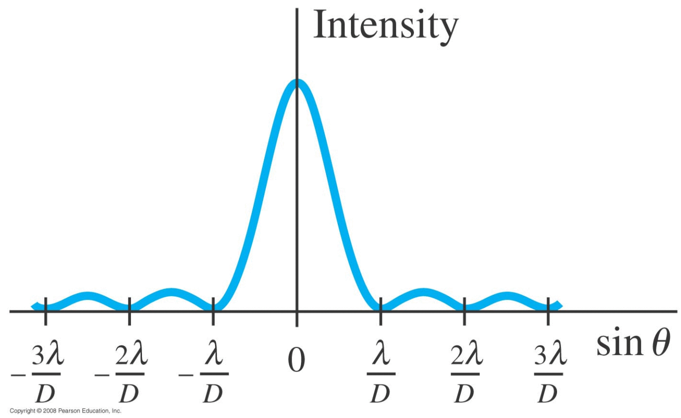
PH2233 Fox : Lecture 20
Chapter 35 : Diffraction

So far we have dealt with diffraction through slit-like geometries (single, double, diffraction gratings) and were able to (at least somewhat) derive what the intensity pattern would be.

For waves passing through a **SINGLE SLIT** of width D, we broke the wave front passing through the slit into an infinite number of infinitesimal sources and paired them up to show that we would have complete cancellation where $D \sin\theta = m\lambda$ for $m = \pm1, \pm2, \ldots$. We also found that the intensity pattern involved the square of the **sinc** function.

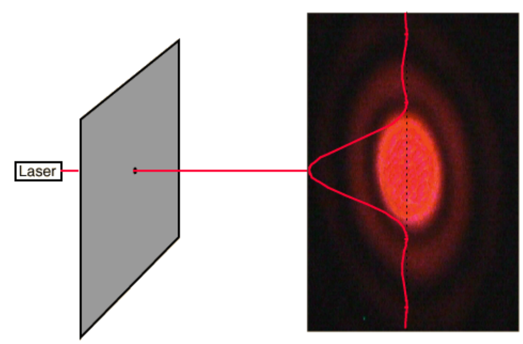


Single-Slit Intensity Pattern

CIRCULAR APERTURES

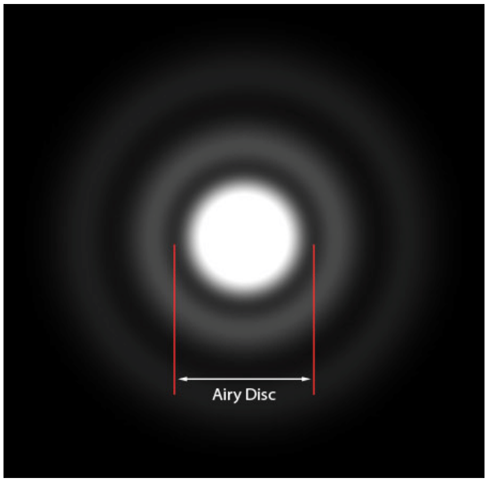
What if the waves are passing through a **circular** aperture instead of a slit? Light passing through a **lens** into a camera, or through our **pupils** into our eyes, and even light reflecting off a **circular mirror** would be examples of this type of geometry.

This pattern is related to one of the Bessel functions, which are solutions to a particular form of differential equation that appears in many situations, including vibrating circular membranes like drumheads, so it's possible you'll encounter them again. The main central 'lobe' (out to the first zero-intensity ring) is called an **Airy disk**, and contains about 97% of the total intensity in the pattern.



Circular Aperture Intensity Pattern

The left figure below shows an idealized version of how a single point source of light gets 'spread out' passing through a circular aperture. The figure on the right is a blown-up picture of a tiny 1 mm by 1 mm area of the image sensor on a digital camera that's pointed at a 'point' source of light.



Idealized

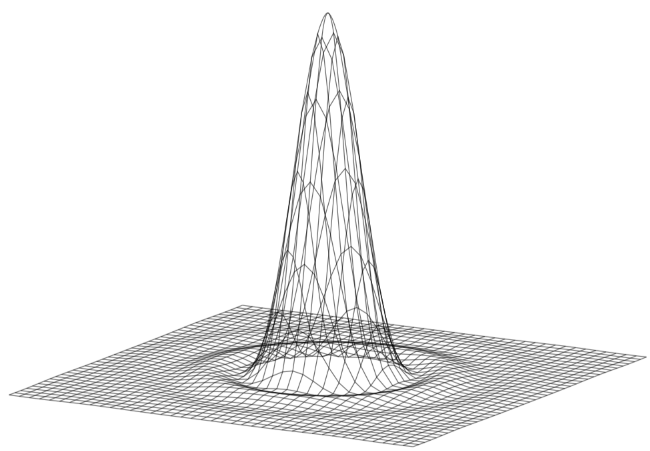
Actual

This function looks somewhat like the *sinc* function for the single-slit intensity pattern, but it's not quite the same, and has its maxima and minima at slightly different locations.

For completeness, the result of doing the integral for the intensity results in:

$$I(\theta) = I_o(\frac{2J_1(\alpha)}{\alpha})^2 \text{ where } \alpha = \frac{\pi D \sin\theta}{\lambda}$$

which looks similar to the single slit sinc function intensity we found, so this is sort of the J_1 version of the sinc function.

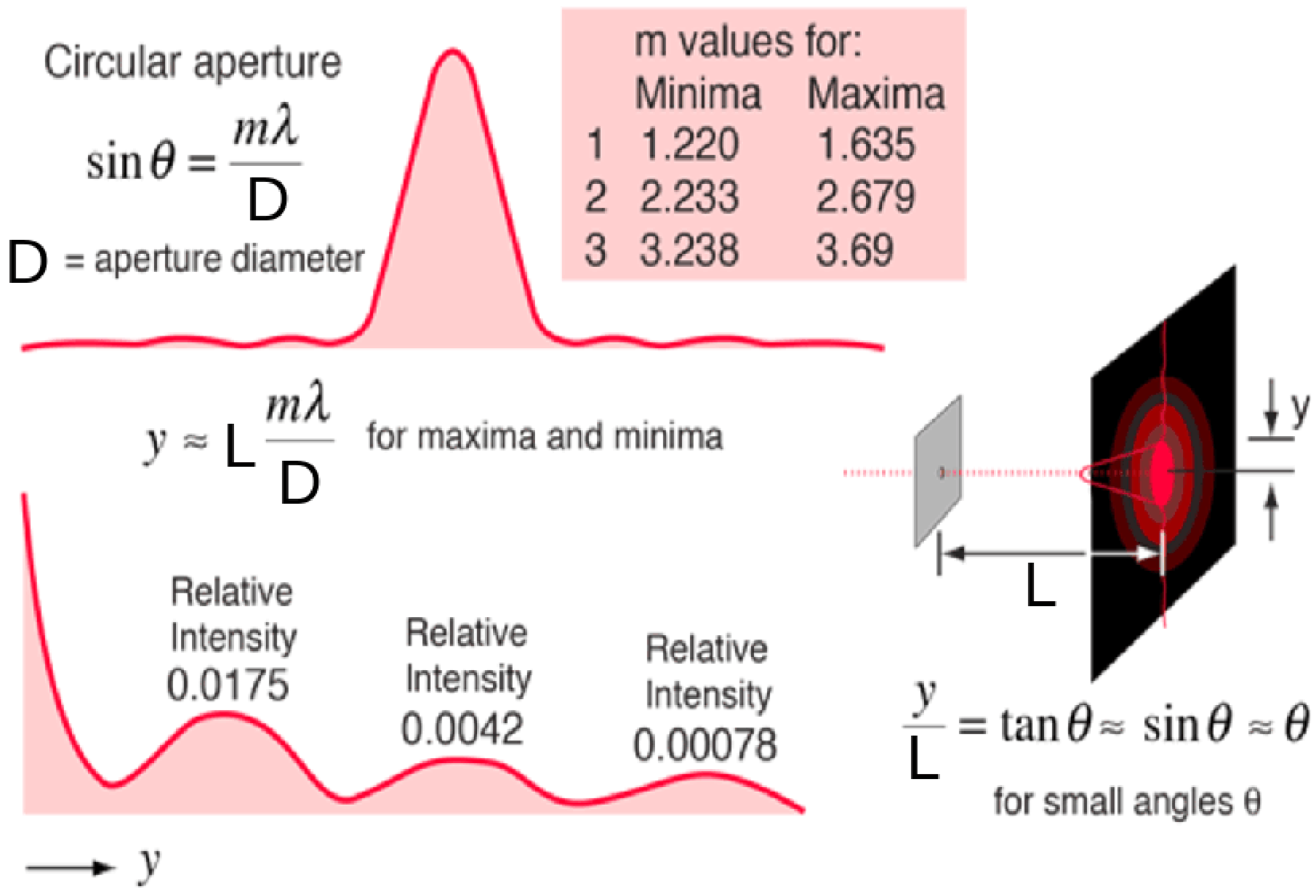


The dark rings (intensity minima) are sometimes still written as $\sin\theta = \mathbf{m}(\lambda/\mathbf{D})$ **but** for circular apertures, these m values are **not integers**. The table below shows the first three of these non-integer 'm' values that denote either the minima or maxima.

If this diffraction pattern is being displayed on a screen (or forming on an image sensor), the physical locations of a given feature would be $y = L\tan\theta$ and if the angles are small (they will be in our examples) then $\tan\theta \approx \theta$ (as long as we use radians) and $\sin\theta \approx \theta$ so we can write these as:

$\theta \approx m\lambda/D$ and $y = L\frac{m\lambda}{D}$

(again, remember that the m values are NOT integers for the circular aperture scenario).



The radii of the rings where complete destructive interference occur are the zeroes of the Bessel function involved here. They're not equally spaced, aren't integers, and aren't any simple fraction of π or anything else nice.

BLURRING DUE TO CIRCULAR APERTURES

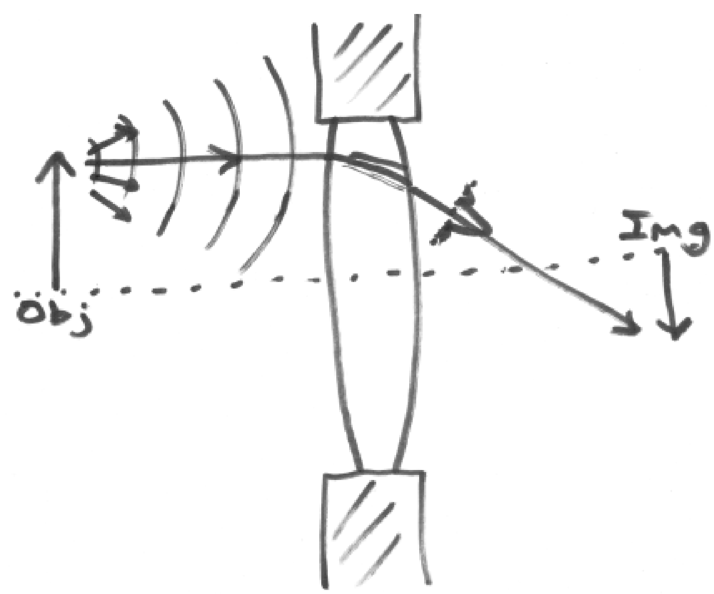
Basically any wave passing through a circular aperture gets **spread out** to some degree. Almost all the energy in the wave (intensity of the light, e.g.) is contained within the central lobe of this pattern, so the first zero, the point where $\sin\theta = 1.22\lambda/D$ is taken to be the (half-)width of the spreading.

When **light** is involved, the wavelength is a few hundred **nanometers** and the aperture (pupil of the eye, diameter of the lens on a camera, microscope, or telescope) is orders of magnitude larger, so the right-hand side there will be incredibly small. That means that (in radians) $\sin\theta \approx \theta$ and this is often simplified to just $\theta = 1.22\lambda/D$ for the amount of angular spreading.

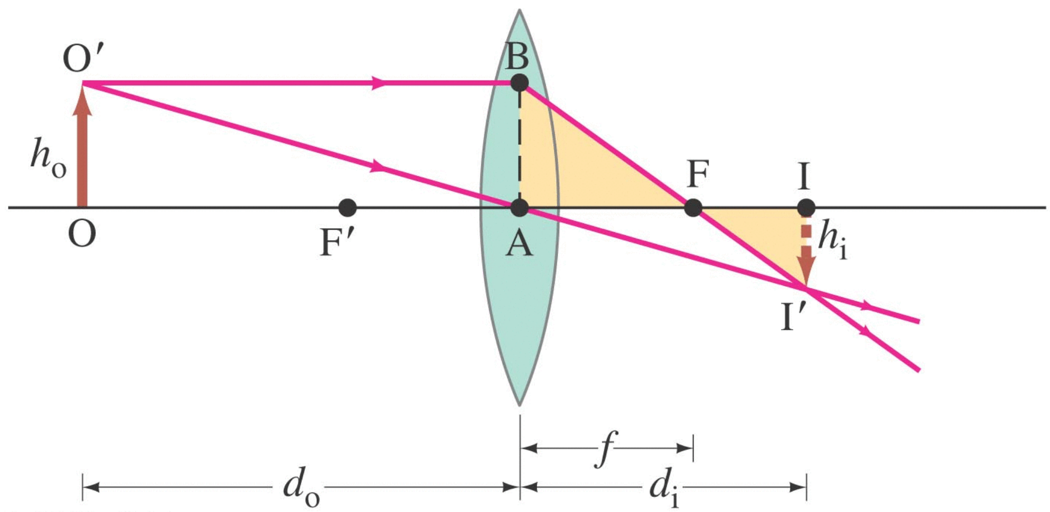
For light around $\lambda = 550\ nm$ (the middle of the visible spectrum) and an aperture represented by the **pupil** of the eye (with a diameter of about $D = 3\ mm$), this central lobe represents an angle of $\theta = (1.22)(550 \times 10^{-9}\ m)/(3 \times 10^{-3}\ m) = 2.24 \times 10^{-4}\ rad$, which is obviously very small.

It's not zero though. Let's look at what happens when light from some object passes through a lens:

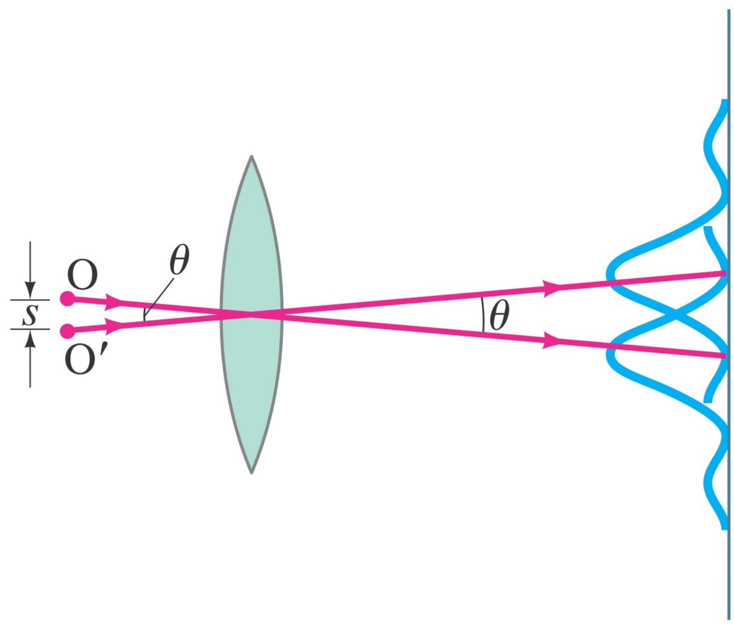
Diffraction and interference are wave phenomena so we need to add this effect when we're doing our ray diagrams. Each ray represents our particle model, following the path a photon takes from the object to the image being formed, but in the wave model we have the light scattering off the object in all directions as waves propagating outward from each point. When the photon (wave) passes through the aperture, it will be spread out due to diffraction. When our 'ray' lands on the screen (film, retina) it won't hit at the exact point in our ray diagrams, but will instead be spread out by an angle $\theta = 1.22\lambda/D$.



Ray diagrams select particular rays we can easily track but it's actually a wavefront passing through the lens and forming a blurred point at the image location.



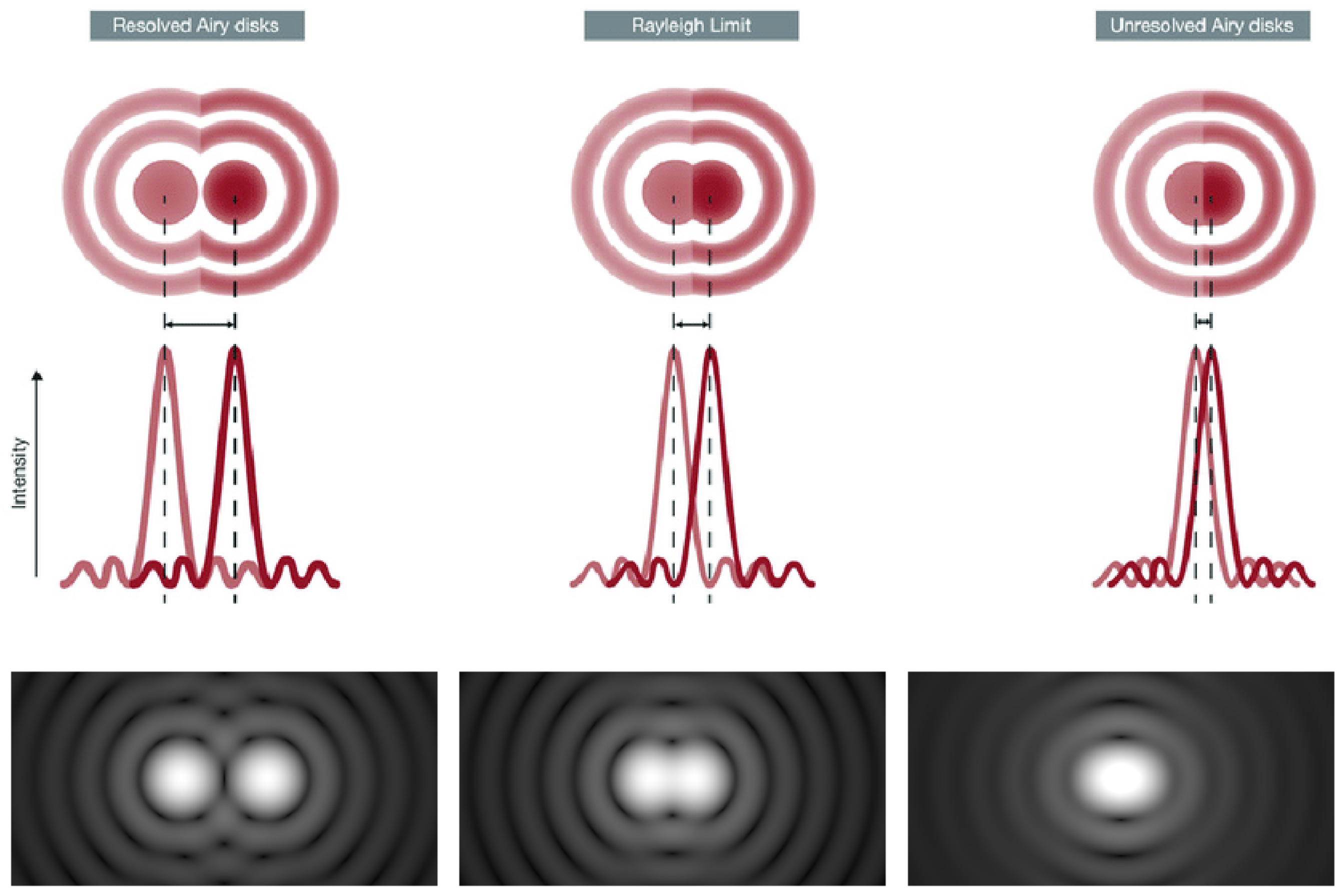
Recall the ray diagrams we did for lenses in chapter 33, in particular where the basic lens equations were derived. In particular, consider the magnification equation: $m = h_i/h_o = -d_i/d_o$



Rearranging terms, we see that $\frac{h_i}{d_i} = -\frac{h_o}{d_o}$.

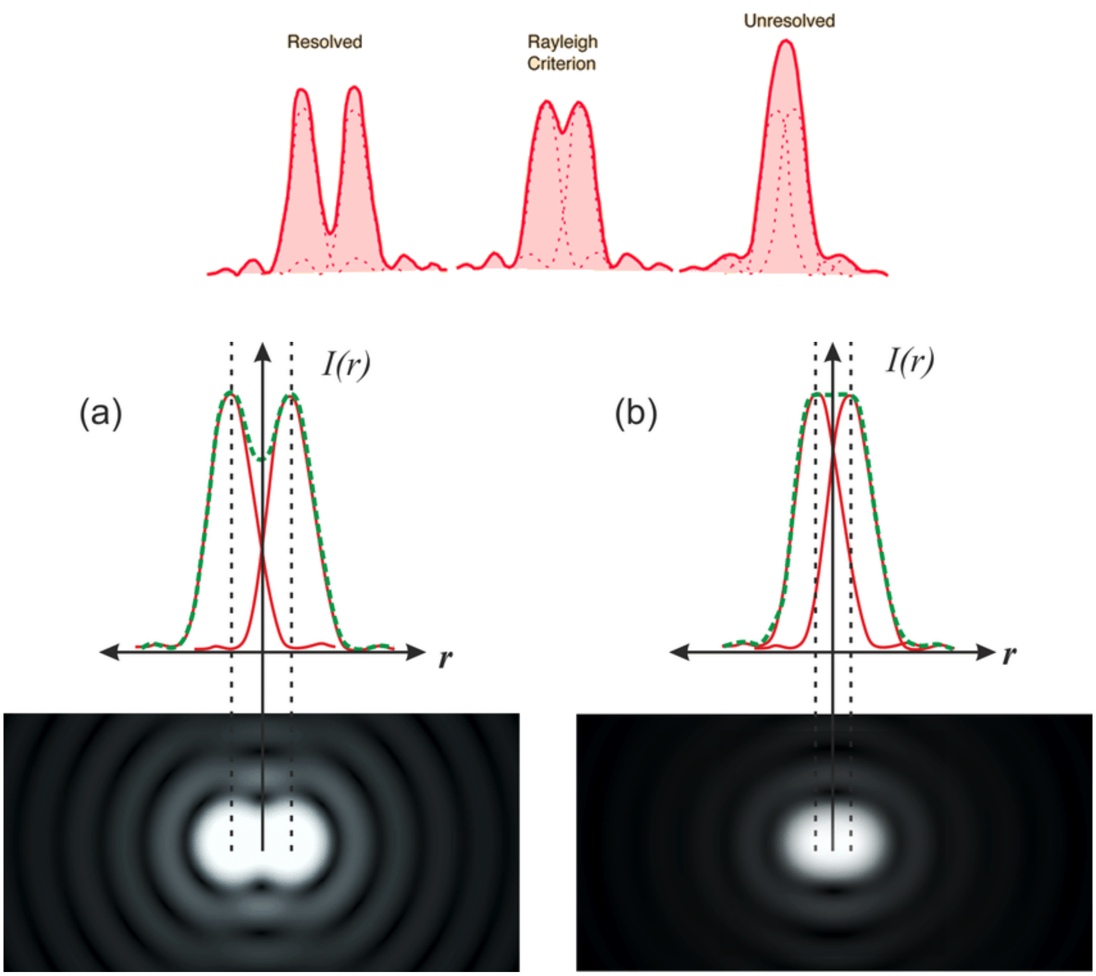
That implies that we have the same angle θ on 'both sides' of the LENS. Put another way, using the vertex of the lens as our origin, **the angular size of the object (relative to the LENS) is the same as the angular size of the image (relative to the LENS)**. (Don't confuse this with the 'apparent magnification' stuff we did with lenses before, where we looked at the angular size of the object and image according to the person (or sensor) viewing them.)

This blurring of the image can clearly be a problem. If the central lobes from two different points on the object overlap enough, we won't be able to tell those points apart.



Rayleigh Criterion

As long as the central peak from point O is 'far enough away' from the central peak from O', we can tell these represent two distinct points. This condition is generally taken to be that the central peak from O is at the first minimum for O' and no closer, and this condition is referred to as the **Rayleigh criterion**. The two points must be separated by at least $sin\theta_{min} = 1.22\lambda/D$ in order for them to be resolved as two unique points on the object. If two points are closer together than that, their blurred images overlap too much and they just look like a single entity. Typically for telescope mirrors, dish-shaped antenna, the pupil of the eye and on and on it's normal for the wavelength to be much smaller than the aperture diameter, so the angle will be small, allowing us to write this as $\boxed{\theta_{min} = 1.22\lambda/D}$ (with θ in radians).



In order for two objects to be seen to be two objects, they must be separated by at least this angle (in radians):

$$sin(\theta_{min}) = 1.22\lambda/D$$

if $\lambda << D$ (the usual case):

$$\theta_{min} = 1.22\lambda/D$$

Hubble Resolving Stars in Andromeda

The 'circular aperture' can take many forms, from a simple hole to a lens or even a circular radio antenna or a circular mirror.

The primary mirror of the orbiting Hubble Space Telescope has a diameter of 2.40 m. Being in orbit, this telescope avoids the degrading effects of atmospheric distortion on its resolution. (a) What is the angle between two just-resolvable point light sources (perhaps two stars)? Assume an average light wavelength of 550 nm. (b) If these two stars are at a distance of 2.5 million light-years, which is the distance of the Andromeda Galaxy, how close together can they be and still be resolved? (A light-year, abbreviated LY, is the distance light travels in 1 year, which is 9.461×10^{15} m).

With the given wavelength and mirror diameter, the Rayleigh criterion for the minimum resolvable angle for the Hubble telescope is $\theta_{min} = 1.22\lambda/D = (1.22)(550\times10^{-9}\ m)/(2.40\ m) = 2.80\times10^{-7}\ rad$.

The distance s between two objects a distance r away and separated by an angle θ is $s = r\theta$ (when the angle is in radians; the usual 'arc-length' formula).

If we're looking $r = 2.5\ million\ LY$ away, the physical separation between the two stars would need to be at least: $s = (2.5 \times 10^6\ LY)(2.80 \times 10^{-7}\ rad) = 0.7\ LY$ in order for Hubble to image them as separate stars.

The average separation distance between stars in a spiral galaxy like Andromeda or the Milky Way is about 5 LY in our part of the Milky Way galaxy which means Hubble can easily resolve stars in Andromeda. We can't see any details, but at least each star is a separate bright spot in the image. In the galactic core, the stars are less than 0.1 LY apart though, so that part of the image will just be a blur.

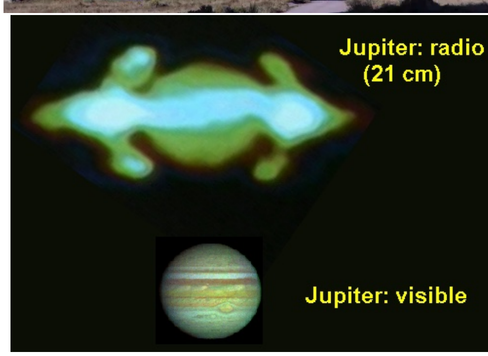
Radio Telescope Observations

Radio astronomy involves detecting radio wave emissions from space: no intelligent communications yet of course but many objects and phenomena in space can emit energy in the radio-wave part of the spectrum. The 'radio-telescopes' are just large circular dish-shaped antennas. The specific wavelengths and frequencies involved cover a very wide range, but one useful radio signal comes from atoms of hydrogen (the most abundent element in the universe) in the form of $\lambda = 21\ cm$ emissions.
The radio telescope shown here is 110 m across.

If it is set to receive $\lambda = 21\ cm$ radio signals and pointed at Jupiter, what will it's angular resolution be?

$\theta_{min} = 1.22\lambda/D = (1.22)(0.21\ m)/(110\ m) = 2.33 \times 10^{-3}\ rad.$

At closest approach, Jupiter is about $625 \times 10^6\ km$ from Earth so how far apart would two points on Jupiter need to be to be resolved? $s = r\theta = (625 \times 10^6\ km)(0.00233\ rad) = 1,450,000\ km$ which is about 10 times the diameter of Jupiter!
Radio astronomy is so blurry that a common technique is to take measurements using an **array** of radio telescopes and then combining the results, which is what is seen here.
For comparison, when Hubble observes Jupiter in optical wavelengths, it's angular resolution of $\theta_{min} = 1.22\lambda/D = (1.22)(550 \times 10^{-9}\ m)/(2.40\ m) = 2.80 \times 10^{-7}\ rad$ means it can resolve features as small as:
$s = r\theta = (625 \times 10^6\ km)(2.80 \times 10^{-7}\ rad) = 175\ km$. Jupiter is about $140,000\ km$ across, so that's about 800 unique samples ('pixels') across the diameter.



Hubble as a spy telescope

If we turn the Hubble towards Earth, what's the smallest feature we can resolve?

The Rayleigh resolution limit for Hubble was found to be $\theta = 2.8 \times 10^{-7}\ rad$. How close would two objects (or two points on an object) be to create such an angle?

$\theta = s/r$ where s is the size of the object, or the distance between the two points of interest. r is the distance from the telescope to the object. Hubble orbits at about 560 km above the Earth, so $s = r\theta = (560,000\ m)(2.8 \times 10^{-7}) = 0.157\ m$ or about 16 cm.

Any two points that close or closer can't be resolved. This effect appears as objects being 'blurred' by a filter of this size. As an example, below is a picture of a car and the same picture after being blurred this way.

$\boxed{\textbf{TV screen pixel resolution}}$

How many pixels on a screen are 'enough'? The higher the pixel density, the closer each pixel is to it's neighboring pixel, so the smaller the angle between them (from the vantage point of the person looking at the screen). Light from the screen has to get through the iris of our eyes, which is essentially a small circular hole with a diameter of about 3 mm. Eventually, neighboring pixels will get so close together that the diffraction blurring caused by the size of this aperture will no longer allow us to perceive them as separate dots - they'll blur together and we'll just see a continuous picture.

Rayleigh limit for the eye

$\sin\theta = 1.22\lambda/D$ where the diameter of the 'hole' (the iris) is about $\boxed{3\ mm}$ and we'll pick a wavelength in the middle of the visible spectrum of $\lambda = 550\ nm$. $\lambda << D$ here, so we'll use the small angle approximation: $\theta = 1.22\lambda/D = (1.22)(550 \times 10^{-9}\ m)/(3 \times 10^{-3}\ m) = 2.2 \times 10^{-4}\ rad$.

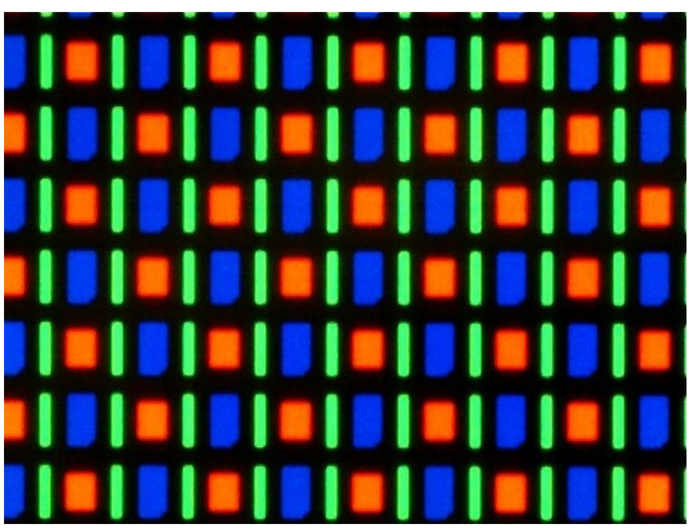
That's the smallest separation angle the eye can resolve.

Pixel separation

Suppose we have a fairly large HDTV (1920x1080) resolution screen that's 1.2 m wide. That means pixels are separated by a distance of $\Delta x = (1.2\ m)/(1920) = 6.25 \times 10^{-4}\ m$ or about 0.6 mm.

If we look at this TV from **3 meters away**, this separation distance represents an angle of: $\theta = s/r = (6.25 \times 10^{-4}\ m)/(3\ m) = 2.1 \times 10^{-4}\ rad$ which is right at the eye's limit. We might just barely be able to detect there are pixels present, instead of a continuous picture.

NOTE: If we look closely enough at each pixel, we see that it's not a single point though. Instead, it's formed from 3 or 4 smaller elements, each one responsible for a single color (red, green, blue, for example), so the separation between elements is slightly smaller than what we computed above, making them even harder to resolve.



Tablet/phone resolution

How many pixels on a phone or tablet screen are enough that we can't see the individual pixels making up the image?

The closer you bring the screen to your eye, the larger the angle between adjacent pixels so the easier it should be to detect their presence, but recall that the 'near-point' is the closest you can focus on something. Any closer and the image forms off the retina and is out of focus, creating it's own form of blurring.

So suppose we hold the screen 25 cm from our eye. What pixel density do we need to not be able to see the individual pixels?

The Rayleigh limit for the eye we found earlier to be 2.2×10^{-4} rad. At a distance of 25 cm that represents a physical separation between pixels of $s = r\theta = (0.25\ m)(2.2 \times 10^{-4}\ rad) = 5.5 \times 10^{-5}\ m$.

How many pixels/meter does this represent? We just found the distance between each pixel, so the pixel density would be the inverse of that or $18,200$ $pixels/meter$ or about 182 $pixels/cm$ or 460 $pixels/inch$.

A typical smartphone screen these days is around 4.5 inches by 2.5 inches which would mean about 2070 pixels by 1150 pixels - pretty close to the 1080p HDTV standard of 1920x1080 pixels.

Typical tablets are larger, but are also held farther away from your eye, so the angle between pixels remains about the same but if you bring the tablet closer to your eye, the pixels can often be clearly seen.

Hubble : Planetary Level of Detail

What level of detail can the Hubble telescope make out when it is pointed at a planet like Mars, Jupiter, or Saturn?

Here, we will compare the Rayleigh angle (the smallest angular feature that can be resolved) to the angular size of the planet (as observed from Earth at its nearest approach). That ratio basically provides the number of unique 'pixels' of information are in the picture. Any higher resolution will provide no additional detail, due to the diffraction introduced by the circular aperture of the telescope.

The Rayleigh resolution limit for Hubble is approximately 2.8×10^{-7} rad. To make the comparisons clearer, all the planet's angular sizes are given using the same 10^{-7} exponent.

This table summarizes the level of detail that images from the Hubble space telescope can resolve on some of the other planets in our solar system, and the Earth's moon.

Given an object's diameter and nearest distance from the Earth, we can determine it's maximum angular size (via $s = r\theta$ basically). If we divide that by Hubble's diffraction-limited angular resolution, we have the number of unique 'samples' (pixels) in the image. For example in the case of Mars, we can only see about 306x306 unique pixels of information. Any higher resolution image is just interpolating data and not revealing any additional detail. The last column gives the physical size of the smallest feature we can resolve on that distant object.

Planetary Level of Detail from Hubble Observations

Planet	Diameter (miles)	Nearest Approach (miles)	Angular Size (radians)	Pixel Equivalent	Smallest Feature
Mars	4200	49 million	800×10^{-7}	306	14 miles
Jupiter	86400	390 million	2200×10^{-7}	800	109 miles
Saturn	72400	800 million	900×10^{-7}	320	224 miles
Moon	2160	239,000	90000×10^{-7}	32,300	0.07 miles 350 feet

NOTE: the resolution of the human eye is limited by not only the size of the pupil but other effects such as the density of cells on the retina. The overall resolution is roughly 5×10^{-4} rad or 5000×10^{-7} rad, which is larger than the angular size of any of the planets, so all we can see with the naked eye is a dot without any features. (In the case of the Moon, we can see features as small as 120 $miles$, so the pixel equivalent is about 20.