PH2233 Fox : Lecture 06
Chapter 15 : Wave Motion

NOTE: some useful videos related to travelling waves, standing waves, and normal modes can be found on the old course website at:

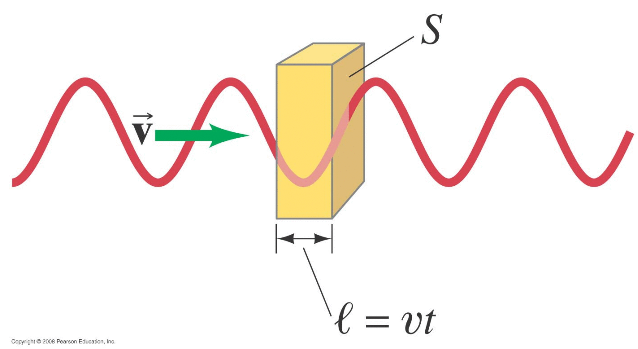
`https://newton.ph.msstate.edu/~fox/ph2233/ch15/index.html`

Or embedded at the appropriate points in the online lecture slides:

`https://newton.ph.msstate.edu/~fox/ph2233/ch15/lecture06.html`

Energy in P-waves

Last time we looked at energy and power in transverse waves on a wire. We can do a similar analysis for a longitudinal wave (a P-wave) passing through a medium like a gas or liquid. These waves aren't constrained to move along a line like the string in the previous example. They are 'wave fronts' (2-dimensional shapes) propagating through the medium, so let's look at the rate at which energy is passing through a particular area in a particular time.



If we look at a particular cross sectional area, how much of the wave will pass through that area in a given amount of time? That will tell us how much energy is passing through the given area per unit of time.

The wave is travelling at a wave speed of v, so in a time interval t the volume of the wave passing through the given area will be $(S)(vt)$ so the mass involved is that volume times the density ρ.

Basically the energy involved in $m = \rho S v t$ worth of 'stuff' vibrating is what's passing through the given area per time.

$E = 2\pi^2[\rho S v t] f^2 A^2$ or $\boxed{E = 2\pi^2 \rho S v t f^2 A^2}$. $(Joules)$

That's fine, but let's look at the rate at which energy is being carried (transported) by the wave. Power is energy/time, so apparently:

$$\boxed{P = E/t = 2\pi^2 \rho S v f^2 A^2} \ (Watts)$$

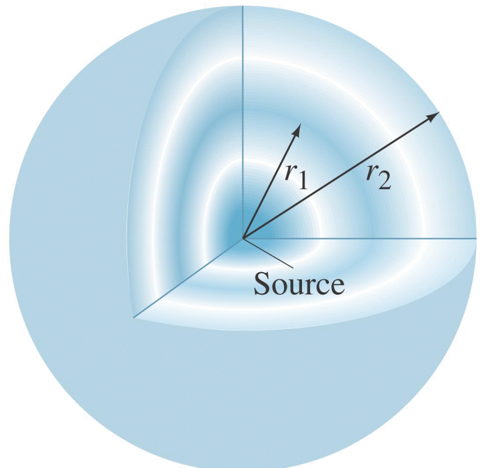
A commonly-used related quantity is the **power per area**, called the **intensity** (in $Watts/m^2$). It's this intensity in the context of sound that our ears respond to, for example, and we'll see in the next chapter that a common unit used for intensity is the **decibel** (dB), related to the base-10 logarithm of the intensity.

Dividing the above equation by the area S:

$$\boxed{I = P/S = 2\pi^2 \rho v f^2 A^2} \ (Watts/m^2)$$

$\boxed{f^2 \text{ effect}}$: the energy/power/intensity are all proportional to the square of the frequency involved, which means that the higher frequencies are more expensive to generate in terms of energy. When objects can vibrate at multiple frequencies, most of the energy ends up being taken up by the lowest frequency (easiest to generate) mode.

$\boxed{1/r^2 \text{ effect}}$: Before we look at some examples, sound and many other sources of wave often send waves out in all directions. A source is putting out some power (watts) but that power gets spread around an expanding surface.



Intensity is power per area, so $I = P/S = P/(4\pi r^2)$
Note that $I \propto 1/r^2$, so if we want to compare the intensity at one distance to the intensity at a different distance:

$$\frac{I_2}{I_1} = \frac{r_1^2}{r_2^2}$$

If we move twice as far away from a sound source, the intensity (W/m^2) drops by a factor of **four**, and looking at the equation for I we see that the wave **amplitude** must drop by a factor of **two**.

Examples

Example - Power vs Frequency) : Compare a high pitched $10,000\ Hz$ signal to a low $200\ Hz$ signal. The higher one has 50x the frequency, so will have $(50)^2$ or 2500 times the intensity (power/area). One reason we're so much more sensitive to higher frequencies - and why the high frequency response of the ear can be obliterated via concerts, ipod's, etc.

Example - Determining A for sound waves in air : Suppose we have a $40\ W$ speaker that's emitting a pure $1000\ Hz$ tone. If we look $2\ m$ away from the speaker, the air molecules are vibrating with what amplitude A?

$P = 2\pi^2 \rho S v f^2 A^2$ so we have everything here to determine A.

First, I should note that this is NOT a realistic scenario. As we saw with the 'equilizer' picture last time, music and voices almost always involve many frequencies, each one carrying a small amount of energy that overall adds up to the total watts of power being emitted by the speaker. If we tried to drive a speaker at a full $40\ W$ concentrated in a single frequency, it would likely destroy itself.

So the amplitude we calculate here will be a considerable OVER-ESTIMATE.

Real speakers aren't omnidirectional, but let's assume this one is, sending sound equally in all directions. At $r = 2\ m$ from the speaker, it's $40\ W$ is being spread evenly over a surface of area $S = (4\pi r^2) = 50.265\ m^2$.

Then $(40) = (2)(\pi)^2(1.2)(50.265)(343)(1000)^2 A^2$. Solving, we find that $A \approx 1 \times 10^{-5}\ m$

So a molecule of air next to our eardrum is oscillating back and forth a distance of only about $0.01\ mm$ (and yet we'll find in the next chapter that a sound intensity this high would cause significant hearing loss).

NOTE: that $0.01\ mm$ looks insignificant to us, but an individual molecule is only about 10^{-10} meters in diameter, so a molecule of air is moving back and forth about $100,000$ times it's diameter as these waves pass by.

Example - Estimating earthquake power : Back in the 60's when my family lived in Okinawa, an undersea volcano about $r = 100\ km$ away generated an earthquake that lasted about 2 minutes. In our neighborhood, I watched houses oscillate side to side with an amplitude of about $A = 10\ cm$ with a frequency of about $f = 0.1\ Hz$ (i.e. about 10 seconds between waves). For rock, v is (very) roughly $4500\ m/s$ and ρ is (very) roughly $3000\ kg/m^3$ so how much power and energy was involved here?

$P = 2\pi^2 \rho S v f^2 A^2$ becomes:

$P \approx (2)(10)(3000)(4\pi 100,000^2)(4500)(0.1)^2(0.1)^2$ or $P = 3.3 \times 10^{15}\ W$. The earthquake lasted about 120 seconds so the total energy released would be roughly $E = Pt = 4 \times 10^{17}\ J$.

1 ton of TNT is equivalent to $4.2 \times 10^9\ J$ so this would be around 94 megatons of TnT. This was a pretty small earthquake, yet it represented more than twice the total energy of the largest hydrogen bomb ever detonated (look up the Tsar Bomba on youtube). Very large earthquakes (like the one that produced the 2004 tsunami) can release thousands of times more energy.

15.5* : the wave equation

See this section for the derivation in the case of a transverse wave on a string/wire. Similar derivations can be done for other types of waves in other materials. The end result is always of the form:

$$\boxed{\frac{\partial^2 D}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 D}{\partial t^2} \quad \text{(wave equation)}}$$

$\boxed{\text{Example}}$: We've been using the form $D(x,t) = A\sin(kx - \omega t)$ to represent a travelling wave, and we've seen that we can build other 'shapes' by combining sines with different amplitudes, wavelengths, and phase shifts.

Let's check that this form satisfies the wave equation.

- $\partial D/\partial x = Ak\cos(kx - \omega t)$ so differentiating again:
- $\partial^2 D/\partial x^2 = -Ak^2\sin(kx - \omega t)$
- $\partial D/\partial t = -A\omega\cos(kx - \omega t)$ and differentiating again:
- $\partial^2 D/\partial t^2 = -A\omega^2\sin(kx - \omega t)$
- Substituting these expressions into the left and right sides of the wave equation:
  $-Ak^2\sin(kx - \omega t) = \frac{1}{v^2}(-A\omega^2\sin(kx - \omega t)$
  Cancelling terms that appear on both sides yields: $k^2 = \frac{1}{v^2}\omega^2$ or $v = \omega/k$ which is in fact the speed of this wave.

$\boxed{\text{General result}}$: Actually **any** function $D(x,t)$ that is of the form $D(x,t) = f(kx - \omega t)$ will satisfy the wave equation (as long as f is continuous and has a first derivative that is also continuous). In english, that means if we have any sufficiently smooth function $f(x)$ (a sine or cosine, a gaussian pulse shape, etc), then replacing its argument x with $kx - \omega t$ (which is $(k)(x - vt)$) creates a travelling wave of the same shape that is a legal solution to the wave equation.

The proof of this involves the chain rule for partial derivatives which is in general a bit more complicated than the chain rule for regular derivatives but in fact ends up working in the 'expected' way for this situation. For example, in order to do $\partial D/\partial x$ the first step will be to take the derivative of f with respect to its argument (the step that turned the sine into a cosine in the first step above), **then** we'll take the partial derivative of that argument (which is $kx - \omega t$) with respect to x, which brings out a factor of k. So $\frac{\partial D}{\partial x} = (f')(k) = kf'$.

Taking another (partial) derivative with respect to x brings out another factor of k leaving us with $\frac{\partial^2 D}{\partial x^2} = k^2 f''$.

The time derivatives proceed in the same fashion. For example, in order to do $\partial D/\partial t$ the first step will be to take the derivative of f with respect to its argument (the step that turned the sine into a cosine in the first step above), **then** we'll take the partial derivative of that argument (which is $kx - \omega t$) with respect to t, which brings out a factor of ω. So $\frac{\partial D}{\partial t} = (f')(\omega) = \omega f'$. (Note this f' is **the same** as we had in the spatial derivative part since it's just the derivative of that function with respect to it's argument (and not with respect to x or t).

So $\frac{\partial D}{\partial t} = \omega f'$ and $\frac{\partial^2 D}{\partial t^2} = \omega^2 f''$.

Putting these expressions back into our original wave equation $\frac{\partial^2 D}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 D}{\partial t^2}$ becomes:

$$k^2 f'' = \frac{1}{v^2}\omega^2 f''$$

As long as f'' exists (which requires f' to be continuous and differentiable, which in turn requires f itself to be so as well), this implies that:

$k^2 = \omega^2/v^2$ or $v = \omega/k$, the known wave speed.

Ultimately then, we can start with any realistic function f with a single argument and replace that argument with $kx - \omega t$ and produce a travelling wave that satisfies the wave equation.

The reason we mention all this here is that this equation appears in many scenarios that don't directly appear to involve waves. It appears in chemistry, crystal growth, fungus and bacteria growth, predator/prey models in biology, and many other scenarios.

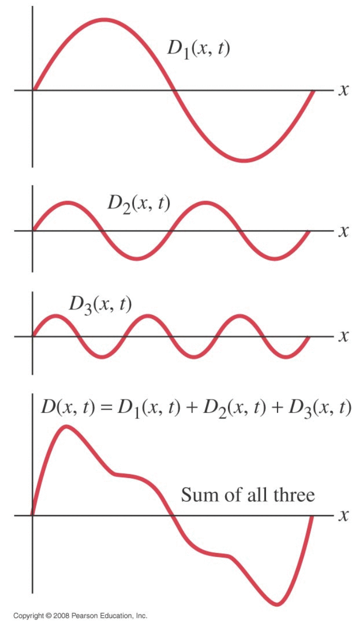
What it implies is that if the analysis of some scenario ends up yielding a differential equation that looks like the wave equation, then wave-like behavior will appear. Just like if the analysis of some scenario yields an equation that looks like $d^2f/dt^2 = -(constant)f$ then periodic (oscillatory) behavior will appear.

15.6 : principle of superposition

The wave equation is linear, so if $D_1(x,t)$ is a solution (maybe a sine wave travelling in the $+x$ direction) and $D_2(x,t)$ is some other solution (maybe a gaussian shape travelling in the opposite direction) then any linear combination, like $D_1 + D_2$, is also a solution.

In a physical sense, this means that the solutions can be superimposed on each other - basically the two (or N) waves are propagating through the medium INDEPENDENTLY.

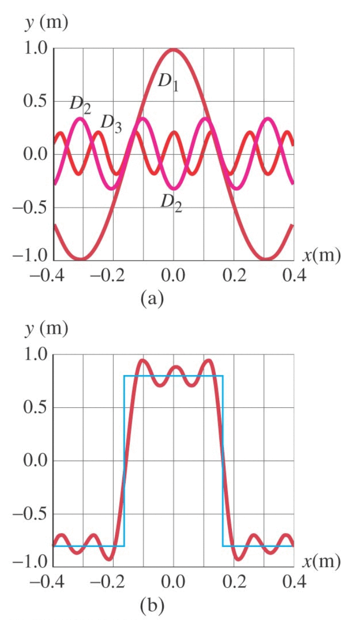
In the top figure, the three sine waves combine to create the shape just below them, so that shape propagating is really made up of the three sine waves all propagating together.



superposition

In this set of figures, we see a collection of cosines or phase-shifted sines being used to try and build a 'square wave'. Again, each sine or cosine will propagate through the medium with the same speed v, so the shape we built from them will maintain that shape and propagate along with that same speed v.

We've already talked about Fourier series - building almost arbitrary shapes from sines and cosines and here we're letting those waves propagate, retaining that arbitrary shape we started with.



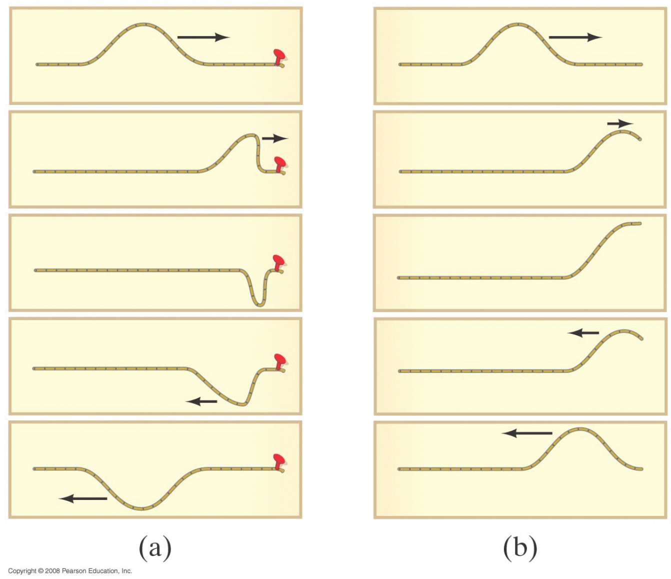
building a square wave

15.7 : Reflection and Transmission

(See the first video in the link given at the top of the first page.)

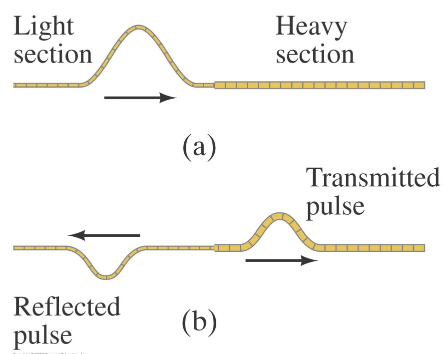
We'll see more on this later, but what happens to a wave that's travelling down a string (or any medium really) that is finite. Eventually we reach the end of the string (or other medium). What happens there?

In strongly depends on the type of medium involves, but in the case of a transverse wave on a string, then:



- If the string is locked in place and can't move, the wave shape 'reflects' but with an inverted amplitude. A positive amplitude (representing a displacement in one direction relative to the string) becomes a negative amplitude (a displacement in the opposite direction) when the pulse 'reflects' from the end.
- If the string is free to move, the pulse deflects that end of the string and reflects with the same sign

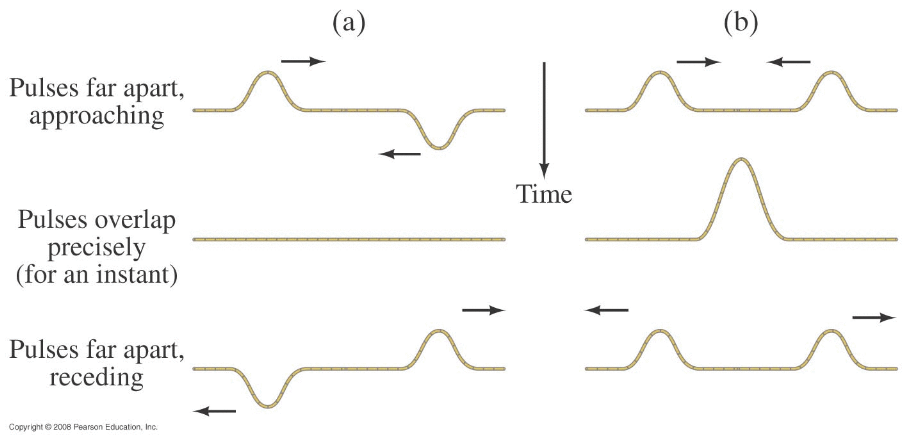
What if the string is connected to another string with a different μ? Maybe a string connected to a metal wire? The tension will be the same in each part of the 'medium' but if μ is different then $v = \sqrt{F_T/\mu}$ will be different.
In this figure a 'light rope' is connected to a 'heavy rope' (the higher μ will mean a slower v for the wave). In this case, part of the wave reflects (and flips sign) and part continues into the new medium (but with a different amplitude and wave speed).



15.8 : Interference

The wave equation tells us that multiple waves can be present in a medium simultaneously and they all behave independently of each other. In the **left** figure below, we have a positive pulse travelling to the right and a negative pulse travelling to the left. They pass through one another (and at one point they appear to have cancelled each other out, but at that point the string may have 0 displacement but the (tranverse) velocity and acceleration along the string isn't zero.

On the **right**, we see how the two pulses (both positive this time) combine to create an extra high amplitude the instant they're both passing through the same point.



Superposition

DIGRESSION: This effect can cause unexpected behavior in structures. If one component of a structure fails, a statics analysis might show that the new configuration is still stable (all the tensions and stresses will be different, but still may be within the tolerances of the components creating the structure). The problem is that the failure would have produced 'waves' of changes in tension that are propagating through the structure, and these waves can constructively interfere with one another and yields much higher displacements (and hence tensions) briefly. Statics programs can't do these simulations so won't be able to alert the user to the possibility of this type of failure in the structure.

15.9 : Standing Waves : Resonance

We know the wave equation is linear, so suppose we have two identical waves (same wavelength and frequency) travelling in opposite directions through some medius:

- Wave travelling in $+X$ direction: $D_1(x,t) = A\sin(kx - \omega t)$
- Wave travelling in $-X$ direction: $D_1(x,t) = A\sin(kx + \omega t)$

The complete wave then would be: $D = D_1 + D_2 = A(\sin(kx - \omega t) + \sin(kx + \omega t))$

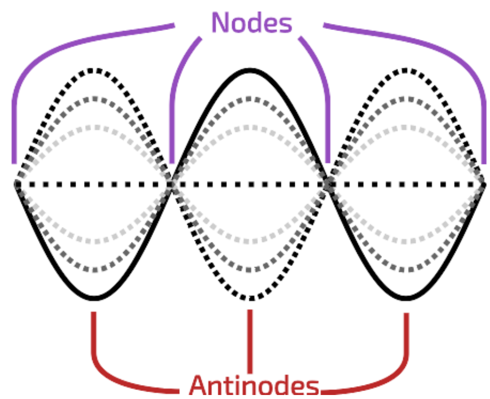
Let's expand out those sines though:

Sine of sum of angles: $\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$ so this becomes:

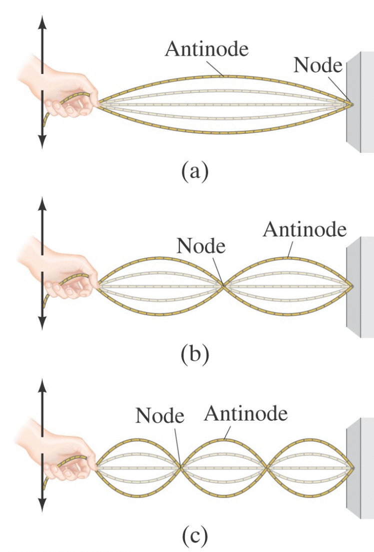
$D(x,t) = A[(\sin(kx)\cos(\omega t) - \cos(kx)\sin(\omega t)) + (\sin(kx)\cos(\omega t) + \cos(kx)\sin(\omega t))]$

Combining terms: $\boxed{D(x,t) = 2\sin(kx)\cos(\omega t)}$

Here's a plot of a particular standing wave. Technically this is still a 'wave' since it's a solution to the wave equation but it's actually TWO identical waves passing through each other in opposite directions. If we look at the spatial part of this 'wave', it's amplitude is ZERO (called a NODE) anywhere that $kx = N\pi$ or $x = N\pi/k$ but $k = 2\pi/\lambda$ so these nodes occur where $x = (N\pi)(\lambda/2\pi) = N(\lambda/2)$. The nodes are located exactly ONE HALF λ apart from one another.
If we look at the time part, the amplitude at any point x is bouncing up and down with an angular frequency of ω which means it's oscillating at the same frequency (or period) as the underlying waves.



Now here's where the magic comes in! Since the points where $x = N(\lambda/2)$ always have a displacement of ZERO, what if we actually lock down the medium (string) at those points?
If the locked-down points are some distance L apart from one another, then the standing waves that will 'fit' in this distance are those where the nodes happen to fall exactly that distance apart from one another.
Top figure : the nodes are L apart, so $L = (1)\frac{\lambda}{2}$
Middle figure : here we have two nodes that are L apart, so $L = (2)\frac{\lambda}{2}$
Bottom figure : here we have three nodes that are L apart, so $L = (3)\frac{\lambda}{2}$



superposition

Rearranging those equations, the standing waves will be of the form: $\boxed{\lambda = \frac{2L}{N}}$

Now $v = \lambda/T = \lambda f$ or $f = v/\lambda$ so let's rewrite that in terms of frequency: $\boxed{f_N = N\frac{v}{2L}}$

This string will vibrate at various integer multiples of the frequency $f_1 = \frac{v}{2L}$ which is called the **fundamental** frequency of the string.

This collection of patterns (standing waves) is commonly referred to as the **normal modes** of oscillation of the medium. The medium can vibrate at these specific frequencies easily. Any other frequencies, representing wavelengths that don't 'fit' in the length of the medium, will quickly die out.

NOTE: with stringed and wind instruments, the normal modes are basically integer multiples of a 'fundamental' (lowest) frequency of vibration. Other objects (baseball bats, wine glasses, ...) can also have normal modes but the frequencies typically aren't simple integer multiples of the lowest frequency.

$\boxed{\text{Piano string example}}$ (from the book, but more accurate numbers...)

On an 88-key piano, key number 28 plays a 'C' note with a frequency of 130.813 Hz. Suppose this wire has a length of $L = 1.10\ m$ and a mass of $M = 9.00\ grams$. How much tension must this string be under in order to produce this frequency as it's fundamental? What other frequencies will this string vibrate at?

$f_1 = \frac{v}{2L}$ so $v = (f_1)(2L) = (130.813\ s^{-1})(2.20\ m) = 287.7886\ m/s$ but $v = \sqrt{F_T/\mu}$ so $F_T = v^2\mu = (287.7886\ m/s)^2(0.009\ kg)/(1.10\ m) = 677.64\ N$.

When we hit this wire, it will most likely vibrate in it's fundamental but other standing waves with higher frequencies will also exist. $f_N = N\frac{v}{2L}$ so these other frequencies are all integer multiples of the fundamental:

- $f_2 = 2f_1 = 261.626\ Hz$ (called C_4 or 'middle C' on the piano)
- $f_3 = 3f_1 = 392.439\ Hz$ (G_4)
- $f_4 = 4f_1 = 523.252\ Hz$ (C_5)
- ... and so on ...

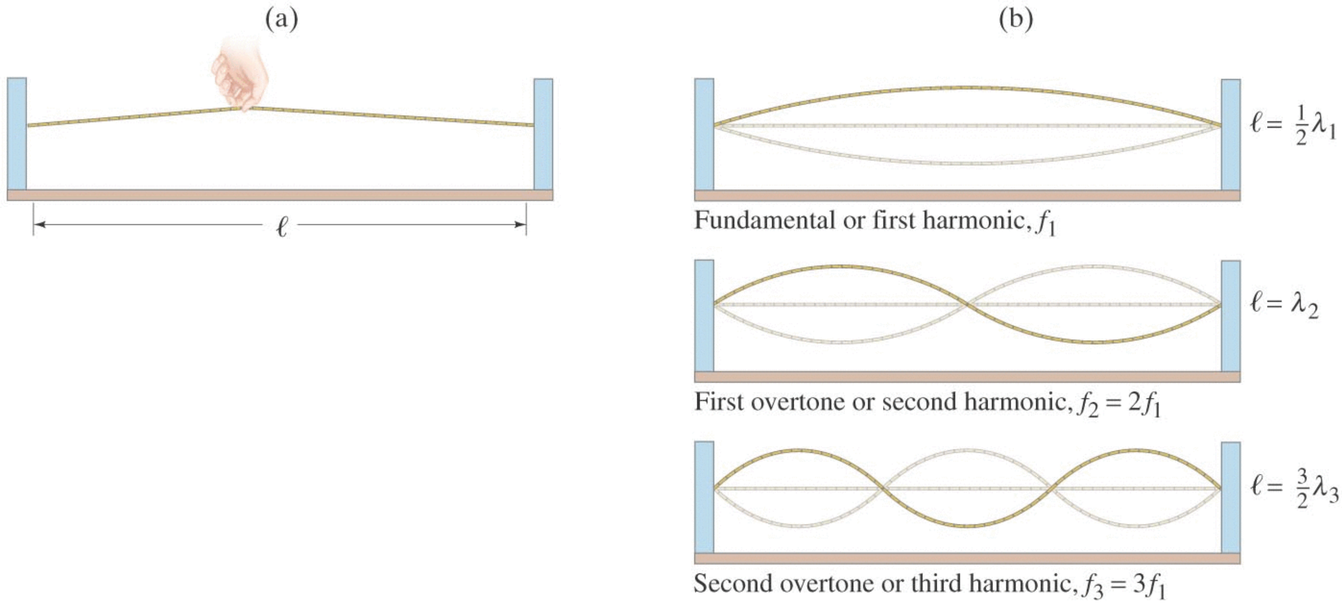
All stringed instruments vibrate at multiple frequencies (and we'll see the same occurs with wind instruments). Electronic instruments that attempt to simulate actual strings need to account for that.

Note that for vibrating strings: $f_1 = \frac{v}{2L}$ with $v = \sqrt{F_T/\mu}$ so $\boxed{f_1 = \frac{1}{2L}\sqrt{\frac{F_T}{\mu}}}$

In the case of a guitar, the 6 strings span frequencies from about 82 Hz to 330 Hz (a factor of a little over 4) and they're all the same length, so this range of frequencies is created by varying the tension in each string and the material used (the μ) for each string.

With pianos, the 88 strings span frequencies from 27.5 Hz to 4186 Hz (a factor of over 150!) and that's too much of a range to achieve just varying the tension and string type - the length of the string also varies considerably, varying from about 5 cm for the highest frequencies up to well over a meter for the lowest frequencies.

We'll see more of this in the next chapter, but these normal modes have TWO naming conventions attached to them.



Normal Modes in other Objects

Here are some interesting links (with animations) showing the 'normal mode' shapes and frequencies for:

- Actual bat : `https://www.youtube.com/watch?v=5F3Q5ErEcfg` (from the PhysicsEverywhere channel)
- Wood and metal baseball bats : `https://www.acs.psu.edu/drussell/bats/batvibes.html`
- An empty beer bottle : `https://www.acs.psu.edu/drussell/Demos/BeerBottle/beerbottle.html`
- Drum head : `https://www.youtube.com/watch?v=v4ELxKKT5Rw`

15.10* : refraction
15.11* : diffraction

(There's a whole chapter on each of these topics later, so we'll skip this for now.)