PH2233 Fox : Lecture 05
Chapter 15 : Wave Motion

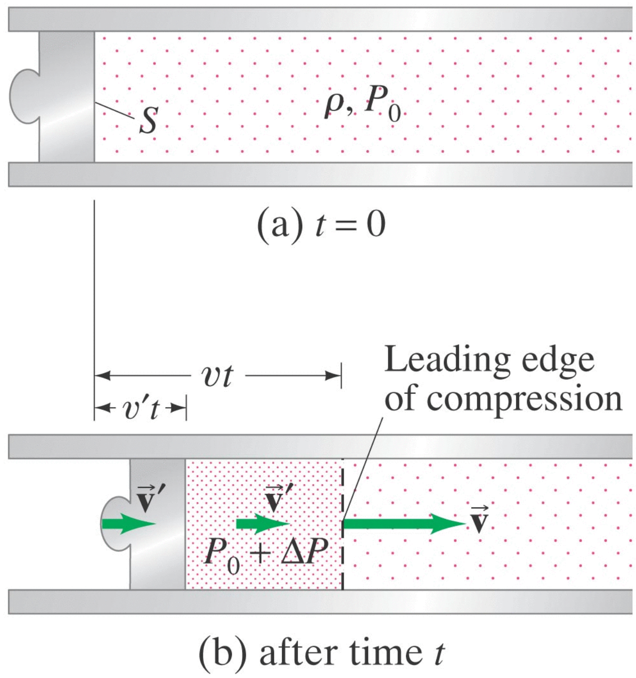
WAVE SPEED : If you make some simplified assumptions about what's going on down at the micro-level, we can derive expressions for the speed.

I'll derive one of these and then just list the other common forms and do a few examples.

In each case though, we'll find that the wave speed v only depends on physical properties of the material but doesn't depend on the amplitude or frequency of the wave. (That turns out to be not entirely correct, as we'll see later with light.)

Wave Speed Derivation
Longitudinal Waves in Gas or Liquid

Symbols:
S : cross-sectional area of piston
ρ : density of medium
P_o : ambient pressure
$P = P_o + \Delta P$: pressure induced by piston
v' : piston velocity
v : wave speed
dashed line : leading edge of compression



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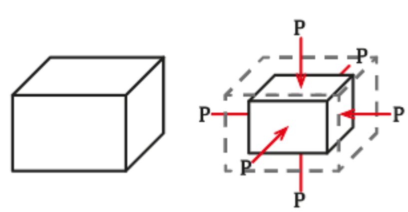
First, we need to introduce the $\boxed{\textbf{Bulk Modulus (B)}}$ idea. If an object is subjected to inward forces from all sides, its volume will decrease: $\Delta V = -\frac{1}{B} V_o \Delta P$ so $B = -\frac{\Delta P}{\Delta V / V_o}$. (Since the volume DECREASES with a pressure increase, that negative sign is thrown in to make the values of B positive.)
Let's rearrange that into a form that will be useful in the wave speed derivation: $\Delta P = -B\frac{\Delta V}{V}$.
See table 12-1 on page 343 for some selected values. (The table is included at the end of these notes too.)



We start off with our gas (or liquid) at some ambient pressure P_o. Now we start pushing the piston towards the right. The gas (or liquid) starts also moving to the right but it also causes a wave disturbance that propagates much faster. In the time Δt, the piston moves a distance of $d = v'\Delta t$ to the right, but the leading edge of the disturbance (i.e. the wave) moves a distance of $d = v\Delta t$ to the right.

The compressed part of the medium is moving along with the piston, so it has picked up a momentum of $p = mv'$. What is the mass involved? It's all the molecules in the medium between the piston and where the leading edge of the compression (the wave) is located at that time. Looking at the upper figure, that would be the original volume times the uncompressed density, so $m = (S)(vt)\rho$.

We can write the momentum then as $p = \rho S v v' \Delta t$. Initially that mass was at rest, so technically we can write this as $\Delta p = \rho S v v' \Delta t$: that's how much the momentum of that amount of material has changed.

We can write that as $\frac{\Delta p}{\Delta t} = \rho S v v'$ but $\frac{\Delta p}{\Delta t}$ is just the **force** that was applied, which itself is the change in pressure ΔP times the area of the piston, so $\rho S v v' = (\Delta P)(S)$. Cancelling the S that appears on both sides: $\boxed{\Delta P = \rho v v'}$.

Let's bring in our bulk modulus now. $\boxed{\Delta P = -B \frac{\Delta V}{V}}$.

The original volume of the material is $V = (S)(v\Delta t)$.

The amount the volume has been changed (reduced) is $\Delta V = (S)v'\Delta t$.

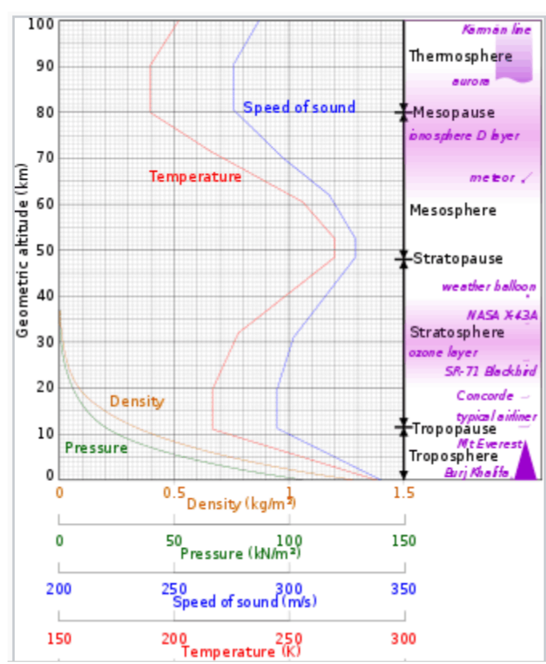
Setting the two boxed equations equal to one another:

$\rho v v' = -B \frac{(-Sv'\Delta t)}{Sv\Delta t}$ and after cancelling some common terms: $\rho v = B/v$ which we can rearrange into $\boxed{v = \sqrt{B/\rho}}$, giving us the wave speed in a liquid or gas.

Example: Speed of Sound in Air

For air (at STP) : $B = 1.42 \times 10^5\ N/m^2$ and $\rho = 1.293\ kg/m^3$ so sound waves should travel through air at about: $v = \sqrt{B/\rho} = \sqrt{(1.42 \times 10^5)/(1.293)} \approx 331\ m/s$.

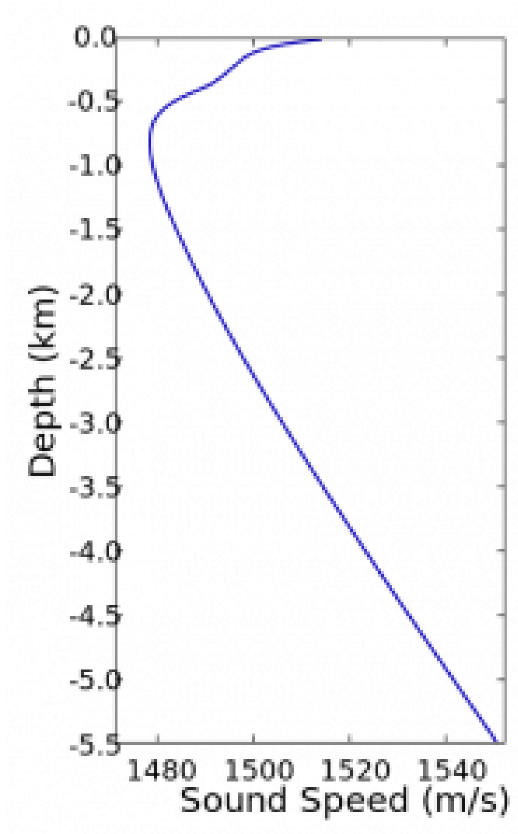
I mentioned in class that we have to be careful with tables of bulk moduli since thermodynamics can mess us up. The table in the book (and included a couple of pages from now) claims that B for air (and other gasses at STP) is $B = 1.01 \times 10^5\ N/m^2$ but there's a little footnote attached noting this is only when the temperature remains constant. That might be arrangeable in a special lab, but out in the real world when a gas is compressed it's temperature goes up. As a sound wave passes through air, the air is alternately compressed and rarified, with its temperature fluctuating up and down slightly. This is NOT an isothermal situation, so a different B is involved, the $B = 1.42 \times 10^5\ N/m^2$ that I used in this example.
Note that the density and bulk modulus vary with pressure and temperature, which vary with altitude, so sound speed also varies with altitude, as seen in this figure.



Example: Speed of Sound in Water

For water (at STP) : $B = 2 \times 10^9\ N/m^2$ and $\rho = 997\ kg/m^3$ so sound waves should travel through water at about: $v = \sqrt{B/\rho} = \sqrt{(2\times10^9)/(997)} \approx 1420\ m/s$.

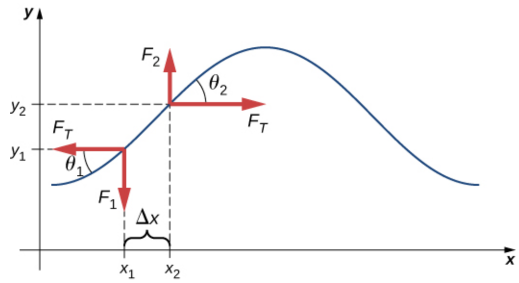
This speed is also affected by factors such as temperature, pressure, and salinity, so the speed of sound underwater varies significantly from this as you go deeper below the surface. In salt water, the sound speed near the surface of the ocean is about 1500 m/s.
We'll talk more about this in the next chapter, but notice how the sound speed reduces with depth until we get down to around 700 m and then rises again after then. This depth is called the 'deep sound channel' or the 'SOFAR' region and we'll find that sounds created around that depth tend to be trapped and can therefore propagate long distances. Whales use that depth to broadcast their songs across entire ocean basins.



Transverse (S) waves on String/Wire

$\boxed{v = \sqrt{F_T/\mu}}$ where F_T is the tension in the wire and μ is the mass/length of the medium.

Note that it's just the mass per length that matters here, so if we double the length of a wire but keep it under the same tension, the wavespeed remains the same. If we use a thinner wire made of the same material, it's mass/length will be less, making the wave speed higher. (We'll see later how this relates to piano wires: the lower notes use thicker strings and/or lower tension, the higher notes use thinner strings and/or higher tension.)



Example : A 20 kg bucket of rocks is hanging at the end of an 80 m long rope that has a mass of 2 kg. If we wiggle the rope, how fast will the disturbance propagate (a) at the bottom of the rope, and (b) at the top of the rope?

(a) At the bottom of the rope, applying Newton's Laws we find that the tension in the rope is equal to the weight of the bucket of rocks, or $F_T = mg = (20\ kg)(9.8\ m/s^2) = 196\ N$.

The mass/length of the rope is $\mu = (2\ kg)/(80\ m) = 0.025\ kg/m$,so:

$v = \sqrt{F_T/\mu} = \sqrt{196/0.025} = 88.544\ m/s$.

(b) At the top of the rope, the tension will be equal to all the weight below that point, which includes both the rope and the bucket of rocks, so $F_T = mg = (22\ kg)(9.8\ m/s^2) = 215.6\ N$, giving us a wave speed of $v = \sqrt{F_T/\mu} = 92.865\ m/s$.

Example : A steel ($\rho = 7800\ kg/m^3$) guitar string is under 46.5 N of tension. If the string is 60 cm long with a diameter of 0.22 mm, what will the wave speed be on this wire?

We have the tension here but need the mass/length $\mu = M/L$. The mass will be the **volume** of the string (which is basically a really long, thin cylinder) times its density, so $M = (volume)(density)$. The volume is just it's length times its cross-sectional area though, so $M = (\pi r^2)(L)(\rho)$. The mass/length then will be $\mu = M/L = (\pi r^2)(\rho)$: the normal 3-D density times its cross-sectional area.

Here then, $\mu = (\pi)(0.11 \times 10^{-3})^2(7800) = 2.965 \times 10^{-4}\ kg/m$.

Finally, $v = \sqrt{F_T/\mu} = \sqrt{46.5/(2.965 \times 10^{-4})} = 396\ m/s$.

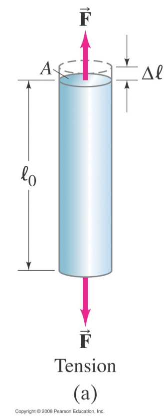
(We'll see soon this translates into the string vibrating at 330 Hz : this is the 'high E' string on this guitar.)

Longitudinal (P) waves (in solids)

$$\boxed{v = \sqrt{E/\rho}}$$

ρ is the density of the medium
E is the **elastic modulus** (also called the **Young's modulus**) of the material.
If we take a solid with some cross sectional area A and apply a force F as shown in the figure, it's length l will change by:

$$\Delta l = \frac{1}{E} l_o \frac{F}{A}$$

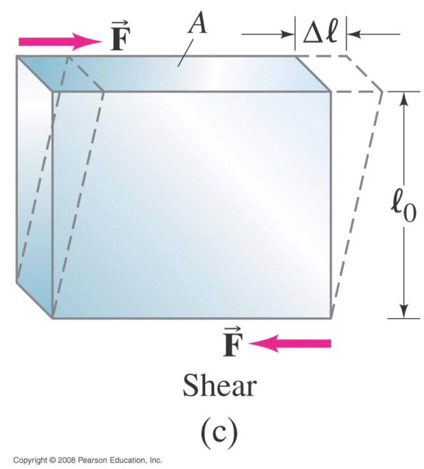


Shear (S) waves (in solids)

$$\boxed{v = \sqrt{G/\rho}}$$

ρ is the density of the material
G is the shear modulus
Here the material is not under tension along it's length (so it's not like the wire under tension above). Here a force is applied laterally, causing the object to deform **laterally** by some Δl. In this case:

$$\Delta l = \frac{1}{G} \frac{F}{A} l_o$$



Example: Speed(s) of Sound in Metal

A train rolling along steel tracks can excite both longitudinal (P) and transverse (S) wave types.

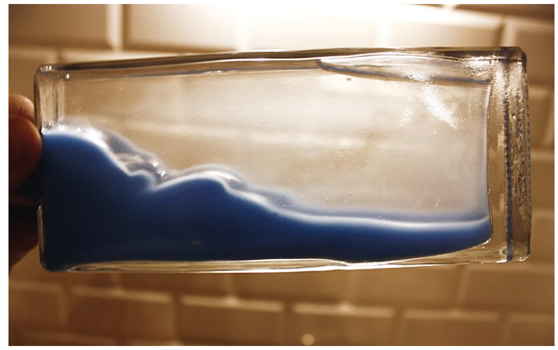
Steel has a density of about $\rho \approx 7800\ kg/m^3$.

The Young's modulus for steel is about $200 \times 10^9\ N/m^2$, yielding a longitudinal speed of $v = \sqrt{E/\rho} = 5060\ m/s$.

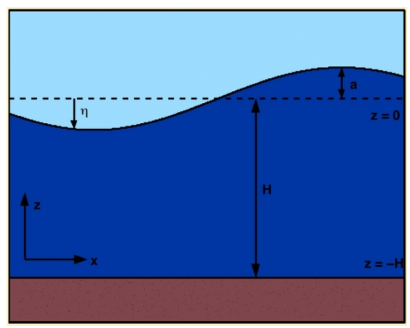
The shear modulus for steel is about $80 \times 10^9\ N/m^2$, yielding a slower shear wave speed of $v = \sqrt{G/\rho} = 3200\ m/s$.

In either case, the wave speed through metals is far higher than that in liquids or gasses.

Surface Waves



Whenever two materials of different density are in contact (like the air-water interface at the surface of a body of water), waves can propagate along that interface. These are more generally called 'surface waves', 'density waves' or 'gravity waves' (not related to rippled in space/time which are also called gravity waves).
A common commercial device that exploits this is created by half-filling a volume with water (of one color) and then putting a layer of lighter oil (with a different dye color in it) on top. Moving the object can create waves that slowly propagate across the interface. (An old lava lamp can show similar behavior.)



The equation for the wave speed is quite complicated in the general case (see `https://uwaterloo.ca/applied-mathematics/current-undergraduates/continuum-and-fluid-mechanics-students/amath-463-students/internal-gravity-waves` but water is about 833 times as dense as air, which leads to a special case (see `https://uwaterloo.ca/applied-mathematics/current-undergraduates/continuum-and-fluid-mechanics-students/amath-463-students/surface-gravity-waves` which can be simplified in a couple of scenarios:

- If the wavelength is much larger than the water depth (the so-called 'shallow water limit'), then $v \approx \sqrt{gd}$ where d is the depth of the water
- If the wavelength is much shorter than the water depth (the so-called 'deep water limit'), then $v \approx \sqrt{g/k}$ (where $k = 2\pi/\lambda$ is the wave number

Example : Tsunami Wave

Early on, we talked about a tsunami with a wavelength of $\lambda = 800\ km$ and period of $T = 3600\ sec$. Estimate the water depth.

First we have to decide is this is a 'shallow water wave' or a 'deep water wave'. Be careful of that naming convention though. Obviously the deep ocean is,'deep' but the important factor is the wavelength compared to the water depth. The ocean is only a few miles deep, so λ is FAR larger than the water depth, making this technically a 'shallow water' wave.

In that case, $v \approx \sqrt{gd}$ so $d \approx v^2/g$. We earlier found the wave speed to be $v = 222\ m/s$, so $d \approx (222\ m/s)^2/(9.8\ m/s^2) = 5000\ meters$. (The Pacific Ocean depth varies quite a bit, but that's not a bad rough estimate.)

TABLE 12–1 Elastic Moduli

Material	Young's Modulus, E (N/m^2)	Shear Modulus, G (N/m^2)	Bulk Modulus, B (N/m^2)
Solids			
Iron, cast	100×10^9	40×10^9	90×10^9
Steel	200×10^9	80×10^9	140×10^9
Brass	100×10^9	35×10^9	80×10^9
Aluminum	70×10^9	25×10^9	70×10^9
Concrete	20×10^9		
Brick	14×10^9		
Marble	50×10^9		70×10^9
Granite	45×10^9		45×10^9
Wood (pine) (parallel to grain)	10×10^9		
(perpendicular to grain)	1×10^9		
Nylon	5×10^9		
Bone (limb)	15×10^9	80×10^9	
Liquids			
Water			2.0×10^9
Alcohol (ethyl)			1.0×10^9
Mercury			2.5×10^9
Gases†			
Air, H_2, He, CO_2			1.01×10^5

†At normal atmospheric pressure; no variation in temperature during process.

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15.3 : Energy Transported by Waves

Suppose we have a wave passing through some medium. If we focus on a tiny calculus-sized infinitesimal piece of the material, it will oscillate with some amplitude A as the wave passes through that point. This little mass element is undergoing simple harmonic motion, so there's some effective spring constant involved, and we found that the total energy related to the motion of this mass is $E = \frac{1}{2}kA^2$ (with that energy sloshing between kinetic and potential energies). Also, $\omega = \sqrt{k/m}$ so $k = m\omega^2$ with $\omega = 2\pi f$ so we can write this as $k = (m)(2\pi f)^2$.

The energy involved in this little fragment of the material then is $E = \frac{1}{2}[4\pi^2 f^2 m]A^2$ or $E = 2\pi^2 m f^2 A^2$.

Consider a transverse wave propagating along a string:

We'll break the string into an infinite number of infinitesimal Δm mass elements where $\Delta m = \mu\Delta x$ and each will then represent a tiny bit of energy: $\boxed{\Delta E = 2\pi^2 f^2 A^2 \mu \Delta x}$

The factor in front of the Δx is constant and represents the **energy-per-meter** of the wave.

How much energy will pass by a fixed point on the string in a time interval Δt? The wave is travelling at a speed of v along the string, so all the energy contained within $\Delta x = v\Delta t$ will pass by this point: $\boxed{\Delta E = 2\pi^2 f^2 A^2 \mu v \Delta t}$

The **rate** at which energy is passing by (i.e. the rate at which energy is being transported along the medium) will be $\boxed{P = \frac{\Delta E}{\Delta t} = 2\pi^2 f^2 A^2 \mu v}$

Example : Power involved in wiggling a rope

A 20 kg bucket of rocks is hanging at the end of an 80 m long rope that has a mass of 2 kg. At the bottom, we're going to wiggle the rope with an amplitude of $A = 5\ cm$ at a frequency of $f = 2\ Hz$. How much power does this require?

$v = \sqrt{F_T/\mu}$ and $\mu = M/L = (2\ kg)/(80\ m) = 0.025\ kg/m$.

At the bottom of the rope $F_T = (20\ kg)(9.8\ m/s^2) = 196\ N$ so $v = 88.543\ m/s$.

The power needed then is $P = 2\pi^2 f^2 A^2 \mu v = (2)(\pi)^2(2)^2(0.05)^2(0.025)(88.543) = 0.4369\ Watts$ (not much).

Now, when we looked at this before, we found that the tension at the top of rope will be higher since that point is supporting not only the bucket of rocks but the mass of the rope itself. At the top of the rope, $F_T = (22\ kg)(9.8\ m/s^2) = 215.6\ N$ so $v = 92.865\ m/s$.

If we assume there aren't any losses here, the power we're putting into the rope at the bottom has to all reach the top, so P is a constant here. We have a different v though so something has to give. μ and f are constants so it looks like the wave amplitude A has to absorb this change. $A^2 v$ must be constant so $(A_{top})^2 v_{top} = A_{bottom})^2 v_{bottom}$ or rearranging:

$A_{top} = A_{bottom}\sqrt{v_{bottom}/v_{top}} = (5\ cm)\sqrt{88.543/92.865} = 4.88\ cm$.