PH2233 Fox : Lecture 09
Chapter 16 : Sound

16.6 : Interference of Sound Waves (beats)

Suppose we have two sound sources that are **slightly** out of tune, producing frequencies that are not quite the same. What will we hear?

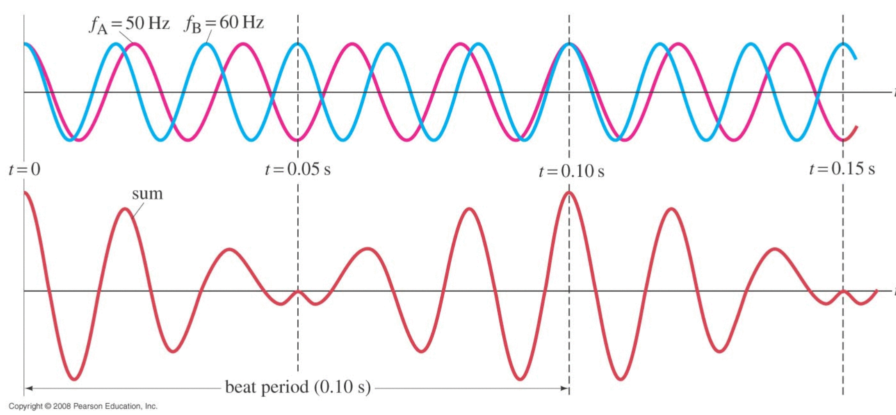
Let's write the first source as $D_1 = A\cos(\omega_1 t)$ and the second as $D_2 = A\cos(\omega_2 t)$ (where $\omega = 2\pi f$).

The combination of these will yield: $D = D_1 + D_2 = A\cos(\omega_1 t) + A\cos(\omega_2 t)$

A useful trig identities (there was a reason you learned these after all!) is:
$\cos(A) + \cos(B) = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

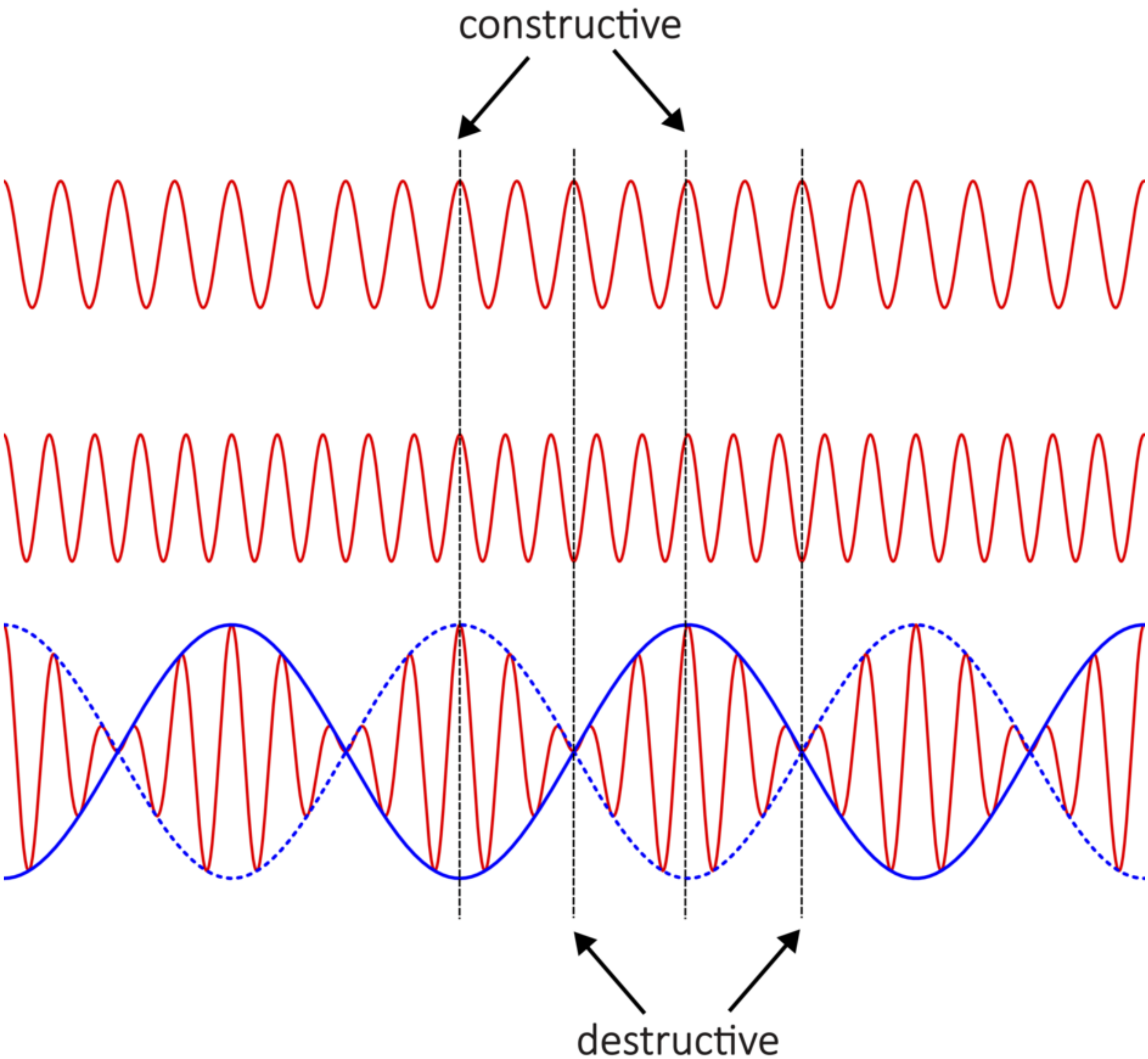
We can thus write the combined time series as:
$D = 2A\cos\frac{(\omega_1+\omega_2)t}{2} \times \cos\frac{(\omega_1-\omega_2)t}{2}$

The figure below shows two such cosine waves with frequencies of $f = 50\ Hz$ and $f = 60\ Hz$ and the result of combining them. At some times, the two waves are exactly in phase yielding a double-amplitude signal but at other times the two waves are exactly out of phase and end up cancelling each other:



See `https://onlinetonegenerator.com/` again here. There are two places where you can create two different frequencies and hear the result: the **Binaural Beats** tab lets you pick two frequencies, and the **Multiple Tone Generator** can do 2 or more and you can turn each on and off easily. If the notes are 'close enough', our auditory system hears them as a single average frequency with a modulated amplitude. At some point though as the two frequencies get far enough apart, we start hearing them as two distinct notes.

Here's a better view of what's going on. The two nearly-equal frequencies combine to produce a tone that has the average of those two, but where it's overall amplitude is changing at the beat frequency. The solid line shows the cosine term that's creating this 'envelope' and notice that we perceive amplitude nodes at twice that frequency.



Essentially when the two sounds have frequencies that are similar, we hear this as a sound a frequency of $f_{avg} = (f_1 + f_2)/2$ but whose amplitude is modulated at a frequency of $f_{beat} = (\Delta f/2) \times 2$.

That final 'times 2' is because we can't hear PHASE. We hear that amplitude modulation going through highes and lows at TWICE the nominal beat frequency. The high amplitude points occur twice in each period of the modulating cosine.

Example : musical instruments out of tune

Suppose a violin is playing a middle C at the correct frequency of $f_1 = 261.6256\ Hz$ but a wind instrument that's warmed up a bit is attempting to produce the same note but is actually 2% higher thanks to the higher temperature so $f_2 = 266.858\ Hz$. (We'll see next that one full note represents a change in frequency of about 6%, so these instruments are already pretty close, with the wind instrument being 'off' by just a third of a full note.

These notes would combine to produce a sound at $f_{avg} = \frac{f_1+f_2}{2} = 264.242\ Hz$ but with a beat frequency of $f_{beat} = f_2 - f_1 = 5.23\ Hz$.

The player of the wind instrument starts making adjustments to bring their instrument into tune with the violin. As they get closer to being in tune, what happens? Suppose the wind instrument is almost there, being 0.2% too high now, so they're producing $f_2 = 262.149\ Hz$. Together, the instruments will produce a sound with $f_{avg} = 261.887\ Hz$ and the beat frequency is now $f_{beat} = f_2 - f_1 = 0.523\ Hz$ (about 2 seconds between the high amplitudes).

The closer they get, the farther apart the amplitude modulations become. Presumably there's some rule of thumb they use to decide they're 'close enough'. Even this close may not be close enough though, if there's a passage in the music where these two instruments would be holding notes for seconds at a time.

Try this with `https://onlinetonegenerator.com`.

In the 'equally temperated chromatic scale' there are **12** steps (notes) in each octave with each step produced by multiplying the previous by a constant factor r.

One octave means an exact doubling of the frequency.

Since it takes 12 steps to exactly double the frequency, $r^{12} = 2$ so $r = 2^{(1/12)} = \mathbf{1.059463}$....

The frequency of each 'note' is almost exactly 6% higher than the previous note in the scale.

TABLE 16–3 Equally Tempered Chromatic Scale†

Note	Frequency (Hz)
C	262
C♯ or D♭	277
D	294
D♯ or E♭	311
E	330
F	349
F♯ or G♭	370
G	392
G♯ or A♭	415
A	440
A♯ or B♭	466
B	494
C′	524

†Only one octave is included.

For stringed instruments, we found that the fundamental frequency was $f_1 = \frac{v}{2L}$ where $v = \sqrt{F_T/\mu}$. If the frequency is 2% too high (like the example above), we can't alter the length of the string so we need to force the wave speed v to change by that percentage.

$v = \sqrt{F_T/\mu}$, so $F_T = \mu v^2$, meaning that the tension is proportional to the **square** of the wave speed. Here, we need to reduce v by a factor 0f 0.98 so the tension must be reduced by a factor $(0.98)^2 = 0.96$... The 2% reduction in the frequency requires a 4% reduction in the string tension.

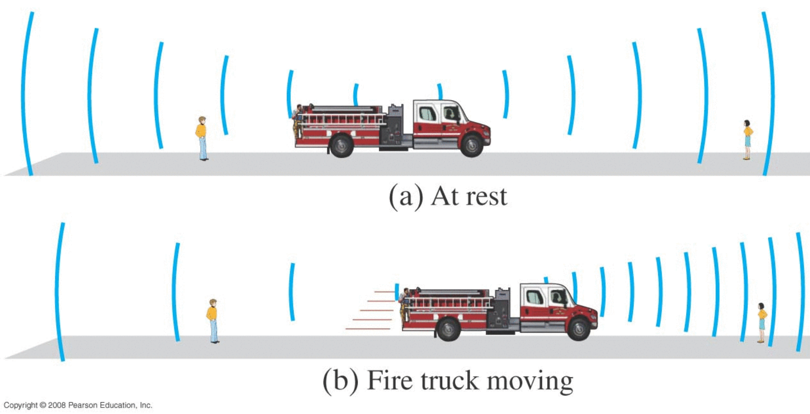
For wind instruments, we found that that the fundamental frequency was $f_1 = \frac{v}{2L}$ where v is the speed of sound in air. We can't realistically adjust that, so we need to alter the **length** of the pipe. Rearranging that equation, $v = 2Lf_1$ so if we need to multiply f_1 by 0.98, we can do that by multiplying L by $1/0.98 = 1.020$.... A 2% reduction in the frequency requires a 2% **increase** in the length of the pipe. (Instead of the **4**% change in the tension for the stringed instrument.)

16.7 : Doppler Effect

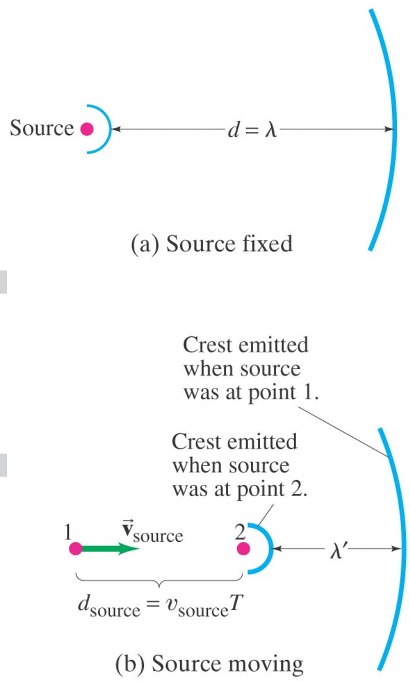
The upper figure below shows a stationary source producing some of some frequency f, and a stationary listener. The sound source is creating travelling (pressure) waves heading off at v_{snd} toward the listener. $v = \lambda/T = \lambda f$ so here the wavelength (distance from peak to peak pressure, say) will be v_{snd}/f meters.

What will the listener 'hear'? Waves of the given wavelength are arriving travelling at v_{snd}. Peaks are hitting their eardrums every time one of the wave peaks arrives. At what rate are they arriving? $f = v/\lambda$ so they hear the same frequency that was emitted.

Now, **what if** the source is moving towards the listener? The source puts out a pulse and T seconds later, that pulse has moved outwards a distance $v_{snd} \times T$. During that same time interval, the source has moved forward a distance of $v_{src} \times T$ where it now emits it's next pulse.



Source moving towards Listener



The distance between each pulse (each peak in the pressure wave) is now $(v_{snd} - v_{src})(T)$, which means the **wavelength** of the waves is now smaller than it used to be.

Peaks that far apart now arrive at the listener. They're bunched up now though. The listener will hear a higher frequency. $f = v/\lambda$ in general, but now λ has been altered, so the listener will be hearing a different frequency. Let's call that f'. Then:

$f' = v_{snd}/\lambda = \frac{v_{snd}}{v_{snd}-v_{src}}\frac{1}{T}$

But $1/T$ is just f (the frequency the SOURCE thinks it's producing) so we can write this as:

$$\boxed{f' = f\frac{v_{snd}}{v_{snd}-v_{src}}}$$ (source moving towards listener)

Example : Suppose the firetruck is emitting sound with a frequency of $f = 1000\ Hz$ and is moving at $20\ m/s$ towards a stationary listener. The frequency heard by the listener will be: $f' = (1000\ Hz)\frac{343}{343-20} = 1061.92\ Hz$. (That's a ratio of about 1.062 which is a littler more than one full NOTE on the western scale.)

Source Moving Away from Stationary Listener : We can re-use the same figure as above but now consider a listener that's behind the source. The source puts out a pulse. Then T seconds (one period) later, that pulse has moved outwards a distance $v_{snd} \times T$ (to the left). The source has moved farther away to the right now though, a distance of $v_{src} \times T$ when it emits it's next pulse.

In this case, the distance between the pulses (or the peaks of pressure), which is the wavelength, has **increased**. The distance between peaks is now $(v_{snd} + v_{src})(T)$

Carrying this through, the frequency heard by the listener will be:
$\boxed{f' = f\frac{v_{snd}}{v_{snd}+v_{src}}}$ (source moving away from listener)

Looking at the firetruck travelling at 20 m/s again: $f' = (1000\ Hz)\frac{343}{343+20} = 944.9\ Hz$. (Again, about 6 percent lower, which is one NOTE lower.)

We can go through this process with a stationary source and moving listener, as well as situations where both source and listener are moving and arrive at a general result:

$$f' = f \cdot \left(\frac{v \pm v_{obs}}{v \mp v_{src}}\right)$$ called the **Doppler Equation**

where:
- v = sound speed
- v_{obs} = observer speed
- v_{src} = source speed.
- Upper sign if moving towards.
- Lower sign if moving away. (**separate analysis for each term**)

I'll repeat a key word here: all those v variables are **speeds** and NOT velocities. The signs are taken care of by the wording below the figure.

(Note I've dropped the subscript snd from the speed of sound term, and also changed from 'listener' to 'observer' which is the term the book uses.)

ALSO: this all assumes that the AIR (the medium supporting the waves) is NOT MOVING. If it is, we need to move into a coordinate system that's moving along with the air and then figure out what the source and observer speeds are relative to the air.

EXAMPLE : Moving Source and Receiver

Suppose the firetruck is travelling to the right at 20 m/s and we're in a car ahead of the truck and also moving to the right but at just 10 m/s.

Here then, $v_{src} = 20$ and $v_{obs} = 10$ (remember, those are speeds and not velocities so they're always positive numbers no matter what direction anything is moving in).

v_{src} sign : The source is moving TOWARDS the listener. TOWARDS means 'upper sign', so we'll use the (-) sign on the v_{src} term.

v_{obs} sign: The observer is moving AWAY from the source (or at least attempting to). AWAY means 'lower sign' for the observer term which means (-) again, so:

$$f' = (1000\ Hz) \cdot \frac{343-10}{343-20} = 1030.96\ Hz$$

Example : DOUBLE DOPPLER

Going back to the first example we did, we had the firetruck moving in some direction at 20 m, emitting a $f = 1000\ Hz$ tone, which a stationary listener 'heard' as $f' = 1061.92\ Hz$.

Suppose that ahead of the truck is some stationary flat surface that will reflect this sound back towards the truck (the back of a large moving van maybe, or a flat wall, ...).

When this 1061.92 Hz sound waves hit the (stationary) wall, they'll reflect back towards the truck at that same frequency. Every pressure pulse hitting the wall becomes a pressure pulse reflected from the wall.

Now we have the firetruck travelling into these pulses. What frequency will someone in the truck here?

For this Doppler step, we consider the wall to be the source, which is stationary and emitting a frequency of 1061.92 Hz.

The firetruck is now the 'observer' and they're moving TOWARDS the source at 20 m/s.

In the Doppler equation then: $f = 1061.92\ Hz$, $v_{src} = 0$ and $v_{obs} = 20$ and since the observer is moving TOWARDS the source, we'd use the 'upper sign' on the v_{obs} term in the equation:

$f' = f \cdot \frac{v \pm v_{obs}}{v \mp v_{src}}$ so here:

$f' = (1061.92\ Hz)\frac{343+20}{343} = 1123.84\ Hz$.

What frequency would a person in the firetruck hear directly from the siren?

In that case, both $v_{src} = 20$ and $v_{obs} = 20$ and they're actually not moving relative to one another so what sign should we pick for the terms here?

Suppose the observer were moving some tiny ϵ faster than the source. Then the observer is moving 'away' from the source (lower sign on the v_{obs} term) and the source is attempting to move towards the listener (so upper sign on the v_{src} term. Then:
$f' = f \cdot \frac{v \pm v_{obs}}{v \mp v_{src}}$ becomes:
$f' = f\frac{343-20}{343-20} = f$.

If we suppose that the observer is moving some tiny ϵ slower then the source, that would swap both signs yielding $f' = f\frac{343+20}{343+20} = f$.

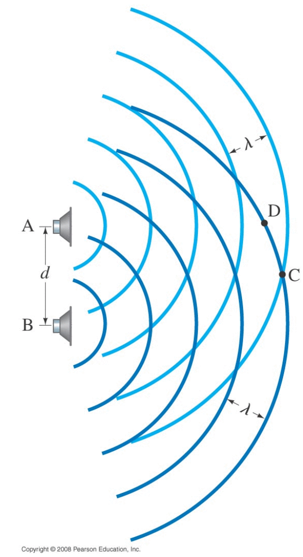
Either way, if the source and observer (listener) are moving in the same direction at the same speed, the observer hears the source frequency unaffected.

Ultimately then, someone sitting in the firetruck would hear the $f = 1000\ Hz$ from the siren directly, plus this 1123.84 Hz sound that bounced off the wall back towards them. Those frequencies are too far apart to be perceived as an average-plus-beat frequency so the listener would just hear two distinct different frequencies.

Interference: Constructive and Destructive (Introducing the idea here but we'll have an entire chapter on it later.)

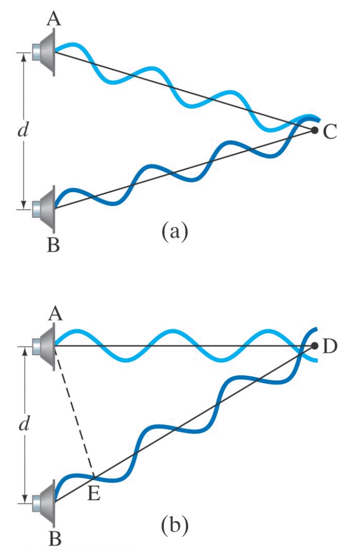
Suppose we have two sources putting out the same frequency f that are separated by some distance. If we stand at some other location, what will we hear?

Assuming the two sources are in phase with one another, the sound will potentially travel a different distance from each source to the listener, meaning the two sine waves will arrive at that point somewhat out of phase, or shifted relative to one another. If the shift works out to be an exact multiple of the wavelength of the sound, the sine waves will arrive 'in phase' with one another and the listener will hear a loud sound at that frequency. BUT, if one signal arrives a half-wavelength out of phase with the other, they'll cancel each other out.
In this figure, we see that point C is exactly 5 wavelengths away from source B, but exactly 4.5 wavelengths away from source A. Sound from source A arrives a half wavelength out of phase with the sound from source B and they'll mostly cancel each other out. (The sound from A is closer, so it's amplitude will be slightly higher than the sound from B, so they won't completely cancel each other out.)



Interference from two sources

Suppose we're in a room where the speakers are mounted on the wall $d = 2\ m$ apart from one another. If we sit somewhere equidistant from each speaker (i.e. along the midline coming out perpendicular to the wall), then the distance from us to each speaker is the same and whatever f is, the waves from those two sources always arrive 'in phase' with one another and we hear a nice loud sound.
What if we move a bit off to one side? Suppose we're 4 m away from the wall, directly in front of speaker A. At what frequencies will constructive interference occur? (I.e. for what frequencies will the distance difference be exactly an integer number of wavelengths?
The distance from A to D is exactly 4 m; the distance from B to D is $\sqrt{(2)^2 + (4)^2} = 4.472136\ m$, so speaker B is 0.472136 m further away from us.
Remember $v = \lambda/T = \lambda f$ so $\lambda = v/f$.



Constructive and Destructive Locations

If the wavelength is some multiple of that, the two waves arrive exactly in phase: $0.472136 = (N)\lambda = (N)(343)/(f)$ or $f = \frac{(N)(343)}{0.472136} = (N)(726.5\ Hz)$. At those frequencies the signals from the two speakers arrive in sync with one another and constructively interfere, creating a sound with (about) twice the amplitude.

If the distance difference happens to be an integer number of wavelengths PLUS ANOTHER HALF WAVELENGTH, then destructive interference will occur and most of the sound at those wavelengths will be cancelled out. That will occur where $f = \frac{(N+0.5)(343)}{0.472136} = (N + 0.5)(726.5\ Hz)$. That occurs at $f = 363.24\ Hz$, $f = 1089.73\ HZ$, $f = 1816.2\ Hz$ and so on.

Speaker Example : continued

Suppose our speakers are separated by 2 meters and we're sitting 4 meters away along the perpendicular bisector of the line connecting the two speakers.

The speakers are each playing a $f = 262\ Hz$ (middle C) tone in phase with one another.

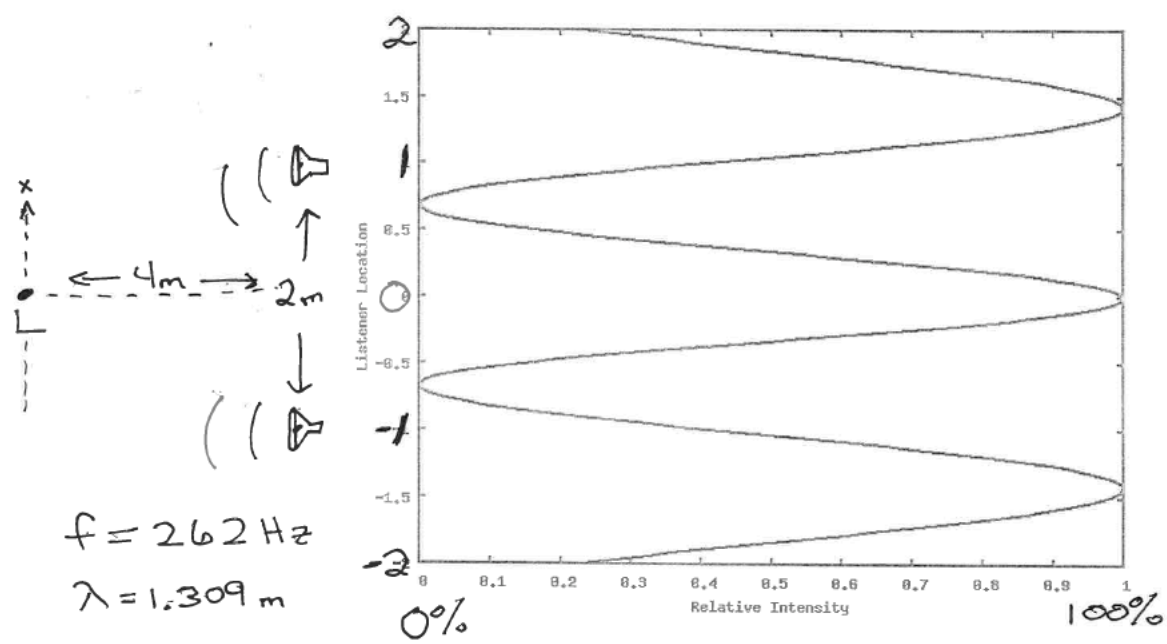
How strong will the signal be if the listener shifts sideways a bit?

Let $x = 0$ be when the person is right on the mid-line, the same distance from each speaker. Then sound travels exactly the same distance from each speaker to their ears and the waves will arrive in phase, leading to a loud tone.

As they move in the $+X$ direction ('up' on the figure), they're getting slightly closer to the upper speaker in the figure and slightly farther from the lower speaker in the figure.

When that distance difference equals exactly half the wavelength of the tone, the sound from the lower speaker arrives exactly a half-wavelength out of phase with the sound from the upper speaker and they'll (nearly) cancel each other out.

Use that `https://onlinetonegenerator.com` website to play that frequency at home and scale all the distance values based on the actual separation distance between your two speakers and see if you can notice this effect. (It'll be less pronounced in the real world because the sound from each speaker actually takes many other paths from the speaker to your ears, bouncing off any smooth surfaces in the vicinity like desktops, walls, and so on...)



Speaker Example: Far-field approximation

Suppose we're very far away from the two speakers. In that case, the paths from each speaker to where we're located are nearly parallel to one another. That let's us short-cut the 'path difference' part of the calculation. In fact, the path difference will just be $\Delta s = d \sin\theta$ which means we can find the locations (angles, really) where constructive and destructive interference occur more easily:

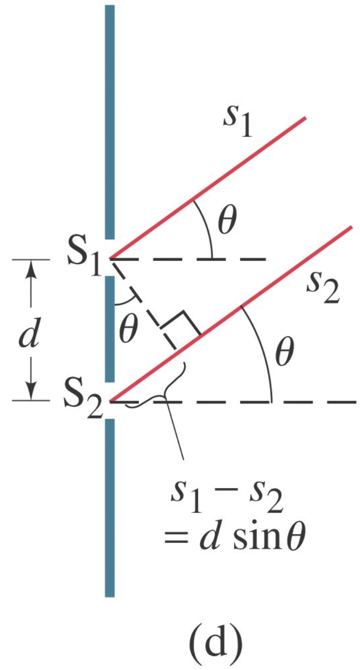
CONSTRUCTIVE interference

$d \sin\theta = m\lambda$ for $m = 0, 1, 2, \cdots$

If the path difference is exactly a multiple of λ PLUS AN ADDITIONAL HALF WAVELENGTH, the two waves arrive perfectly OUT of phase and cancel each other. Mathematically we can write this as:

DESTRUCTIVE interference

$d \sin\theta = (m + \frac{1}{2})\lambda$ for $m = 0, 1, 2, \cdots$



In the previous example the speakers were located $d = 2\ m$ apart. If they're emitting a frequency of $f = 262\ Hz$, at what angles will constructive and destructive interference occur (when we're far away)?

$v = \lambda/T = \lambda f$ so $\lambda = v/f = (343\ m/s)/(262\ Hz) = 1.309\ m$.

Constructive interference : $d \sin\theta = m\lambda$ for $m = 0, 1, 2, \cdots$ so here:

$(2.0) \sin\theta = 1.309m$

Rearranging we have: $\sin\theta = 0.65458m$.

- $m = 0$ yields $\theta = 0^o$
- $m = \pm 1$ yields $\theta = \pm 40.9^o$
- no other solutions

Destructive interference : $d \sin\theta = (m + \frac{1}{2})\lambda$ for $m = 0, 1, 2, \cdots$ so here:

$(2.0) \sin\theta = 1.309(m + \frac{1}{2})$

Rearranging we have: $\sin\theta = 0.65458(m + \frac{1}{2})$.

- $m = 0$ yields $\theta = +19.1^o$ and $m = -1$ yields $\theta = -19.1^o$.
- $m = 1$ yields $\theta = +79.1^o$ and $m = -2$ yields $\theta = -79.1^o$
- no other solutions

The main point here is that wherever you are, near or far, there are going to be some frequencies that are extra loud and some that are extra quiet: **except** right along a line representing the perpendicular bisector coming out from the midpoint between the two speakers. Always sit in the middle seat in a theater!