(300\ N/m)(A)^2$ from which $A = 0.014\ m$. (So apparently the toy is only oscillating with an amplitude of $1.4\ cm$ which is only just barely past the point where we measured its position and velocity at $1.2\ cm$ as described in the statement of the problem.)

(c) The maximum speed of the toy occurs when the spring is neither compressed nor extended (i.e. at $x = 0$). At that point, all the available energy is in the form of the kinetic energy of the moving object, so at that point $\frac{1}{2}mv^2 = 0.02835\ J$ so $(0.5)(0.150\ kg)(v)^2 = 0.02835\ J$ from which $v = 0.615\ m/s$.

Problem 30 : A proud deep-sea fisherman hangs a 65.0 kg fish from an ideal spring having negligible mass. The fish stretches the spring 0.120 m. (a) Find the force constant of the spring. The fish is now pulled down 5.00 cm and released. (b) What is the period of oscillation of the fish? (c) What is the maximum speed it will reach?

(a) The hanging fish will pull down the spring until the weight of the fish (i.e. the force of gravity, or mg) is exactly balanced by the restoring force of the spring. At that position, $kd = mg$ so $(k)(0.120\ m) = (65.0\ kg)(9.8\ m/s^2)$ from which $k = 5308\ N/m$.

(b) Now that we know the spring constant, we can compute the angular frequency $\omega = \sqrt{k/m} = \sqrt{5308/65} = 9.04\ rad/sec$ at which the fish will oscillate up and down. The period is related to ω by $T = 2\pi/\omega = (2)(\pi)/(9.04\ rad/sec) = 0.695\ s$.

(c) The fish will oscillate about its initial position with an amplitude of 5 cm (i.e. 0.05 m). For SHM, the maximum velocity can be found from $|v_{max}| = \omega A = (9.04\ rad/s)(0.05\ m) = 0.452\ m/s$.

(Note that the angular frequency ω and therefore f and T only depend on the spring constant and the mass of the fish, not on the amplitude of the oscillation. The maximum speed, on the other hand, does depend on the amplitude. So if you pulled the fish down further, it would still oscillate with the same frequency and period, but would pass through the equilibrium position faster.)

Problem 32 : A thrill-seeking cat with mass 4.00 kg is attached by a harness to an ideal spring of negligible mass and oscillates vertically in SHM. The amplitude is 0.050 m, and at the highest point of the motion the spring has its natural unstretched length. Calculate the elastic potential energy of the spring (take it to be zero for the unstretched spring), the kinetic energy of the cat, the gravitational potential energy of the system relative to its lowest point of the motion, and the sum of these three energies (i.e. the total mechanical energy of the system) when the cat is (a) at it highest point, (b) at its lowest point, and (c) at its equilibrium position.

We'll look at this from the point of view of conservation of energy. At the highest point in its motion, the cat has restored the spring to it's original unstretched length (i.e. the length it had when no cat was attached). At the bottom, it has stretched the spring by a total of $2A$ or 0.10 m.

(a) At the top of the motion, the cat is not moving (so no kinetic energy), the spring is unstretched (so $U_{el} = 0$) but the cat is 0.10 m above its lowest point, so is 'storing' gravitational potential energy of $U_g = mgy = (4.00\ kg)(9.8\ m/s^2)(0.10\ m) = 3.92\ J$.

There are no 'other' forces at work here (other than gravity and the spring), so this will be the total mechanical energy in the system and will remain exactly 3.92 J at any point in the motion.

(b) At the lowest point, all the gravitational potential energy is gone so $U_g = 0$ and the cat is again not moving so $K = 0$, so all the 3.92 J of energy we have must be stored in the spring, so $U_{el} = 3.92\ J$. Note at this point we could compute the spring constant if we wanted to since $U_{el} = \frac{1}{2}kx^2$ and here the spring has been stretched by a length of $x = 0.10\ m$ relative to its unstretched (i.e. un-cat) length. So $\frac{1}{2}(k)(0.1)^2 = 3.92\ J$ and $k = 784\ N/m$. Turns out we don't need this at any point to solve this problem though.

(c) At the cat's equilibrium position (i.e. exactly in the middle between the highest and lowest points), we have energy in all three forms. The spring is stretched 0.05 m from its rest length, the cat is 0.05 m above its lowest point, so it has some gravitational potential energy. Whatever remains will be the kinetic energy of the cat. So at this midpoint, $U_g = mgy = (4.00\ kg)(9.8\ m/s^2)(0.05\ m) = 1.96\ J$. To find the elastic potential energy, note that U_{el} has the form $\frac{1}{2}kx^2$. Since the spring is HALF it's fully extended length, is will be storing ONE QUARTER as much energy as it was at that point, so $U_{el} = \frac{1}{4}(3.92\ J) = 0.98\ J$. The total energy still has to be conserved, though, so $K + U_{el} + U_g = 3.92\ J$ becomes $K + 0.98J + 1.96J = 3.92J$ or $K = 0.98\ J$. (From which we could calculate the speed of that cat at this point from $K = \frac{1}{2}mv^2$ or $0.98J = (0.5)(4.00\ kg)(v)^2$ or $|v| = 0.7\ m/s$. So the cat will pass through the midpoint with a speed of 0.7 m/s.

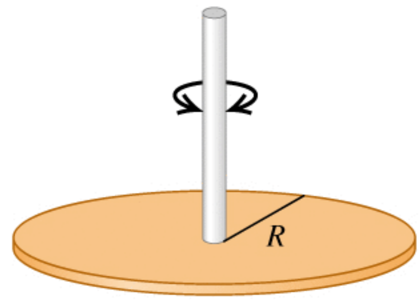
Problem 34 : A uniform, solid metal disk of mass 6.50 kg and diameter 24.0 cm hangs in a horizontal plane, supported at its center by a vertical metal wire. You find that it requires a horizontal force of 4.23 N tangent to the rim of the disk to turn it by 3.34^o, thus twisting the wire. You now remove this force and release the disk from rest. (a) What is the torsion constant for the metal wire? (b) What is the frequency and period of the torsional oscillations of the disk? (c) Write an equation of motion for $\theta(t)$ of the disk.

(a) Similar to the case of a spring where we have $F = -kx$, for a torsion spring, we have $\tau = -\kappa\theta$ where θ is measured in RADIANS, not DEGREES. Since we are applying the force tangent to the rim of the disk, the restoring torque being produced by the wire will be just $\tau = -FR$. (Why negative? Let's say the force we applied caused the disk to rotate through a positive number of degrees. Then the torque we applied is coming out of the paper (in the +k direction) so the torque the wire is generating is in the opposite direction. So: $\tau = -\kappa\theta$ becomes $-FR = -\kappa\theta$ or $\kappa = FR/\theta$. To convert from degrees to radians, we note that 2π radians is the same as 360^o so $3.34^o = 3.34^o \times \frac{2\pi}{360^o} = 0.0583\ rad$. Finally, $\kappa = FR/\theta = (4.23\ N)(0.120\ m)/(0.0583\ rad) = 8.71 N \cdot m/rad$. (Note at this step we needed the RADIUS of the disk but they gave us the DIAMETER, so we divided that number by 2, and converted from cm to m.)

(b) The frequency of oscillation for such a torsion spring is given by $f = \frac{1}{2\pi}\sqrt{\kappa/I}$ where I is the moment of inertia of the disk. For a solid, uniform disk, the moment of inertia is $I = \frac{1}{2}MR^2 = (0.5)(6.50\ kg)(0.120\ m)^2 = 0.0468\ kg\ m^2$, so $f = \frac{1}{2\pi}\sqrt{\kappa/I}$ becomes $f = \frac{1}{2\pi}\sqrt{8.71/0.0468} = 2.17\ Hz$. The period $T = 1/f = 0.461\ s$. We'll need ω for the next part, so we'll go ahead and compute that here: $\omega = 2\pi f = 13.6\ rad/s$.

(c) For angular motion, our generic equation is $\theta(t) = \Theta\cos(\omega t + \phi)$. Θ is the amplitude of the motion and ϕ the initial phase. In this case, we twisted the disk, then let it go, so the motion is starting with its maximum amplitude already, so $\Theta = 3.34^o$ and $\phi = 0$ so $\theta(t) = (3.34^o)\cos(13.6\ rad/s\ t)$

Problem 36 : A thin metal disk of mass $2.00 \times 10^{-3} kg$ and radius 2.20 cm is attached at its center to a long fiber. The disk, when twisted and released, oscillates with a period of 1.00 s. (a) Find the torsion constant of the fiber.



The angular frequency $\omega = \sqrt{\kappa/I}$. Ultimately we want to solve for the value of κ so if we square both sides of this equation and then multiply through by I, we arrive at $\kappa = I\omega^2$. The problem gave us T, not ω. We can certainly convert, but let's go ahead and make some substitutions so we can end up with an equation that directly relates κ to T. (Who knows, we may end up having to do a lot of these, so rather than go through the conversions all the time we'll just do it once.) $\omega = 2\pi f$ and $f = 1/T$ so $\omega = 2\pi/T$ leaving us finally with: $\kappa = I\omega^2 = 4\pi^2 I/T^2$.

The moment of inertia of a disk is $I = \frac{1}{2}MR^2$ so here $I = (0.5)(.002\ kg)(0.022\ m)^2 = 4.84 \times 10^{-7}\ kg\ m^2$. Substituting in this value, and the known period of 1.00 s we arrive at $\kappa = 1.91 \times 10^{-5}\ N\ m/rad$.

Problem 37 : You want to find the moment of inertia of a complicated machine part about an axis through its center of mass. You suspend it from a wire along this axis. The wire has a torsion constant of 0.450 $N\ m/rad$. You twist the part a small amount about this axis and let it go, timing 125 oscillations in a time of 265 s. (a) What is the moment of inertia you want to find?

The angular frequency $\omega = \sqrt{\kappa/I}$. Squaring both sides, $\omega^2 = \kappa/I$ or rearranging: $I = \kappa/\omega^2$. But $\omega = 2\pi f$ so we can also write this as $I = \kappa/(2\pi f)^2$.

The object goes through 125 oscillations in 265 seconds, so the frequency is $f = (125\ oscillations)/(265\ s) = 0.4717\ Hz$. Finally $I = \kappa/(2\pi f)^2 = (0.450)/(2\pi \times 0.4171)^2 = 0.0512\ kg\ m^2$.

Problem 41 : You pull a simple pendulum of length 0.240 m to the side through an angle of 3.50^o and release it. (a) How much time does it take the pendulum bob to reach its highest speed? (b) How much time does it take if the pendulum is released at an angle of 1.75^o instead of 3.50^o?

The highest speed is reached when the pendulum is just passing through its lowest point. At this point, the pendulum has moved through one quarter of a complete period. So if we can find the period for a full oscillation, we can divided by four to find the time it takes to reach this 'maximum speed' point.

The period for a complete swing is given by $T = 2\pi\sqrt{L/g}$ so here $T = 2\pi\sqrt{(0.240\ m)/(9.8\ m/s^2)} = 0.983\ s$. To reach the bottom of the arc (the point where the speed is maximum) we have moved through 1/4 of a period, or $(0.983\ s)/4 = 0.246\ s$.

Note that the period for small angle oscillations ONLY depended on the length of the rod (and the constant g). Nowhere did the initial angle enter into the formula for the period. This approximation breaks down as the angle gets larger, but for tiny angles like this, the period doesn't depend on the initial angle at all. So the answer to (b) is identical to the answer for (a).

Problem 43 : A building in San Francisco has light fixtures consisting of small 2.35 kg bulbs with shades hanging from the ceiling at the end of light thin cords 1.50 m long. (a) If a minor earthquake occurs, how many swings per second will these fixtures make?

The bulbs are described as 'small' and the length of the cord is 1.5 m, so this has the look of a 'simple pendulum' where we can treat the mass as if it were all concentrated at the end of the cord. We don't have much choice really, since otherwise we would need the moment of inertia of the bulbs and we don't have the information we need here to calculate it.

The period of oscillation for a (simple) pendulum is $T = 2\pi\sqrt{L/g}$ which does not depend on the mass of the object at all. For the given length of cord, the period will be $T = 2\pi\sqrt{(1.50\ m)/(9.8\ m/s^2)} = 2.46\ s$. The problem is asking for the number of swings per second, which is the frequency. $f = 1/T$ so $f = 1/(2.46\ s) = 0.407$ swings/second.

Problem 44 : A certain simple pendulum has a period on the earth of 1.60 s. (a) What is its period on the surface of Mars, where $g = 3.71\ m/s^2$?

We don't know the length of the pendulum (although we could compute it since we're given the period of oscillation on the earth). Let's turn this into a ratio problem so we don't need to compute L at all.

$T = 2\pi\sqrt{L/g}$. Let's use subscripts of e and m to indicate the earth and mars respectively. Then: $T_e = 2\pi\sqrt{L/g_e}$ and $T_m = 2\pi\sqrt{L/g_m}$. Dividing the second equation by the first: $T_m/T_e = \sqrt{L/g_m}/\sqrt{L/g_e} = \sqrt{g_e/g_m}$ Finally, $T_m = T_e\sqrt{g_e/g_m}$.

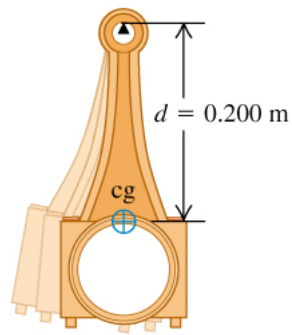
For our particular situation, $g_e = 9.8\ m/s^2$, $g_m = 3.71\ m/s^2$, and $T_e = 1.60\ s$ so $T_m = (1.60\ s)\sqrt{9.8/3.71} = 2.60\ s$. (So a grandfather clock on Mars would run quite a bit more slowly than one on Earth.)

Problem 47 : After landing on an unfamiliar planet, a space explorer constructs a simple pendulum of length 50.0 cm. She finds that the pendulum makes 100 complete swings in a time of 136 s. (a) What is the value of g on this planet?

The period of a pendulum is given by $T = 2\pi\sqrt{L/g}$. Here, the pendulum completes 100 swings in 136 seconds, so each swing must be 1.36 seconds long (i.e. $T = 1.36\ s$).

Squaring our equation for the period and rearranging to solve for g, we have: $g = 4\pi^2 L/T^2$ so here $g = 4\pi^2(0.50\ m)/(1.36\ s)^2$ or $g = 10.67\ m/s^2$.

Problem 49 : A 1.80 kg connecting rod from a car engine is pivoted about a horizontal knife edge as shown in the figure. The center of gravity of the rod was located by balancing and is 0.200 m from the pivot. When it is set into small amplitude oscillation, the rod makes 100 complete swings in 120 s. (a) Calculate the moment of inertia of the rod about the rotation axis through the pivot?



For a physical pendulum like this (i.e. a pendulum where we can't assume the mass is all located at a point, but instead represents an extended object), the period of motion is related to the moment of inertia about the pivot and the distance from that pivot to the center of mass d by: $T = 2\pi\sqrt{I/(mgd)}$.

Here, we know the object made 100 swings in 120 seconds, so the period for one swing would be 1.2 seconds. We also know the mass and the value of d so we know everything in this equation except the moment of inertia I. Rearranging the equation to solve for I: $I = mgd(\frac{T}{2\pi})^2$ so here $I = (1.80\ kg)(9.8\ m/s^2)(0.20\ m)(\frac{1.2\ s}{2\pi})^2 = 0.129\ kg\ m^2$.

Problem 50 : We want to support a thin hoop by a horizontal nail and have the hoop make one complete small-angle oscillation each 2.0 s. (a) What must the hoop's radius be?

The moment of inertia of a ring about its center is $I = MR^2$, where M is the mass of the ring (or hoop) and R is its radius. But here we want the ring to pivot about a point on its circumference, so we need to use the parallel axis theorem to 'move' the value of I to this point. $I = I_{CM} + Md^2$ where d is the distance between the center of mass and the new point we're rotating around. The center of mass of the ring will be at its center, so here $d = R$. Then the moment of inertia about this pivot point becomes: $I = MR^2 + MR^2 = 2MR^2$.

For a physical pendulum like this (i.e. a pendulum where we can't assume the mass is all located at a point, but instead represents an extended object), the period of motion is related to the moment of inertia about the pivot and the distance from that pivot to the center of mass d by: $T = 2\pi\sqrt{I/(Mgd)}$. Here, $d = R$ so $T = 2\pi\sqrt{(2MR^2)/(MgR)} = 2\pi\sqrt{2R/g}$. Note at this point that the mass of the ring has completely disappeared and the period of oscillations only depends on the radius of the ring. (As well, note that if we shrunk the hoop down and concentrated all its mass at its center of mass, then this becomes a simple pendulum with a length of L which would give a period of $2\pi\sqrt{R/g}$ so the hoop has a period that is $\sqrt{2}$ times longer than that simple pendulum.)

Anyway, going back to the problem at hand, squaring both sides we get $T^2 = (2\pi)^2 \times 2R/g$ or $R = gT^2/(8\pi^2)$

For the values here, $T = 2.0\ s$ so $R = (9.8)(2.0)^2/(8\pi^2) = 0.496\ m$.

(If you have an old hula-hoop around somewhere, I think those had a diameter of about 1 meter, so $R = 0.5\ m$. If you balance it on your finger and let it oscillate, it should have a period of $T = 2\pi\sqrt{2R/g} = (2)(\pi)\sqrt{(2)(0.5)/(9.8)} = 2.01\ s$ which closely matches the numbers in this problem.)

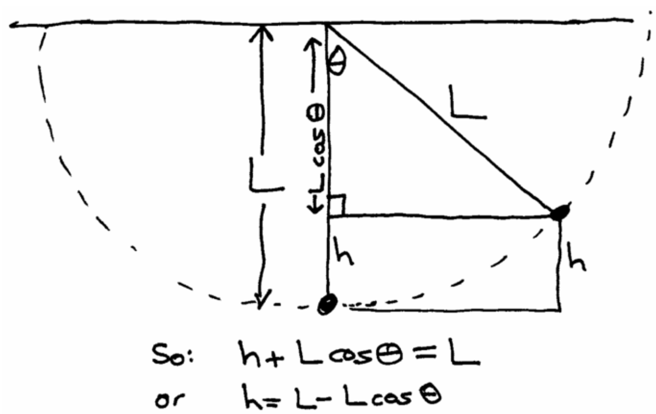
Problem 52 : A 1.80 kg monkey wrench is pivoted 0.250 m from its center of mass and allowed to swing as a physical pendulum. The period for small-angle oscillations is 0.940 s. (a) What is the moment of inertia of the wrench about an axis through the pivot? (b) If the wrench is initially displaced 0.400 rad from its equilibrium position, what is the angular speed of the wrench as it passes through the equilibrium position?

(a) The period of a pendulum formed from an extended physical object (instead of just a tiny point mass at the end of a massless string) involves the moment of inertia of the object: $T = 2\pi\sqrt{I/(mgd)}$. In this problem, we know the mass ($m = 1.80\ kg$) and the period ($T = 0.940\ s$), and also the distance between the pivot point and the center of mass of the object ($d = 0.250\ m$) so we can directly use this equation to solve for I.

First, let's rearrange this equation to isolate I. Squaring both sides we have $T^2 = (2\pi)^2 I/(mgd)$ or $I = (mgdT^2)/(2\pi)^2$. For this particular object then, $I = (1.8)(9.8)(0.25)(0.940)^2/(2\pi)^2 = 0.0987\ kg\ m^2$.

(b) In more real-life terms, 0.400 rad is an angle of $(0.400\ rad) \times \frac{360^o}{2\pi\ rad} = 22.9^o$ which is NOT a small angle, so the small angle approximations are not accurate. We can use energy considerations though to solve this. At the initial position, the center of mass of the object is elevated above its lowest point, so we are storing some gravitational potential energy. At the bottom of the arc, we have an object with moment of inertia I rotating at an angular speed of Ω which means it has a kinetic energy of $K_r = \frac{1}{2}I\Omega^2$. Why are we using the capital Ω here? During the derivation of SHM for a swinging object, we calculated $\omega = \sqrt{mgd/I}$ which is some constant value that represents the angular frequency of the oscillations of the pendulum. It is the ω that appears in $\theta(t) = \Theta\cos(\omega t + \phi)$. As the object swings, it has some angular speed that we can compute from $d\theta/dt$ and that value is NOT constant. It is zero when the object has swung out to its maximum angle then it increases to some maximum value as the object swings through the bottom of its arc. So whatever this is, it's not constant and we've already 'used up' the symbol ω so we define the instantaneous angular frequency as $d\theta/dt = \Omega$ and just use the capital letter version of the symbol instead of the lower case one.

So getting back to the problem at hand if you draw out a figure of the object when its swung out at the starting angle, we need to determine the elevation of the object (well, the center of mass of the object) relative to its lowest point. Turns out this is $U_g = mgd(1 - \cos\theta_o)$ where d is the length of the 'string' and θ_o is the initial angle. (The figure shows the calculation for a point mass. For an extended object, instead of L, the appropriate value becomes the distance from the pivot to the location of the center of mass d.)



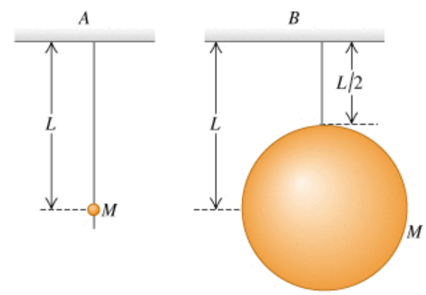
From conservation of energy, the gravitational potential energy the object has when it has swung out to its maximum angle will be equal to the (rotational) kinetic energy it has at the bottom of the swing. So: $mgd(1 - \cos\theta_o) = \frac{1}{2}I\Omega^2$ where Ω represents the angular speed of the object as it swings through the bottom of the arc: that is, it is the maximum angular speed (which is what the problem is asking for). We can simplify this further by multiplying both sides by $2/I$ so we can isolate Ω: $\Omega^2 = 2(mgd/I)(1 - \cos\theta_o)$. But $mgd/I = \omega^2$ so $\Omega^2 = 2\omega^2(1 - \cos\theta_o)$ or $\Omega = \omega\sqrt{2 - 2\cos\theta_o}$. $\omega = 2\pi/T = 6.684\ rad/sec$ and $\theta_o = 0.4\ rad$ so $\Omega = (6.684\ rad/sec)\sqrt{2 - 2\cos(0.4)} = 2.66\ rad/sec$.

Problem 54 : A holiday ornament in the shape of a hollow sphere with mass $2.0 \times 10^{-2} kg$ and radius $5.0 \times 10^{-2} m$ is hung from a tree limb by a small loop of wire attached to the surface of the sphere. If the ornament is displaced a small distance and released, it swings back and forth as a physical pendulum. (a) Calculate its period. (You can ignore friction at the pivot. The moment of inertia of the sphere about the pivot at the tree limb is $5MR^2/3$).

The ornament is a physical pendulum, so $T = 2\pi\sqrt{I/mgd}$. The moment of inertia of a sphere about a point on its edge is already given here to be $I = 5MR^2/3$. d is the distance between the pivot point and the center of mass, which will be located at the center of the sphere (ornament) so here $d = R$. Making these substitutions, we get: $T = 2\pi\sqrt{\frac{5MR^2/3}{MgR}}$. The mass cancels out, as does one factor of R, leaving us with just: $T = 2\pi\sqrt{(5R)(3g)}$. (If we shrink the ornament so all its mass is located at the center of mass, we would have gotten a period of $T = 2\pi\sqrt{R/g}$, so the spherical ornament will apparently be swinging with a slightly longer period.) Note that yet again, the actual mass of the ornament does not enter into the equation for the period. An ornament made of lead or tin foil, as long as it is completely hollow, will oscillate with the same period - at least for small angle oscillations.

Entering the specific numbers we have for this problem, $T = 2\pi\sqrt{(5 * 0.05)/(3 * 9.8)} = 0.58\ s$.

Problem 55 : The two pendulums shown in the figure each consist of a uniform solid ball of mass M supported by a massless string, but the ball for pendulum A is very tiny while the ball for pendulum B is much larger. Find the period of each pendulum for small displacements. Which ball takes longer to swing?



Pendulum A can be treated as a simple pendulum for which $T = 2\pi\sqrt{L/g}$.

Pendulum B is a physical pendulum. Given the figure, we can see that this sphere has a radius $R = L/2$, and the distance from the center of mass to the pivot point is $d = L$. The moment of inertia of a solid sphere about its center of mass is $I_{cm} = \frac{2}{5}MR^2$ or here $I_{cm} = \frac{2}{5}M(L/2)^2 = ML^2/10$. We need the moment of inertia about the pivot point though, so using the parallel axis theorem: $I = I_{cm} + Md^2 = ML^2/10 + ML^2 = 11ML^2/10$.

The period for a physical pendulum is $T = 2\pi\sqrt{I/(mgd)}$ or here $T = 2\pi\sqrt{(11ML^2/10)/(MgL)} = 2\pi\sqrt{(11L)/(10g)}$. Factoring out terms a bit more: $T = \sqrt{11/10} \times 2\pi\sqrt{L/g} = \sqrt{11/10}T_a$ where T_a is the period of the simple pendulum. So this particular physical pendulum has a period that is $\sqrt{11/10} = 1.049$ times longer than that of the simple pendulum.

Problem 56 : A 2.20 kg mass oscillates on a spring of force constant 250.0 N/m with a period of 0.615 s. (a) Is this system damped or not? How do you know? If it is damped, find the damping constant b. (b) Is the system undamped, underdamped, critically damped, or overdamped? How do you know?

The mass is oscillating, so that means right away that the system is NOT critically damped or overdamped (since in those cases, the object does not oscillate at all - it simply exponentially drops towards the equilibrium position).

If there were no damping at all, the mass should be oscillating with a period of $T = 2\pi\sqrt{m/k} = 0.589\ s$. It's oscillating more slowly than that, so we do have some damping going on. It's not critical or overdamping, so must be underdamping.

With underdamping, the new frequency is given by $\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$ The mass is seen to be oscillating with a period of 0.615 s which represents a frequency of $\omega' = 2\pi/T = 10.22\ rad/s$. Rearranging our equation to solve for b: $b = 2m\sqrt{\frac{k}{m} - (\omega')^2}$ so here $b = (2)(2.20\ kg)\sqrt{(250\ N/m)/(2.2\ kg) - (10.22\ rad/s)^2}$ or finally $b = 13.33 kg/s$.

(Look at the original equation for damped motion: $F = -kx - bv$. F has units of Newtons which in basic terms is units of mass times acceleration or $(kg)(m)/(s)^2$. v has units of m/s so b must have units of kg/s for the product of bv to end up being in the units of a force.)

Problem 64 : Four passengers with a combined mass of 250 kg compress the springs of a car with worn-out shock absorbers by 4.00 cm when they enter it. Model the car and passengers as a single body on a single ideal spring. (a) If the loaded car has a period of vibration of 1.08 s, what is the period of vibration of the empty car?

We know that a 250 kg mass caused the 'spring' to compress by 0.04 m, so we can find the spring constant involved. $mg = kd$ so $(250\ kg)(9.8\ m/s^2) = (k)(0.04\ m)$ or $k = 61250\ N/m$.

The period of oscillation for a mass bouncing up and down on a spring is $T = 2\pi\sqrt{m/k}$. We were given the period for the situation where the people were in the car, and we now know the value of k, so we can use this to find the mass (which represents the mass of the car plus the mass of the passengers). Rearranging the equation, we have $m = (\frac{T}{2\pi})^2 k = 1810\ kg$. That was the combined mass of the car and the passengers though, so the car itself must have a mass of $1810kg - 250kg = 1560\ kg$.

The period for the car alone bouncing up and down on its springs then would be $T = 2\pi\sqrt{m/k} = (2)(\pi)\sqrt{1560/61250} = 1.003\ s$.

Problem 74 : A hanging wire is 1.80 m long. When a 60.0 kg steel ball is suspended from the wire, the wire stretches by 2.00 mm. If the ball is pulled down a small additional distance and released, at what frequency will it vibrate? Assume the stress on the wire is less than the proportional limit (section 11.5). (Basically this is just saying that the wire is acting like a spring being extended without any permanent deformation going on. If we add too much weight, the wire will stretch out permanently.)

The wire is essentially acting like a very stiff spring. Attaching the weight to the end of the wire caused it to extend slightly. At that position, we have the weight of ball down balanced by the 'spring force' of the (slightly) stretched wire pulling upward. From $mg = kd$, we can find an effective spring constant for the wire to be $k = mg/d$, where d is the tiny distance the wire was stretched.

When the ball is pulled down further and starts oscillating, the frequency will be $f = \frac{1}{2\pi}\sqrt{k/m}$ but since $k = mg/d$ we find that $f = \frac{1}{2\pi}\sqrt{(mg/d)/m} = \frac{1}{2\pi}\sqrt{g/d} = \frac{1}{2\pi}\sqrt{\frac{9.8}{.002}} = 11.1 Hz$ (Note in that last step we converted the value of d from 2.0 mm to 0.002 m to have consistent metric units throughout.)

(Note here we never actually needed to calculate the effective spring constant for the wire, but if we did, from $mg = kd$ we find that $k = mg/d = 294,000\ N/m$: a very stiff spring indeed.)

This problem produced an interesting result. Note that the final equation for the frequency of oscillation was $f = \frac{1}{2\pi}\sqrt{g/d}$. Since $T = 1/f$ we could write this as $T = 2\pi\sqrt{d/g}$. These equations ONLY depend on d. So if I hang an object on a string (rope, wire, etc) and notice that it causes the string to extend by a distance d, then I immediately know what the period (or frequency) this object will oscillate up and down with on this string.

Problem 79 : On the planet Newtonia, a simple pendulum having a bob with mass 1.25 kg and a length of 185.0 cm takes 1.42 s, when released from rest, to swing through an angle of 12.5^o, where it again has zero speed. The circumference of Newtonia is measured to be $51,400\ km$. What is the mass of the planet?

The description of the motion basically represents one half of a complete cycle: it's starting at rest out at some angle, then it swings over to the other side and (momentarily) stops again. This represents half of a complete cycle (so a complete period, with the pendulum ending up back where it started, must be $T = 2 \times 1.42s = 2.84\ s$. During this time interval, the object swung through a total angle of 12.5^o, which means it started off on one side displaced half this much or 6.25^o from the vertical. This is a small enough angle that we can use the simple pendulum equation to find the period.

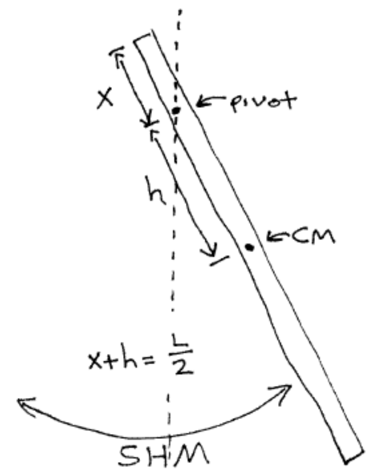
The period for a simple pendulum is given by $T = 2\pi\sqrt{L/g}$ so rearranging this to solve for g: $g = L(\frac{2\pi}{T})^2$. For this particular pendulum then: $g = (1.85)(\frac{2\pi}{2.84})^2 = 9.055\ m/s^2$.

From Physics 2, the gravitational acceleration at the surface of a planet of mass M and radius R is $g = GM/R^2$ or rearranging this we find that $M = gR^2/G$. We have the circumference of the planet: $C = 5.14 \times 10^7\ m$ (note we need to convert everything to standard metric units first) so we can find its radius from $C = 2\pi R$. Here, $R = 8.18 \times 10^6\ m$. Looking up the value of the gravitational constant G then, we have $M = gR^2/G = (9.055)(8.18 \times 10^6)^2/(6.67 \times 10^{-11})$ or $M = 9.08 \times 10^{24}\ kg$.

Comparing this planet to the earth: the radius of the earth is $6.4 \times 10^6\ m$ and the mass of the earth is about $6 \times 10^{24}\ kg$, so this planet is heavier than the earth but is also larger. Since R appears squared and in the denominator of $g = GM/R^2$, the net effect is that the acceleration due to gravity on the surface of this planet ends up being a bit less than on the earth.

Problem 80 : A student wants to use a meter stick as a pendulum. She plans to drill a small hole through the meter stick and suspend it from a smooth pin attached to the wall (figure). Where in the meter stick should she drill the hole to obtain the shortest possible period? How short an oscillation period can she obtain with a meter stick in this way?

This is a physical pendulum, so the period of oscillation is given by $T = 2\pi\sqrt{\frac{I}{mgh}}$ where I is the moment of inertia about the rotation point and h is the distance from the rotation point to the center of mass. Both of those are changing here since we're moving the point of rotation in order to make the period as small as possible.
We'll treat this as a thin rod. The center of mass (CM) of the rod will be right in the middle. Let l be the length of the rod, and h be the distance from the CM to the point where we've drilled the hole.
Then $I = I_{cm} + mh^2 = \frac{1}{12}mL^2 + mh^2$ and substituting this into our equation for T, we find that: $T = 2\pi\sqrt{(\frac{1}{12}mL^2 + mh^2)\ /\ (mgh)}$ (Note that the mass m will cancel here.)



As we change the position of the hole, h changes, in turn changing the value of the period T. We're trying to minimize the value of T so this has turned into a calculus problem: the minimum (well, and/or maximum) value will occur when $dT/dh = 0$.

Collecting terms a bit, we can write our equation as:

$T = \frac{2\pi}{\sqrt{g}}(\frac{1}{12}\frac{L^2}{h} + h)^{1/2}$

Differentiating:

$\frac{dT}{dh} = \frac{2\pi}{\sqrt{g}}\frac{1}{2}(\frac{1}{12}\frac{L^2}{h} + h)^{-1/2}(-\frac{1}{12}\frac{L^2}{h^2} + 1)$ which we now need to set equal to zero.

The constants out front we can ignore so one of those two remaining terms must be zero in order for their product to be zero:

First term: This term is raised to the $-1/2$ power, so to make this term zero, the contents inside the parentheses need to be infinity. Well we can get that by setting $h = 0$. If we go back to our original equation for the period though, $T = 2\pi\sqrt{I/(mgh)}$ so the $h = 0$ solution represents an infinitely long period. Looking at the figure, $h = 0$ means the rotation axis is right at the location of the center of mass: i.e. we've drilled the hole right at the midpoint of the meter stick. Well at that point, there is no net torque and the meter stick won't rotate at all. So this is apparently not the solution we're looking for. It represents the **maximum** period (infinity in this case), not the **minimum** period we want.

Second term: setting this term to zero implies that $(-\frac{1}{12}\frac{L^2}{h^2} + 1) = 0$.

Rearranging, we find: $\boxed{h = \frac{1}{\sqrt{12}}L}$

Up to this point, this has been generic for any long thin rod. For our meter stick $L = 1\ m$ so $h = \frac{1}{\sqrt{12}}(1\ m) = 0.288675\ m$.

This is h though, not the x that we're looking for. h is a measurement from the midpoint of the rod out to the hole. x is a measurement from the end of the rod in to the hole. So $x + h = \frac{1}{2}L$ or $x = \frac{L}{2} - h$ and finally $x = 0.5 - 0.288675 = 0.2113..\ m$. If we want the meter stick to swing with the shortest period, we should drill a hole about 21 cm in from the upper end of the stick.

Now that we have h, we earlier derived an equation for T as a function of h so we can go back and compute the actual period now to be $T = 1.53\ s$.

Let's compare this to the period if we have the hole right at the very end of the meter stick. In that case $h = L/2$ and $I = I_{cm} + mh^2$ becomes $\frac{1}{12}mL^2 + \frac{1}{4}mL^2$ or $I = \frac{1}{3}mL^2$. $T = 2\pi\sqrt{I/(mgh)}$ becomes: $T = 2\pi\sqrt{(\frac{1}{3}mL^2)/(mg\frac{L}{2})} = 2\pi\sqrt{\frac{2L}{3g}}$ which for $L = 1\ m$ becomes $T = 1.64\ s$ (which is only slightly longer that what we found for the shortest period).

If we graph $T(h)$ or $T(x)$ we see that if the hole is right on the end (i.e. where $x = 0$ and $h = L/2$) the period is $1.64\ s$. As we move the hole in to $x = 0.2213..\ m$, the period reaches a minimum at $T = 1.53\ s$, then as we keep moving the hole further in (closer to the CM), the period rises eventually to infinity at $x = 0.5$ ($h = 0$).
This graph shows $T(x)$ as x varies from 0 to 49 centimeters. The period is remarkably stable until we get pretty close to the midpoint.