PH2233 Fox : Lecture 14
Chapter 33 : Lenses and Optical Instruments

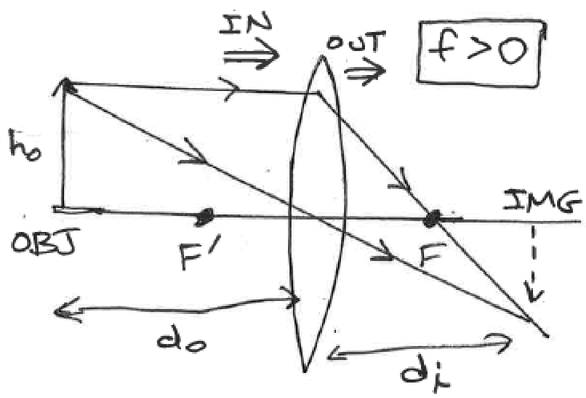
(We'll start by looking at the lens equation at the end of the previous lecture notes, and also the 'lens power' convention used by folks in the field.)

Lens Power: $P = 1/f$ when f is measured in meters, yielding P in units of Diopters (really just m^{-1}).

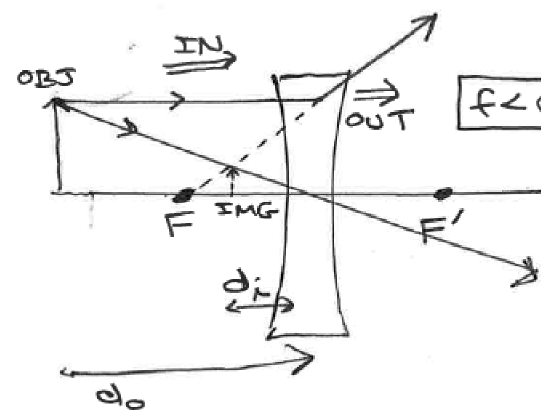
Lens equation (same as mirror equation):

$$\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f} \qquad m \equiv h_i/h_o = -d_i/d_o$$

Sign Rules : review from previous lecture notes.



Converging Lens



Diverging Lens

The easiest way to deal with this is to consider the actual photon paths. Photons leave the object and enter the lens on the left side in these figures, making that the 'incoming ray' or 'incoming photon' side of the lens. The lens is clear, so those photons exit on the other side, defining the 'outgoing ray' or 'outgoing photon' side of the lens.

If the **object** is on the 'correct' side (the incoming ray side) then d_o is positive. (If we only have a single lens, this will always be the case, but sometimes not when more than one lens is involved.)

If the **image** is on the 'correct' side (the outgoing ray side) then d_i is positive. (Otherwise, d_i is negative.) the lens

EXAMPLE : Classroom Projector

Let's apply what we've learned to the projector in the classroom. (This is out of date now, but originally these projectors basically had a small screen inside that was VERY bright and a lens was used to create a magnified image on the projector screen at the front of the classroom. These particular projectors actually use lasers to directly draw an image on the screen.)

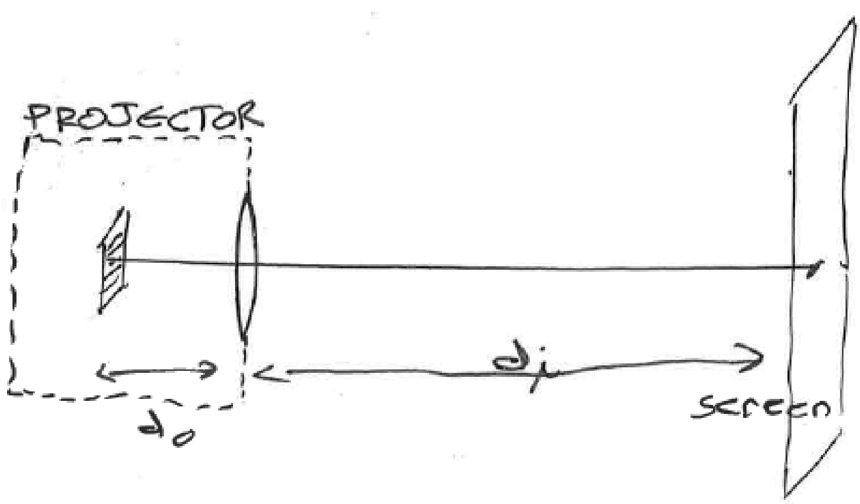
d_i : from lens to screen : about 5 m

d_o : object will be a little display screen inside the projector but where? There's a lot of electronics in there, power supplies and so on. I just guessed that the internal display screen is about 20 cm away from the lens. Using centimeters in all our measurements here:

$d_i = 20$ implies $f = (d_o d_i)/(d_o + d_i) = (500)(20)/(500 + 20) = +19.23\ cm$. (A **converging** lens, since $f > 0$. Also note that the internal display screen is less than a centimeter away from the focal length of the lens.)

$m = -d_i/d_o = -(500)/(20) = -25$ so the image on the projection screen is 25X the size of the little screen inside the projector, but note it's also negative. That means the projected image will be upside down relative to the internal screen, but we can fix that easily enough but mounting the internal screen upside down I guess.

How large must the little internal display screen be? The image forming on the projector screen is about 3 m across, so the little screen inside the projector would be 1/25 of that or about 12 cm across. There's enough room for that, so that could be correct.



(33-3) : Combinations of Lenses

Note: our 'sign conventions' will come into play big time now.

EXAMPLE 1 : Let's place an object 30 cm to the left of a $f = 20\ cm$ converging lens. Determine the location, size and nature (real or virtual) of the image.

- Object on same side as rays incoming to the lens, so $d_o = +30\ cm$
- $d_i = \frac{d_o f}{d_o - f} = \frac{(30)(20)}{30-20} = +60\ cm$ That's positive, so the object is on the 'outgoing ray' side, making it a REAL image.
- $m = -d_i/d_o = -(60)/(30) = -2$: the image will be larger, but inverted

Let's now place another identical $f = 20\ cm$ converging lens 30 cm to the **right** of that image. **The IMAGE created by the first lens is now the OBJECT for the second lens.** Determine the location, size and nature (real or virtual) of the final image these two lenses in combination have produced.

- The **image** from the first lens becomes the **object** for the second lens. This lens is 30 cm to the right of the image formed by the first lens.
- The object is on the same side as rays incoming to the (second) lens, so $d_o = +30\ cm$
- $d_i = \frac{d_o f}{d_o - f} = \frac{(30)(20)}{30-20} = +60\ cm$ That's positive, so the object is on the 'outgoing ray' side, making it a REAL image.
- $m = -d_i/d_o = -(60)/(30) = -2$: the image will be larger, but inverted

Overall Result : The first lens multiplied the object height by $m = -2$. The second lens treats that image as it's object and further multiplies it by another factor of -2. Overall then, the final image has a multiplication factor of $m = m_1 \times m_2 = (-2)(-2) = +4$ relative to the original (actual) object present here.

The final image is four times larger than the original object, and is now UPRIGHT (and real).

QUESTION : Is there any way we could achieve this (a real, upright image) using a single lens of any type?

The easy answer here is NO. An 'upright image' implies that the magnification factor m needs to be a positive number, BUT $m = -d_i/d_o$. With a positive d_o, the only way to end up with a positive m is if we have a NEGATIVE d_i, which means the image will have to be VIRTUAL, not REAL...

EXAMPLE 2 : Let's move the second lens so it's now just 30 cm to the right of the first lens. Determine the location, size and nature (real or virtual) of the final image these two lenses in combination have produced.

- From above, the first lens created an image at $d_i = +60\ cm$ and that lens introduced a multiplication factor of $m_1 = -2$.
- The **image** from the first lens becomes the **object** for the second lens. This lens is 30 cm to the LEFT of the image formed by the first lens. It's NOT on the 'incoming ray side' for the second lens, which means that $\boxed{d_o = -30\ cm}$ for the 2nd lens.
- $d_i = \frac{d_o f}{d_o - f} = \frac{(-30)(20)}{(-30)-(20)} = +12\ cm$.
- Since this d_i is POSITIVE, this image is forming on the outgoing ray side for this lens, making this a REAL image.
- $m = -d_i/d_o = -(12)/(-30) = +0.4$

The first lens multiplied the object height by $m = -2$. The second lens now multiplies it's 'object' (the image created by the 1st lens) and further multiplies it by a factor of $+0.4$. Overall then, the final image has a multiplication factor of $m = m_1 \times m_2 = (-2)(+0.4) = -0.8$.

The final image is a little smaller than the original object, REAL, and INVERTED. (I couldn't come up with a plausible real-world scenario where this setup would be useful, since we can create inverted real images easily with a single lens too.)

EXAMPLE 3 : Let's replace the second lens in the previous example with a **diverging** lens with $f = -20\ cm$ now. Determine the location, size and nature (real or virtual) of the final image these two lenses in combination have produced.

- From above, the first lens created an image at $d_i = +60\ cm$, so 60 cm to the right of the that lens, and it introduced a multiplication factor of $m_1 = -2$.
- The **image** from the first lens becomes the **object** for the second lens. Again, the 'object' for the 2nd lens is on the 'wrong side' of that lens (NOT on the 'incoming ray' side), so $d_o = -30\ cm$.
- $d_i = \frac{d_o f}{d_o - f} = \frac{(-30)(-20)}{(-30)-(-20)} = -60\ cm$
- Since d_i is NEGATIVE, this image is forming NOT on the outgoing ray side making it a VIRTUAL image.
- $m = -d_i/d_o = -(-60.0)/(-30) = -2.0$

The first lens multiplied the object height by $m = -2$. The second lens now multiplies it's 'object' (the image created by the 1st lens) and further multiplies it by a factor of -2. Overall then, the final image has a multiplication factor of $m = m_1 \times m_2 = (-2)(-2) = +4$

The final image is four times larger than the original object, and is now UPRIGHT and VIRTUAL.

This is the most common geometry for **binoculars** and **spyglass** type telescopes, basically a combination of an initial converging lens, then a diverging lens placed just outside the focal point of the first lens. The result is typically an upright, magnified, and virtual image. (The 'virtual image' part is kind of a requirement for devices like this since the person's eye will be right up against that second lens.)

See near the end of the **examples33.pdf** file on Canvas for a more detailed exampled of such a spyglass geometry. Canvas also has a **twolenses.pdf** example like we did here with the 2nd lens moving around and changing type.

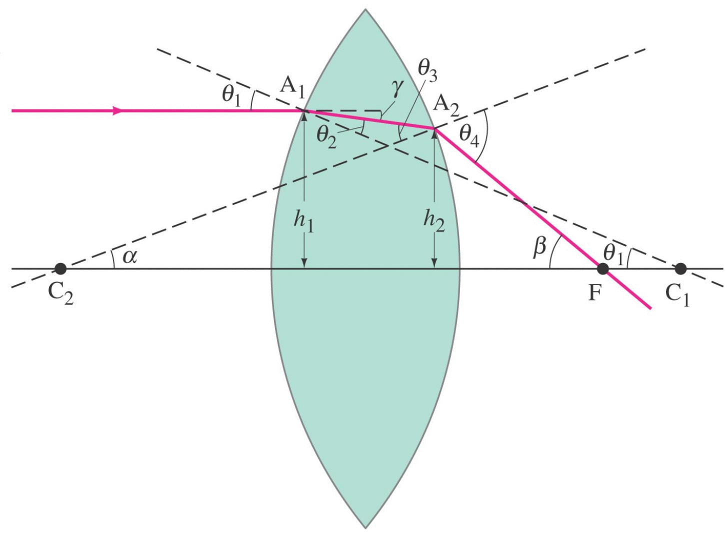
(33-4) : Lensmaker's equation

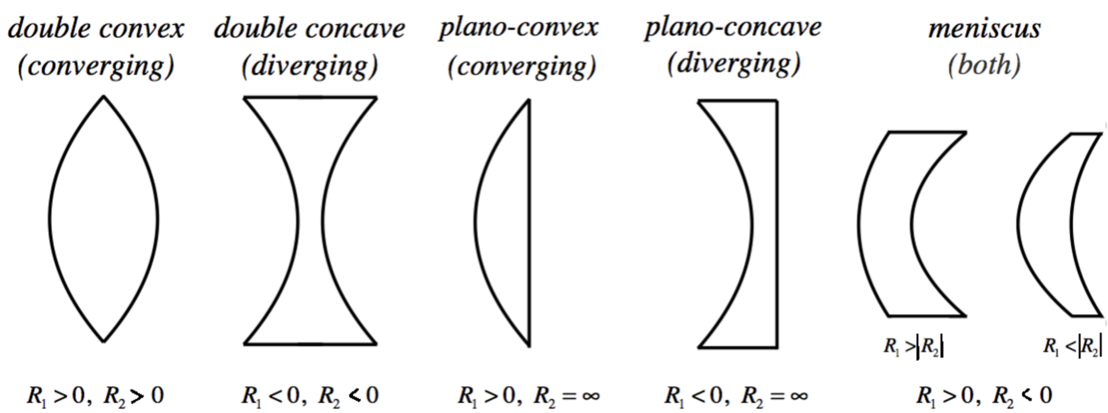
Suppose we have a (double convex) lens made of a material of index of refraction n that is embedded in a medium with an index of refraction n_o.

Judiciously propagating angles and using approximations like $\sin\theta \approx \tan\theta \approx \theta$, we find (see the book for details of the derivation!) that:

$$\frac{1}{f} = \left(\frac{n-n_o}{n_o}\right)\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

which is called the **Lensmaker's Equation**. The radii follow a **sign convention** illustrated below. Basically this book (and others, but **not all**) use the convention that for a double-convex lens (as shown here), both R values are taken to be **positive**. For each side, the **center of curvature** for that side is over on the other side of the lens. If that side's center is NOT on the 'correct' side, that side's R is taken to be negative.





Sign Conventions for R

Let's look at each of these shapes and determine their focal lengths. (I won't do every one in class, but will leave all of them here.)

In each case, we'll assume these are glass ($n = 1.5$) lenses operating in air ($n_o = 1.0$). We can rearrange the Lensmaker's equation a bit to eliminate most of the $1/xx$ operations:

$f = \frac{n_o}{n-n_o} \times \frac{R_1 R_2}{R_1+R_2}$ and in these examples, $n_o = 1$ so: $\boxed{f = \frac{1}{n-1} \times \frac{R_1 R_2}{R_1+R_2}}$

- **Double Convex** : Suppose $R_1 = R_2 = +20\ cm$. $f = \frac{1}{1.5-1} \times \frac{(20)(20)}{20+20} = +20\ cm$. (Positive, so this is a converging lens.)

- **Double Concave** : for each side of this lens, the center of curvature is on the 'wrong' side, so both of these R values will be negative. Suppose $R_1 = R_2 = -20\ cm$. $f = \frac{1}{1.5-1} \times \frac{(-20)(-20)}{(-20)+(-20)} = -20\ cm$. (Negative, so this is a diverging lens.)

- **Plano-convex** : one side is flat, the other is convex. The center of curvature for the left side is over on the right where it 'belongs', so $R_1 = +20\ cm$. The right side is flat, so it's center of curvature is essentially at $R_2 = \infty$. We'll have to leave the Lensmaker's equation in its original form for this one:

  $\frac{1}{f} = \frac{n-n_o}{n_o}(\frac{1}{R_1} + \frac{1}{R_2})$ so: $\frac{1}{f} = \frac{1.5-1}{1} \times (\frac{1}{20} + \frac{1}{\infty}) = (0.5)(0.05+0) = 0.025$ from which $f = +40\ cm$. (Positive, so this is a converging lens.)

- **Plano-concave** : one side is flat, the other is concave. The center of curvature for the left side is over on the left, which is the 'wrong' side, so $R_1 = -20\ cm$ now. The right side is flat, so it's center of curvature is essentially at $R_2 = \infty$. We'll have to leave the Lensmaker's equation in its original form for this one:

  $\frac{1}{f} = \frac{n-n_o}{n_o}(\frac{1}{R_1} + \frac{1}{R_2})$ so: $\frac{1}{f} = \frac{1.5-1}{1} \times (\frac{1}{-20} + \frac{1}{\infty}) = (0.5)(-0.05+0) = -0.025$ from which $f = -40\ cm$. (Negative, so this is a diverging lens.)

- **Meniscus (2nd to last figure above)** : Here the left side has it's center of curvature over on the right, which is the 'correct' side, so R_1 will be a positive number. It's 'flatter' than the right side though, so let's say $R_1 = +40\ cm$ here. The right side of the mirror has it's center of curvature over to the right, which is the 'wrong' side for that face, so let's say $R_2 = -20\ cm$.

  $f = \frac{1}{1.5-1} \times \frac{(40)(-20)}{(40)+(-20)} = -80\ cm$.

  This is negative, so this is a diverging lens. Notice we were able to create a much larger focal length this time, compared to the second example above. A large (negative) f using that 'double concave' geometry would result in a lens that is very much thinner in the middle than around its edges. The current shape would be much more 'robust'.

- **Meniscus (last figure above)** : Here the left side has it's center of curvature over on the right, which is the 'correct' side, so R_1 will be a positive number. Let's say $R_1 = +20\ cm$. The right side of the mirror has it's center of curvature over to the right, which is the 'wrong' side for that face, so it will be a negative number. It's also 'flatter' than the left face, so let's say $R_2 = -40\ cm$.

  $f = \frac{1}{1.5-1} \times \frac{(20)(-40)}{(20)+(-40)} = +80\ cm$.

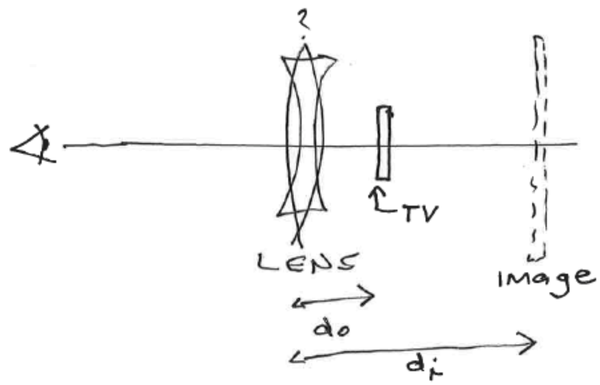
  This is positive, so this is a converging lens. Notice we were able to create a much larger focal length this time, compared to the first example above. A large (positive) f using that 'double convex' geometry would result in a lens that is very much thicker in the middle than around its edges (and remember all our equations require 'thin lenses', so a very thick double-convex lens would likely create a distorted image. This new shape would be a better option.

Note in general that any lens that is THICKER IN THE MIDDLE will be a CONVERGING lens. Any lens that is THINNER IN THE MIDDLE is a DIVERGING lens.

Lens Design Example : TV Magnifier

In the early days of television, the screens were very small but one 'solution' was to add a large lens in front of the screen to create a magnified image.

We don't want to have to flip the TV upside down, so we'll need this lens to create an upright, magnified image. $m = -d_i/d_o$ so since d_o will be positive, we'll need d_i to be negative. This lens needs to create a virtual image.



Let's say we want $m = +1.5$ with the lens sitting 15 cm in front of the TV. That means $d_i = -md_o = -(1.5)(15\ cm) = -22.5\ cm$.

Apparent Magnification : Before we go further, will this image **appear** to be larger than the screen? We're creating a magnified image, but it's also farther away, so what will the **apparent** (angular) magnification be?

Suppose we're sitting 2 m (200 cm) away from the TV. Then $\theta_{obs} = (size)/(distance) = (h_o)/200$.

The image is 1.5 times larger, but is 22.5 cm behind the lens - which is $200 - 15 = 185$ cm from our eye, so the image will be $185 + 22.5 = 207.5$ cm away from out eye. $\theta_{img} = (size)/(distance) = (1.5h_o)/207.5$.

$M_{app} = \frac{\theta_{img}}{\theta_{obs}} = \frac{1.5h_o/207.5}{h_o/200} = 1.446$ so it certainly looks bigger, just not quite 1.5 times bigger.

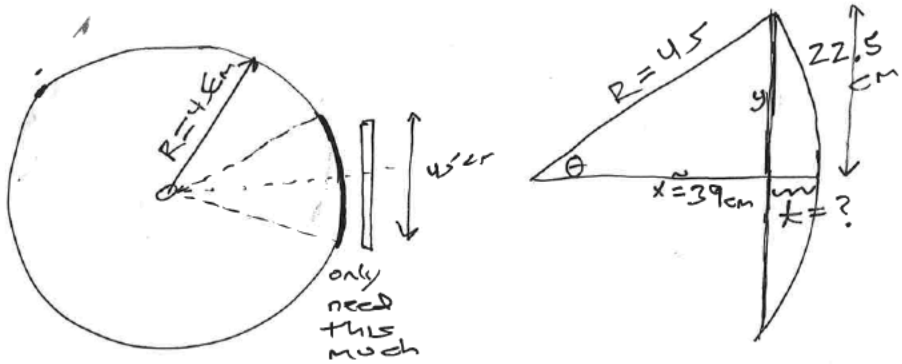
Focal Length : What focal length does this lens need to have? Leaving everything in centimeters:

$f = \frac{d_o d_i}{d_o - d_i} = \frac{(15)(-22.5)}{15+(-22.5)} = +45\ cm$. That's positive, so we're dealing with a **converging** lens (i.e. one that's thicker in the middle).

Suppose we use a **double convex** shape, with the same R on each side, and we'll make this from glass or plastic which will have an index of refraction around 1.5 so we'll let $n = 1.5$ for this design.

$f = \frac{1}{n-1}\frac{R_1 R_2}{R_1 + R + 2} = \frac{1}{0.5}\frac{(R)(R)}{2R} = R$ so ultimately, the radius of curvature of our lens will be $R = f = 45$ cm (i.e. our lens is essentially a slice cut from a 90 cm diameter sphere, with its flat side 'glued' to another such slice).

How thick will this lens end up being? The lens needs to be big enough to display the whole image, which is supposed to be 1.5 times larger than the actual TV screen. I used to have a little TV with a 12" screen, so say $h_o \approx 30\ cm$ making $h_i \approx 45\ cm$.



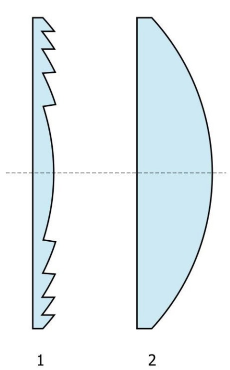
This figure shows how we can determine the lens thickness. The left side shows a circle of radius $R = 45\ cm$ but we only need a slice of this circle that can cover 45 cm (maybe a little less), so I've highlighted that with a thick black line. Using the right figure, we find that each side of this lens will 'stick out' about $t = 6\ cm$, for a total thickness of 12 cm at the middle of the lens. Hm. That's almost **5 inches thick**. That's a **lot** of glass. This lens would weigh over 50 *pounds* which wouldn't be either cheap or convenient.

Can we still make this practical somehow? The trick here is that really the important thing a lens does (refraction) is happening right at the surface. The only thing we 'need' about this lens is to have its surface have the right amount of curvature for each ray (photon) that hits it. We don't need the 'insides' of the lens, just it's surfaces, so one solution is to carve out most of the glass inside the lens and push the two faces together. This is called a **Fresnel Lens** and goes back over 200 years.

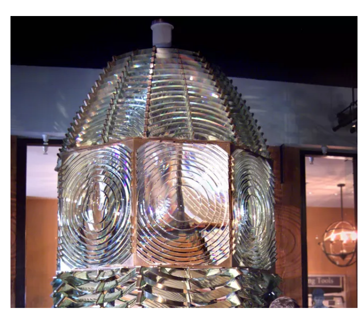
The left figure below shows a common Fresnel Lens that's 'equivalent' to a much thicker plano-convex lens (flat on one side, convex on the other).

Middle figure: The downside is all those edges (rings, really) aren't invisible to the eye and can be annoying...

These lenses are commonly used in lighthouses (rightmost figure).



Fresnel Lens



You can buy lenses made like this from cheap plastic all over the place. Here are a couple advertised on Amazon for reading books or magnifying phone and tablet screens.

They're cheap enough to be given away at conferences and tradeshows. Here's one similar to the one I showed in class:

These are usually plano-convex shapes, with $R_1 = R$ and $R_2 = \infty$. The one I showed has $f = 26\ cm$ so what radius of curvature would be needed for a 'normal' glass lens with this focal length?

$\frac{1}{f} = (n-1)(\frac{1}{R_1} + \frac{1}{R_2} = (1.5-1)(\frac{1}{R} + 0) = \frac{0.5}{R}$ so basically $f = R/0.5 = 2R$ or finally $R = f/2 = (26\ cm)/2 = 13\ cm$. Mine covers $9\ cm$ so using the top figure on the previous page with $R = 13$ and $y = 9/2 = 4.5\ cm$ we find $t = 0.8\ cm$, so a traditional plano-convex lens would be almost a centimeter thick at the center. (Definitely not very functional as a bookmark...)