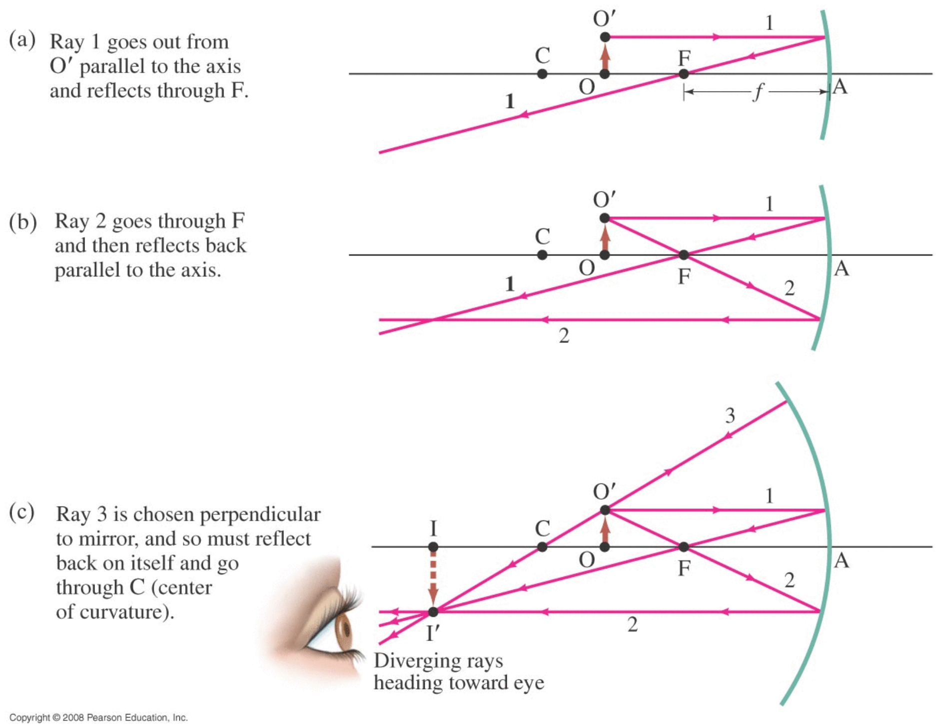
PH2233 Fox : Lecture 11
Chapter 32 : Light Reflection and Refraction

Ray Diagrams for Spherical Mirrors

Let's place an object in front of a concave mirror. Light hits the object and scatters photons in all directions from every point on the object. Let's follow a few of the photons leaving the top of the arrowhead (the point marked O' in the figure):



The photons starting at the point O' all nominally pass through the point I', so to our eye, they appear to be coming from that location. That is where the image of the object will form.

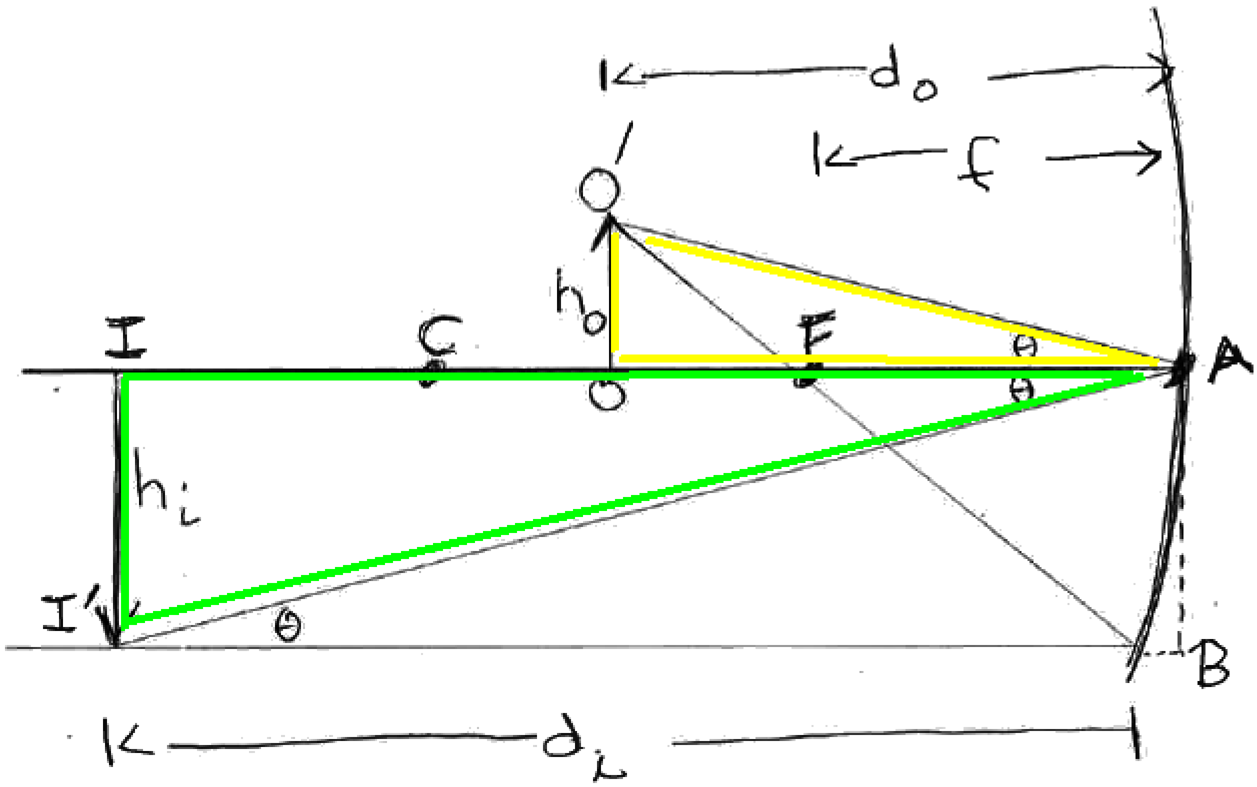
(Another ray we can draw: a photon that leaves O' and hits the vertex of the mirror (labelled A in the figure, but usually denoted with the letter V) will bounce back at the same angle below the axis, and it also passes through I'.)

These are called the **principal rays** but there are an infinity of paths starting at O' and they all pass through I'.

These **ray diagrams** give us a simple geometric way of analyzing mirrors.

Mirror Equation and Magnification

Let's convert this geometric process into an equation.



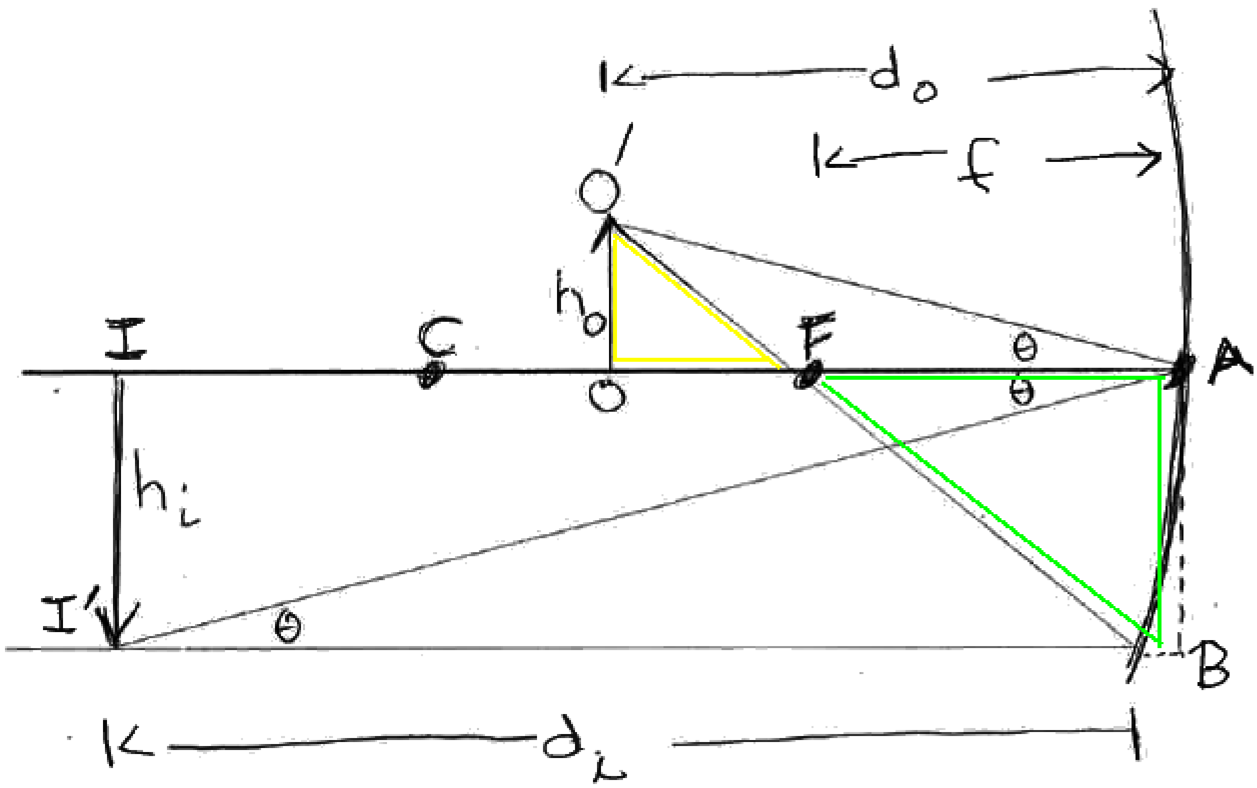
(We need the results here to simplify the full 'mirror equation' derivation on the next page.)

Similar Triangles: O-O'-A and I-I'-A (If the colors come though on your pdf viewer, the first triangle is highlighted in yellow, the second in green.)

- $\frac{h_o}{d_o} = \frac{|h_i|}{d_i}$
- Sign convention: image inverted here, so $h_i < 0$. Thus: $|h_i| = -h_i$
- Making that substitution: $\frac{h_o}{d_o} = \frac{-h_i}{d_i}$
- Rearrange: $\frac{h_i}{h_o} = -\frac{d_i}{d_o}$
- The ratio of the image to object height is called the magnification of the mirror, so here:

$$\boxed{m = \frac{h_i}{h_o} = -\frac{d_o}{d_i}}$$ **MAGNIFICATION FACTOR**

Where will this image form? See next page...



Similar Triangles: O-O'-F and A-B-F (almost)

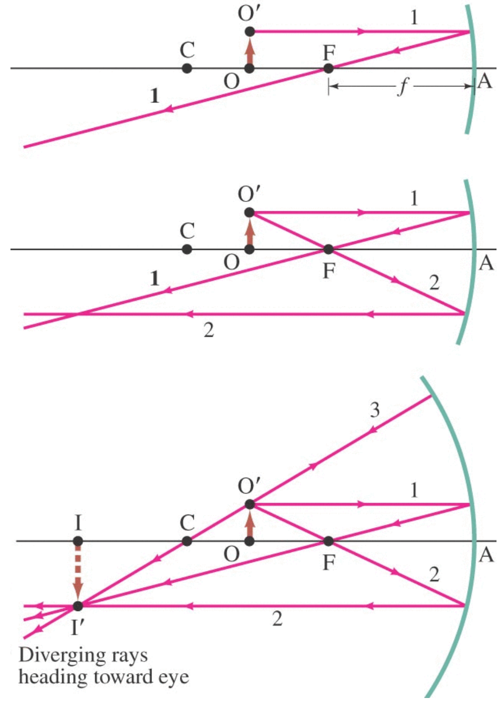
Note that these are only similar as long as the mirror is nearly flat ('small angle' or 'thin mirror' approximation).

- Comparing the height to the base for each triangle: $\frac{h_o}{d_o - f} = \frac{-h_i}{f}$
- Rearrange: $\frac{h_i}{h_o} = -\frac{f}{d_o - f}$
- But the left-hand side is also equal to $-\frac{d_i}{d_o}$ from previous figure
- Thus: $\frac{d_i}{d_o} = \frac{f}{d_o - f}$ or: $\frac{d_o}{d_i} = \frac{d_o - f}{f}$
- Multiply both sides by $\frac{1}{d_o}$:
  $\frac{1}{d_i} = \frac{d_o - f}{d_o f} = \frac{1}{f} - \frac{1}{d_o}$
- Finally, rearrange slightly: $\boxed{\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f}}$ **MIRROR EQUATION**

This is an approximation, but in reality it ends up being a very good approximation since real telescope mirrors have such large radii of curvature that they appear nearly flat.

$\boxed{\textbf{Mirror Equation Example (1)}}$

Let's take the picture we started with and apply our mirror equations to it. Let's say that the focal length here is $f = 20\ cm$. If it's $20\ cm$ from the vertex of the mirror (the point labelled A in these figures, but usually given the letter V), then I'd say the object is located about $30\ cm$ away from the mirror, so let's use $d_o = +30\ cm$.



Where will the image form? The **ray diagram** shows about where it should be so let's use the mirror equations to get the 'exact' location and size.

$$\frac{1}{f} = \frac{1}{d_o} + \frac{1}{d_i}$$

First let's rearrange this to solve for d_i:

$$\frac{1}{d_i} = \frac{1}{f} - \frac{1}{d_o}$$

Let's put the RHS of this equation over a common denominator:

$$\frac{1}{d_i} = \frac{d_o}{fd_o} - \frac{f}{fd_o} \text{ or:}$$

$$\frac{1}{d_i} = \frac{d_o - f}{d_o f}.$$

Flipping the whole equation over: $\boxed{d_i = \frac{d_o f}{d_o - f}}$

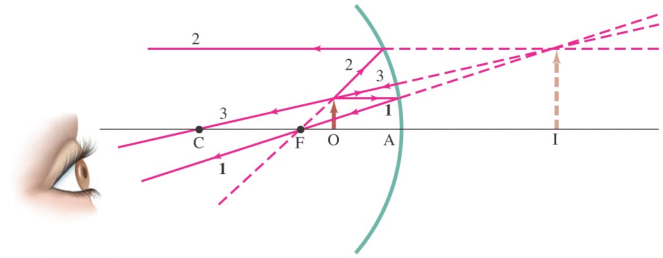
Here, we have $d_o = +30\ cm$ and $f = +20\ cm$ so $d_i = \frac{(30)(20)}{30-20} = 60/10 = +60\ cm$ (that puts the image on the same side as the outgoing photons: a **real** image).

Magnification: $m = -d_o/d_i = -(60)/(30) = -2.0$ (so an inverted or upside down image).

Note again: this is a REAL image. The photons leaving the mirror actually pass through where the image has formed. That means we can put a piece of film (or an image sensor like your phone has) right there and capture that image as the photons land on it.

Mirror Equation Example (2) : Object inside focal length

In the previous example, the object was located between the focal point and the mirror and we ended up creating an inverted but real image floating in front of the mirror.
What if we move our object close enough to the mirror that it's now located INSIDE the focal point of the mirror?



Here we have a 1 cm tall object placed 10 cm from a concave mirror that has a radius of curvature of 30 cm. Let's follow three of the many photons spewing out from a point on the object.

Ray 1 : a photon (ray) heading off parallel to the axis reflects off the mirror heading towards F.

Ray 2 : If we draw a line from F to the point on our object, that ray will bounce off the mirror and turn into a ray travelling parallel to the axis (because a ray coming in parallel to the axis along the same path in the opposite direction would reflect off the mirror and head to F, passing through that point on our object).

Ray 3 : Draw a line from the center of the sphere out to the mirror. Now, a photon travelling along that **radial** will bounce straight back from the mirror along the same path since it's a radial: it's a line that hits perpendicular to the sphere.

These three rays intersect at the point over on the right in the figure, so that's where the light appears to be coming from when we look there. We've created a **virtual** image (it's behind the mirror - no photons **actually** come from or pass through that point), and it's upright this time.

Let's verify this with our mirror equations and sign conventions.

The center of the sphere is in front of the mirror, so $r = +30\ cm$ and $f = r/2 = +15\ cm$.

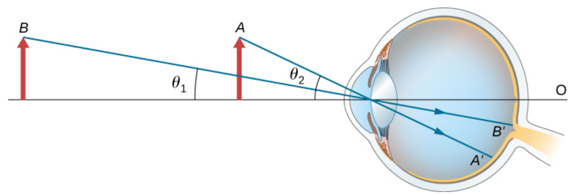
The object is in front of the mirror (on the photon side) so $d_o = +10\ cm$. The object is 1 cm tall, so $h_o = +1\ cm$.

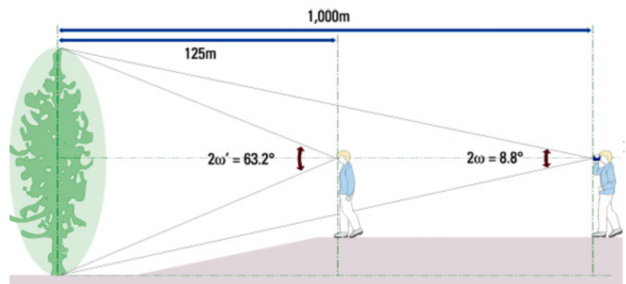
Then $d_i = \frac{d_o f}{d_o - f} = \frac{(10)(15)}{10-(15)} = \frac{150}{-5} = -30\ cm$ and $m = -d_i/d_o = -(-30\ cm)/(10\ cm) = +3$ so the image height will be $h_i = m h_o = (3)(1\ cm) = +3\ cm$.

We've created a virtual image that's 3 times larger than the object. But look at the figure: that image is farther away from our eye, so does it actually look larger to us? Let's look at this effect.

Apparent or Angular Magnification

Below, the same object is viewed at different distances showing how the angle the object takes up in our field-of-view changes.





It's fairly common to use the $\theta = (size)/(distance)$ approximation (giving θ in radians here remember), basically coming from the arc-length $s = r\theta$ expression (which requires θ to be in radians).

Technically a more accurate result can be done using trig. On the right, suppose the tree has a height of h and we're located some distance d away from it. Then cutting the tree in half, we have $\tan\theta = (h/2)/d = h/(2d)$. The half-angle (from the middle of the object to the top, say) would be $\theta = tan^{-1}(h/2d)$ so the full angle the object subtends in our field of vision is $2\theta = 2tan^{-1}(h/2d)$ (with θ coming out in radians of course). It's normally not worth that extra effort and I'm sure there are contexts where it's done, but for purposes of this class the simple $\theta = (size)/(distance)$ approach is fine.

Let's look at the problem we did on the previous page and determine the apparent magnification. Does the object actually 'look' bigger to the person viewing it?

Based on the dimensions given in the problem, the person's eye is about 50 cm from the vertex of the mirror, or about 40 cm from the object.

What angular size is the object? Let's use the 'arclength' formula $s = r\theta$ so $\theta = s/r = (1.0\,cm)/(40\,cm) = 0.025\ rad$ or about $\mathbf{1.4^o}$. (That sounds pretty small, but the Moon in the sky only takes up about $1/2\ deg$ so it's about 3 times larger than the Moon appears to us, in an angular sense.)

What angular size is the image? The image formed 30 cm behind the lens, so it's 80 cm from the person's eye but it's also 3 cm tall. $\theta = s/r = (3\ cm)/(80\ cm) = 0.0375\ rad$ or about $\mathbf{2.15^o}$.

That means the image 'looks' $M = (0.0375\ rad)/(0.0250\ rad) = 1.5$ times larger. This **angular** (also called **apparent**) magnification (using the symbol M instead of m) is what we see really. A small object close up may appear larger than a big object farther away, or vice versa.

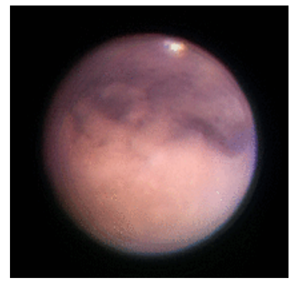
It's important to note the difference in these two 'magnifications' involved in mirrors (and lenses). The $m = h_i/h_o$ is basically the mathematical relationship between the (physical) sizes of the image and object, but since the image is closer or farther away from our eyes, the **angular** or **apparent** magnification is more relevant to us!

(Apparently some stores have used this 'trick', blocking your direct view of an actual object so you can only see the image, which 'appears' larger.)

Concave Mirrors in Telescopes

At it's closest approach, Mars is about 5.58×10^7 km from the Earth and has a diameter of about 6784 km. Let's see what sort of image a couple of different telescopes will create? Rearranging the mirror equation: $\frac{1}{d_i} = \frac{1}{f} - \frac{1}{d_o}$. Now f will be a few (or a few tens of) meters, but the object we're looking at is many orders of magnitude farther away. That means $d_o >>>> f$ and $1/d_o$ will be infinitesimal compared to $1/f$. Essentially, $d_i = f$ to many significant figures.

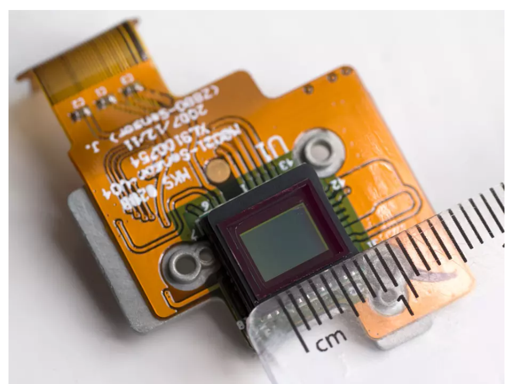
Backyard Telescope
Suppose we have a backyard telescope with $f = +2.80$ m and we turn this telescope on Mars at closest approach. Where will the image form, and how large will this image of Mars be?



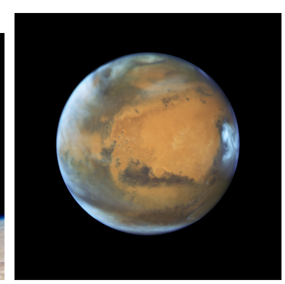
Using the argument above, the image will form at $d_i = +2.80$ m with a magnification of $m = -d_i/d_o = -(2.80\ m)/(5.58 \times 10^{10}\ m) = -5.02 \times 10^{-11}$

The height of the image of Mars this telescope creates will be $h_i = mh_o = (-5.02 \times 10^{-11})(6784000\ m) = -3.04 \times 10^{-4}$ m or about -0.304 mm.

NOTE: the digital cameras in phones use image sensors where each pixel is about 1.1 to 8.4 microns across, depending on the image sensor used (the more 'megapixels' in the image, the smaller the pixels in the image sensor itself will be).
The picture shows a $3264X2448$ (8 megapixel) sensor that's a rectangle roughly 6 mm by 4 mm, implying that each pixel is only about 1.8 μm or 1.8×10^{-6} m in size. Our Mars image was 0.304 mm across, which is 304 microns, or $(304\ microns)/(1.8\ microns/pixel) = 170\ pixels$. Mars has a diameter of about 6784 km, so each pixel represents 40 km of 'stuff' averaged together.



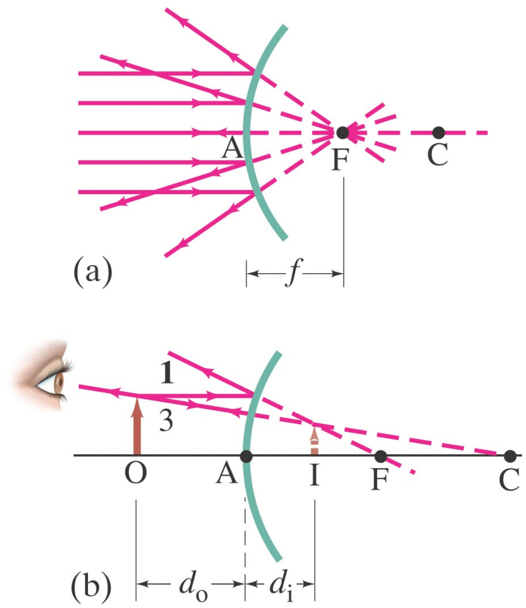
Hubble Space Telescope
The Hubble telescope has a focal length of $f = 57.6$ m. It turns out to achieve this using TWO mirrors instead of one and we'll see how that works later. For now, let's just treat this as a single concave mirror with the given focal length.



Repeating the above process: $d_i = f = 57.6$ m, so $m = -d_i/d_o = -(57.6\ m)/(5.58 \times 10^{10}\ m) = -1.03 \times 10^{-9}$ and $h_i = mh_o = (-1.03 \times 10^{-9})(6784000\ m) = -7.0 \times 10^{-3}$ m or -7 mm. That's about 23 times larger than the image the backyard telescope created, making a much larger image on our film or image sensor, allowing for smaller details to be seen. And our 7 mm image size now represents $(7000\ microns)/(1.8\ pixels/micron) = 3900\ pixels$. Mars has a diameter of about 6784 km, so each pixel now represents about 1.7 km of features averaged together.

Convex Mirrors

We can go through the same sort of 'similar triangles' argument for convex mirrors and create the same equations but with some signs flipped around, but if we chose a particular **convention** for our signs, we can end up using the exact same equation we have for concave mirrors.



The key is to think of the side of the mirror where the photons actually travel as defining positive coordinates. In the case of a convex mirror, the center of the sphere is over on the other side, so let's call that a **negative** radius, resulting in a **negative** focal length $f = r/2$. If an image forms over there, that would represent a negative value for d_i.
Picking numbers that are roughly to scale with the lower figure, suppose $r = -40\ cm$, making $f = -20\ cm$ and we have an object located at $d_o = +20\ cm$.

Then using our rearranged equation: $d_i = \frac{d_o f}{d_o - f} = \frac{(20)(-20)}{20-(-20)} = -400/40 = -10\ cm$.

That implies the image should form 10 cm **behind** the mirror. (That 'looks' about right since the distance from the vertex of the mirror to the focal point F we're claiming to be 20 cm and the image is forming about midway between those two points.)

The magnification will be $m = -d_i/d_o = -(-10\ cm)/(20\ cm) = +0.5$ so the image should be half as large as the object, and upright since $m > 0$. Again, that looks consistent with the ray diagram version.

Example: Mirrored Sphere Yard Ornament

Occasionally you see mirrored sphered on pedestals in yards around town.
Suppose we have such a sphere with a diameter of 20 cm. If we stand 2 m away from the surface of the sphere and are 1.8 m tall, where and how large is our image?
The center is inside the sphere (i.e. behind the mirror), so $r = -10\ cm$ making $f = -5\ cm$.
We're 2 m from this mirror, so $d_o = +200\ cm$. That makes $d_i = \frac{d_o f}{d_o - f} = \frac{(200)(-5)}{200-(-5)} = \frac{-1000}{205} = -4.878\ cm$.

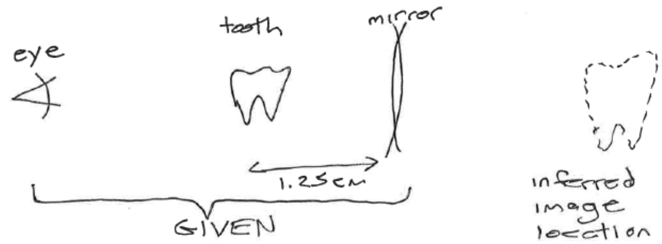
The magnification factor here is $m = -d_i/d_o = -(-4.878\ cm)/(+200\ cm) = +0.02439$ making the image height $h_i = mh_o = (+0.02439)(1.8\ m) = +0.0439\ m$ or about 4.4 cm tall.

What's the angular size of the image? We're $200 + 4.878$ centimeters away from this image, so $\theta_{image} = (size)/(distance) = (4.39\ cm)/(204.88\ cm) = 0.0214\ rad$ or about 1.23 deg.

If the mirror were perfectly flat instead, the image would form 2 m behind the mirror so we'd be 4 m away from our image, which would be 1.8 m tall (same as our real height) giving an angular size of $\theta = (size)/(distance) = (1.8\ m)/(4\ m) = 0.45\ radian$ or about 25.8 deg, so you might say the 'angular magnification' here is about $1.23/25.8 = 0.048$ or about 1/21. Definitely a small image (you can see the image of the photographer in the mirrored sphere here).

Dental Mirror

A dentist uses a curved mirror to view teeth on the upper side of the mouth. Suppose she wants an upright image with a magnification of 2.00 when the mirror is 1.25 cm from a tooth. (Treat this problem as though the object and image lie along a straight line.)



(a) What kind of mirror (concave or convex) is needed? (b) What must be the focal length and radius of curvature of this mirror? (c) Draw a principal-ray diagram to check your answer in part (b).

Leaving everything in units of centimeters:

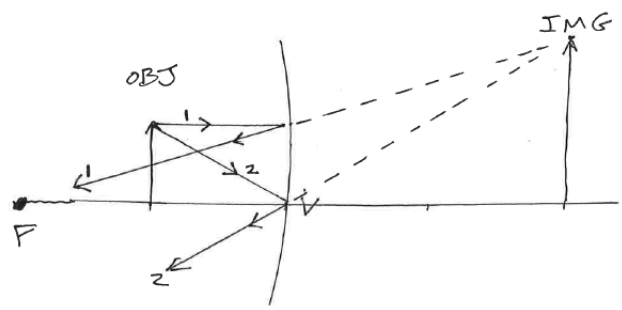
Here we have $d_o = 1.25\ cm$ and $m = +2.00$.

$m = -d_i/d_o$ so $d_i = -md_o = -(2.00)(1.25\ cm) = -2.50\ cm$. That means the image will be behind the mirror: a **virtual** image.

Both types of mirrors can potentially do this, so let's use the information we have now to determine the focal length:

$\frac{1}{f} = \frac{1}{d_o} + \frac{1}{d_i} = \frac{d_i + d_o}{d_o d_i}$ so $f = \frac{d_o d_i}{d_o + d_i} = \frac{(1.25\ cm)(-2.50\ cm)}{(1.25\ cm)+(-2.50\ cm)} = +2.5\ cm$

$f = R/2$ so the radius of curvature here is $R = +5.00\ cm$. The positive focal length (and radius of curvature) means we have a **concave** mirror. A ray diagram of the situation is shown on the right.



Suppose the dentist's eye is 20 cm from the tooth. What will the apparent magnification of the tooth's image be?

We don't know the size of the tooth, so let's just call it h_o (and we'll leave all our measurements in centimeters still). Then:

- $\theta_{obs} = (size)/(distance) = (h_o)/(20)$
- $\theta_{img} = (size)/(distance) = (h_i)/(20 + 1.25 + 2.50) = h_i/23.75$ BUT:
- $h_i = mh_o = 2h_o$ so $\theta_{img} = 2h_o/23.75 = h_o/11.875$
- Finally: $M = \theta_{img}/\theta_{obs} = \frac{h_o}{11.875} \times \frac{20}{h_o} = 1.6842...$

The image definitely does appear larger, just not quite 'twice as large'. m and M are different things: m is the mathematical magnification of the image; M is how it **appears** to us (how much of our field of view it takes up).

Example 32-7 : Convex Rearview Mirror

Suppose an external rearview car mirror is convex with a radius of curvature of 80 cm. Determine the location of the image and its size (both mathematical and apparent) if a truck is 10 m from the mirror.
For the 'apparent' magnification part, assume your eye is 50 cm from the mirror.

The truck distance was given in meters, so let's use meters for all our measurements.

This is a convex lens ao $r = -0.80\ m$, making $f = r/2 = -0.40\ m$.

The object is at $d_o = 10\ m$ so the image will form at $d_i = \frac{d_o f}{d_o - f} = \frac{(10)(-0.4)}{10-(-0.4)} = \frac{-4}{10.4} = -0.3846\ m$.

That's negative, which means the image forms about 38 cm behind the mirror.

Image size: $m = -d_i/d_o = -(-0.3846\ m)/(10\ m) = +0.03846$. The image height will be the actual height of the truck (probably several meters) times that factor, which is positive so the image will be upright (convenient!).

What about the angular or apparent magnification though? I don't know the actual height of the truck, so let's just leave that symbolic for this analysis since the actual height ends up not mattering (well, within reason anyway). If the truck has a height of h_o, the image will have a height of $0.03846h_o$.

If we just turned around and looked at the truck, what angle would it subtend? We're 50 cm from the mirror, so if we turn around the truck would be about 9.5 m away from us. The angular size of the object would be $\theta_{object} = (size)/(distance) = (h_o)/9.5 = 0.1053h_o$

How about the angular size of the image? It's 38.46 cm behind the mirror, and our eyes are 50 cm in front of the mirror, so the image is 88.46 cm away from our eye. The angular size of the image would be $\theta_{image} = (size)/(distance) = (0.03846h_o)/(0.8846) = 0.04348h_o$.

The apparent (or angular) magnification then is $\theta_{image}/\theta_{object} = \frac{0.04348h_o}{0.1053h_o} = 0.413$.

The image we're looking at in the mirror is less than half the (angular) size the truck itself would have if we looked directly at it. A truck twice as far away from us would only take up half as much angle in our vision, so the image **looks** like a vehicle that's over twice as far away from us as it actually it. Or put another way, the object is actually less than half as far away as it appears to be in the mirror! Hence the warning printed on these mirrors about objects being closer than they appear to be.