Physics 2233 : Chapter 35 Examples : Diffraction

Single-slit diffraction

- complete destructive interference occurs at $D \sin\theta = m\lambda$ for $m = \pm1, \pm2, ...$
- constructive interference occurs at $\theta = 0$ and somewhere between each destructive line above; exact angles are where $\tan(\beta/2) = \beta/2$ where $\beta = \frac{2\pi D \sin\theta}{\lambda}$ which is a nonlinear equation in θ without any simple general solution

Intensity: $I = I_o snc^2(\beta/2)$ where snc is the 'sinc' function: $snc(x) = \frac{\sin(x)}{x}$ and β is defined above

Intensity at secondary maxima (approximate) : $I \approx \frac{I_o}{(m+\frac{1}{2})^2\pi^2}$
(If the main peak has an intensity of I_o, the next ones will have intensities of $0.4053 I_o$, $0.0450 I_o$, $0.0162 I_o$ and so on, so only the first secondary peak has a significant amplitude.)

Resolution

Light from each point of an 'object' is spread out via diffraction as it passes through an aperture. The Rayleigh criterion claims we can resolve two things as being separate when the peak intensity from one is no closer than the first diffraction minimum of the other.

- **Thin Slit case** : the central 'blob' (between the $m = -1$ and the $m = +1$ destructive interference points) has an angular half-width of θ where $\sin\theta = \lambda/D$ (the spot then subtends angles between $-\theta$ and $+\theta$). If $\lambda << D$ the angle will be small so $\theta \approx \lambda/D$
- **Circular Aperture** (camera lens, iris of the eye, etc), of diameter D, the angular half-width is approximately $\sin\theta \approx 1.22\lambda/D$ (the spot is now circular with intensity dropping to zero at points where θ reaches this value). If $\lambda << D$ the angle will be small so $\theta \approx 1.22\lambda/D$

Diffraction Gratings

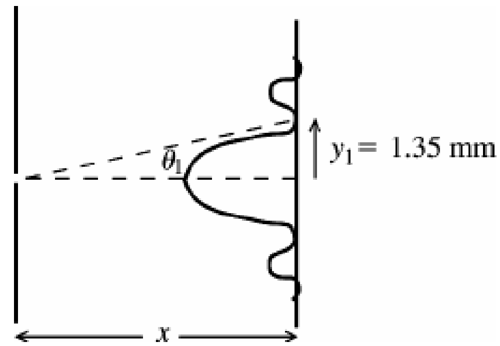
Formed from a large number of equally spaced parallel slits (or reflectors) with d being the distance from one slit to the next.

High intensity at: $\sin\theta = m\frac{\lambda}{d}$ for $m = 0, \pm1, \pm2, ...$

Between these points, intensity is reduced in relation to the number of slits or reflectors. A large number, such as in a typical diffraction grating, results in nearly zero intensity everywhere except at the above angles.

Example 10 Monochromatic light from a distant source is incident on a slit 0.750 mm wide. On a screen 2.00 m away, the distance from the central maximum of the diffraction pattern to the first minimum is measured to be 1.35 mm. Calculate the wavelength of the light.

The text describes the location of the first dark fringe in the pattern. This feature is at an angle of $\tan\theta = \frac{y_1}{x} = \frac{1.35\times10^{-3}\ m}{2.00\ m}$ or $\theta = 6.75\times10^{-4}\ rad$. The dark fringes are located at a series of angles where $\sin\theta = \frac{m\lambda}{D}$ for $m = \pm1, \pm2, ...$ where D is the width of the slit. We just calculated the angle to the first dark fringe ($m = 1$) so for that feature, we have the angle, the order m, and the width of the slit, so can solve for the wavelength: $\sin 6.75\times10^{-4} rad = \frac{(1)(\lambda)}{0.750\times10^{-3}\ m}$ or $\lambda = 5.06\times10^{-7}\ m$ or $506\times10^{-9}\ m$ which is 506 nm.



Example 11 Light of wavelength 585 nm falls on a slit 0.0666 mm wide. (a) On a very large distant screen, how many *totally* dark fringes (indicating complete cancellation) will there be, including both sides of the central bright spot? Solve this problem without calculating all the angles. (Hint: What is the largest that $\sin\theta$ can be? What does this tell you is the largest that m can be? (b) At what angle will the dark fringe that is most distant from the central bright fringe occur?

(a) The dark fringes are located at angles θ that satisfy $\sin\theta = \frac{m\lambda}{D}$ for $m = \pm1, \pm2, ...$ The largest value that $|sin|$ can have is 1.00, so we see that eventually we'll reach a value of m that will exceed this, and only m values smaller than that will provide real solutions.

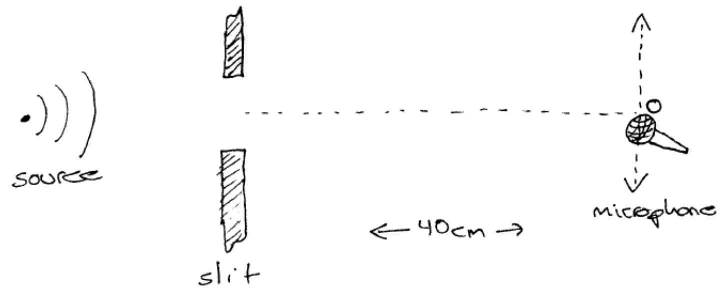
Looking for this extreme case: $1 = \frac{m\lambda}{D}$ or $m = D/\lambda = (0.0666\times10^{-3})/(585\times10^{-9}) = 113.8$. Well m is an integer, so 113 is the highest it can be. Once we reach 114, the result will something larger than 1.

So we have 113 dark fringes on one side (leading to the sine having a value of just short of +1), and another 113 dark fringes on the other side (leading to the sine having a value of just short of -1). In all, we should see 226 dark fringes. (The intensity of the bright fringes drops off as the square of the order m, so this last one will be down a factor of around $13,000$ compared to the central bright line, so realistically it would be difficult to see those outer bright and dark fringes...)

(b) For the very last fringe, $m = \pm113$, so $\sin\theta = \frac{m\lambda}{D}$ becomes $\sin\theta = \frac{(\pm113)(585\times10^{-9})}{0.0666\times10^{-3}} = \pm0.99257$ or $\theta = \pm83.0^o$.

Example 12 Diffraction occurs for all types of waves, including sound waves. High-frequency sound from a distant source with wavelength 9.00 cm passes through a narrow slit 12.0 cm wide. A microphone is placed 40.0 cm directly in front of the center of the slit, corresponding to point O in Fig. 36.5a. The microphone is then moved in a direction perpendicular to the line from the center of the slit to point O. At what distances from O will the intensity detected by the microphone be zero? (Pretend this is 'far field' even though it isn't, but don't assume small angles.)

The description here leaves something to be desired, but basically if we have a vertical slit and we are facing it, we are moving the mic left and right keeping it 40 cm from the plane of the surface into which the slit was cut. (Think of this being a door that is just slightly open, leaving a 12 cm wide 'slit', and we are stepping back and forth sideways, holding the microphone always 40 cm from the opening.)



The minima will be located where $\sin\theta = \frac{m\lambda}{D}$, for $m = \pm1, \pm2, ...$ Here $\lambda = 9.0\ cm$ and the slit width (the width of the slit) is 12.0 cm.

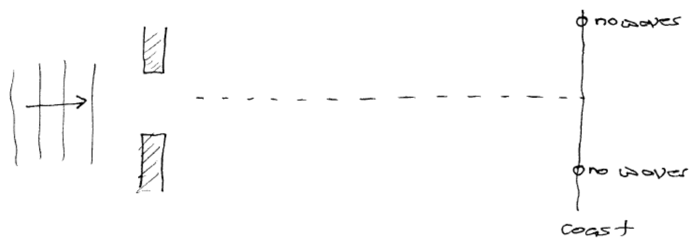
For $m = 1$ this becomes: $\sin\theta = \frac{(1)(9.0\ cm)}{12\ cm} = 0.75$ from which $\theta = 48.6^o$. The distance from the central maximum to this first minimum then will be $\tan\theta = y_1/x$ or $y_1 = \tan(48.6^o)(40.0\ cm) = 45.4\ cm$.

For $m = -1$ this becomes: $\sin\theta = \frac{(-1)(9.0\ cm)}{12\ cm} = -0.75$ from which $\theta = -48.6^o$. The distance from the central maximum to this first minimum then will be $\tan\theta = y_1/x$ or $y_1 = \tan(-48.6^o)(40.0\ cm) = -45.4\ cm$.

Note that in general, the right hand side in $\sin\theta = \frac{m\lambda}{D}$ becomes $(0.75)(m)$ when we plug in the specific values we have here for the wavelength and slit size. The absolute value of this side cannot exceed 1 (if it does, we end up with having to find some angle for which the sine function is larger than 1 and no such (real) solution exists.) So the solutions we found for $m = \pm1$ are the only angles for which the sound is minimized.

Example 14 A series of parallel linear water wave fronts are traveling directly toward shore at 15.0 cm/s on an otherwise placid lake. A long concrete barrier that runs parallel to the shore at a distance of 3.20 m away has a hole in it. You count the wave crests and observe that 75.0 of them pass by each minute, and you also observe that no waves reach the shore at ± 61.3 cm from the point directly opposite the hole, but waves do reach the shore everywhere within this distance. (a) How wide is the hole in the barrier? (b) At what other angles do you find no waves hitting the shore?

We can model the hole in the concrete barrier as a single slit that will produce a single-slit diffraction pattern of the water waves on the shore. For single-slit diffraction, the minima (angles at which destructive interference occurs) are given by: $\sin\theta = \frac{m\lambda}{D}$, for $m = \pm 1, \pm 2, ...$



We need the wavelength in this equation, but we have the wave speed and the frequency, so we can calculate the wavelength from $v = \lambda f$, or $\lambda = v/f$.

We recorded that 75 waves passed in one minute (60 seconds) so the frequency is $(75\ cycles)/(60\ seconds) = 1.25\ Hz$.

The wavelength, then is; $\lambda = v/f = (0.15\ m/s)/(1.25\ s^{-1}) = 0.12\ m$. (These are pretty small, so these 'waves' are more like ripples, not big ocean swells.)

Based on the text, we were basically given the location of the first minimum. We can convert this to an angle using $\tan\theta = (0.613\ m)/(3.20\ m) = 0.1916$ from which $\theta = 10.84^o$.

The location of the minima is related to the size of the 'slit': $\sin\theta = \frac{m\lambda}{D}$, for $m = \pm 1, \pm 2, ...$ Here, for $m = 1$ we have $\sin(10.84^o) = \frac{(1)(0.12\ m)}{a}$ which provides $D = 0.638\ m$. The hole is apparently 63.8 cm wide.

Now we have all the information we need to calculate the angles of all the other minima: $\sin\theta = \frac{m\lambda}{D}$, for $m = \pm 1, \pm 2, ...$ so $\sin\theta = m\frac{0.120\ m}{0.638\ m} = (m)(0.188)$. We already have the value for $m = 1$ (10.84^o). Since the sine function is odd, we basically just have to calculate this for positive values of m and then the corresponding angle for negative m will just be the negative of the same angle we got for the positive value of m: $m = \pm 1$ produces $\theta = \pm 10.84^o$
$m = \pm 2$ produces $\theta = \pm 22.1^o$
$m = \pm 3$ produces $\theta = \pm 34.3^o$
$m = \pm 4$ produces $\theta = \pm 48.8^o$
$m = \pm 5$ produces $\theta = \pm 70.1^o$
Any larger values of m give a right-hand-side whose magnitude exceeds 1, so no further solutions exist. (And note that these were large angles, so we couldn't use the small-angle approximation of θ_m being about equal to $m\lambda/a$.)

Example 16 Sound of frequency 1250 Hz leaves a room through a 1.00 m wide doorway (see 36.5). At which angles relative to the center-line perpendicular to the doorway will someone outside the room hear no sound? Use 344 m/s for the speed of sound in air, and assume that the source and listener are both far enough from the doorway for Fraunhofer diffraction to apply. You can ignore effects of reflections.

Here we have **single-slit diffraction**, even if the 'slit' is huge compared to the ones we've been dealing with for light waves. Diffraction will produce minima at angles for which $\sin\theta = m\lambda/a$. Here the slit width D is the width of the door ($D = 1.00\ m$). We need the wavelength to apply this equation. The wavelength and frequency are connected to the wave speed: $v = \lambda f$ and we have the speed (344 m/s) and the frequency (1250 Hz) so can compute the wavelength: $\lambda = v/f = (344\ m/s)/(1250\ Hz) = 0.2752\ m$, so the doorway is about three wavelengths wide.

So now: $\sin\theta = m\lambda/a = m(0.2752)/(1.00) = 0.2752m$ will tell us the angles where the sound will be gone. As with the last couple of problems, we see that the number of solutions is limited, since the right-hand side cannot be allowed to exceed a magnitude of 1. Running through plus and minus values of m until we can go no further:

$m = \pm 1$ provides $\theta = \pm 16.0^o$
$m = \pm 2$ provides $\theta = \pm 33.4^o$
$m = \pm 3$ provides $\theta = \pm 55.6^o$
and no further solutions are possible.

(So if we rig up a speaker to be playing a pure tone, we should be able to stand outside the door and move back and forth and hear the sound vanish at certain angles. The same thing happens with sound passing through the spaces between buildings, cars, and whatever. This may be one reason why emergency sirens put out sounds with varying frequencies, running from low to high to low and so on, although I suspect it was done just to make them less ignorable than to deal with diffraction effects...)

Example 20 A laser beam of wavelength $\lambda = 632.8\ nm$ shines at normal incidence on the reflective side of a compact disc. The tracks of tiny pits in which information is coded onto the CD are $1.60\mu m$ apart. For what angles of reflection (measured from the normal) will the intensity of light be maximum?

Here we essentially have a **reflection diffraction grating** with little 'mirrors' located d apart. The intensity maximum will occur where we have constructive interference from the light reflected from all the little mirrors. For a reflection interferometer, we have maxima where $d \sin\theta = m\lambda$ ($m = 0, \pm 1, \pm 2, ...$) (where d is the separation between the centers of successive 'mirrors) or since we are interested in the angles: $\sin\theta = m\lambda/d$. Here $d = 1.60 \times 10^{-6}$ (meters) and $\lambda = 632.8 \times 10^{-9}$ (meters) so $\sin\theta = m(632.8 \times 10^{-9})/(1.60 \times 10^{-6}) = (0.396)(m)$.

Technically, $m = 0$ gives the solution $\theta = 0$ but when we hold the CD at this angle, our head is blocking the light source so the answer key didn't include that angle as a solution. I suppose if you were wearing one of those little head-mounted flashlights...

For $m = \pm 1$, $\sin\theta = \pm(1)(0.396)$ from which $\theta = 23.3^o$.
For $m = \pm 2$, $\sin\theta = \pm(2)(0.396)$ from which $\theta = 52.3^o$.
No further solutions are possible since the right hand side exceeds 1 for (integer) $|m| > 2$.

If we generalize here, holding the CD at some angle θ, there will be SOME frequency which will be constructively interfering. Rearranging the equation to solve for λ: $\lambda = \frac{d}{m} \sin\theta$. No matter what the angle is, for a given value of d, we can find some m which produces a wavelength in the visible range. So THAT wavelength will be constructively interfering, and the CD will appear primarily that color. As the angle changes, a different wavelength will constructively interfere. If I hold the CD fairly close (even at arm's length), the angle my eye makes with various spots on the CD is changing, so again looking at different parts we'll see different colors accentuated and attenuated. Many fish, animals, and birds have tiny linear structures on their scales/feathers/etc that have the same effect.

Example 20 (continued)

The previous example claimed that the spacing between the shiny tracks on a CD is $1.6\mu m$, which is $1600\ nm$.

How did they get this?

A CD encodes information on a series of concentric tracks of the same width. An old fashioned audio CD player turns at an average angular speed of around 300 revolutions/minute and will play for 84 minutes. Each revolution represents a laser reading the contents of one complete track, so apparently it's reading (on average) 300 tracks per minute and can do so for 84 minutes. That means there must be $(300\ tracks/min)(84\ min) = 25,200$ concentric rings making up the CD. The distance between the inner and outer tracks is $3.7\ cm$. Each track is some width (the separation distance 'd' we need in our diffraction grating equations, so apparently $25200d = 3.7\ cm$ or $d = 1.47 \times 10^{-4}\ cm$.

Converting: $d = (1.17 \times 10^{-4}\ cm) \times \frac{1\ m}{100\ cm} \times \frac{1 \times 10^9\ nm}{1\ m} = 1470\ nm$.

That's a bit off from the $1600\ nm$ the previous problem used, and I checked around and the 'standard' track spacing on CD's is 1.6 *micron* (i.e. the same $1.6\mu m$ that the previous problem used).

The difference here may be due to the assumption that the average (angular) speed of the CD is $300\ rev/min$. It's actually not constant: the motor turns more slowly when the laser is reading the outer tracks (remember, the tangential speed $v = r\omega$ so if it kept ω constant, the tracks on the outer edge would be flying under the laser much faster than the tracks on the inner edge).

We ended up being off by only about 10 percent, so I'm guessing it's the 300 revolutions/min number that's the culprit here...

Example 40 : Detecting extrasolar planets

The Earth orbits the Sun roughly in a circle with radius 150 million kilometers. The nearest star (alpha Centauri) is about 4 light-years away. Would the Hubble telescope be able to 'see' a planet in the same orbit around that star?

When we point the telescope at alpha Centauri, at what angle would the planet be located? To be able to see it, that angle will need to be the Rayleigh angle (or larger).

What is the Rayleigh resolution limit for Hubble?

Each point of light passing through an aperature gets spread out into a particular intensity pattern we derived. The Rayleigh criterion says that if the central spot of this intensity pattern from one object is located at the first minimum in the intensity pattern from a second object (or any further apart), we can be sure there are two objects. If the angle is smaller, the two blobs overlap too much and we can't tell there are actually two.

The first minimum occurs at: $\sin\theta = 1.22\lambda/D$ which we can use to find the minimum angle the two 'things' need to be separated by in order to have confidence there actually are two things present.

Visible light varies over a few hundred nanometers, but let's pick a wavelength in the middle of that spectrum, say $\lambda = 550\ nm$, and see how much such light is spread out.

The open end of the Hubble telescope has a diameter of 2.4 m, so our Rayleigh angle will be: $\sin\theta = (1.22)(550 \times 10^{-9}\ m)/(2.4\ m)$. The right-hand side here is so small we can approximate $\sin\theta \approx \theta$ so $\theta = (1.22)(550 \times 10^{-9}\ m)/(2.4\ m) = 2.8 \times 10^{-7}\ rad$.

Objects that are that close or farther apart can be resolved as two objects; otherwise they're just a single blob...

Compare with the angle between alpha Centauri and this hypothetical planet.

If s is the separation distance between two objects we want to resolve, then we can use $s = r\theta$ if the objects are far enough away that the angle is small. Here, $r = 4\ lightyears$ which we'll need to convert to standard units first. One light year (distance light would travel in a year) is about $9.5 \times 10^{15}\ m$, so alpha Centauri (and it's planets, if any) at 4 light-years from us represents $r = 4 \times 9.5 \times 10^{15}\ m = 38 \times 10^{15}\ m$.

Depending on the orientation of the orbit of this hypothetical planet, the distance from the star to the planet would vary but the maximum separation would be the orbit radius. If this planet is the same distance from its sun as ours is, then $s = 150 \times 10^9\ m$. Thus the angle between the star and this planet would be $s = r\theta\ \rightarrow\ \theta = s/r = (150 \times 10^9)/(38 \times 10^{15}) = 3.9 \times 10^{-6}\ rad$. By shifting the exponent, we can write this as $\theta = 39 \times 10^{-7}\ rad$ to make it easier to compare with the Rayleigh resolution limit for the Hubble telescope, which was $2.8 \times 10^{-7}\ rad$. The star and the planet are separated by 14 times the resolution of the telescope, so can easily be resolved. We wouldn't be able to make out any details (see the next problem) but at least we would be able to tell the planet is there. At this point, thousands of extra-solar planets have been identified via this method and others.

Example 42 : Resolving Features on stars and extrasolar planets

Here we extend the previous problem and determine if we can 'see' any details about this hypothetical planet, or if we can see details on the surface of alpha Centauri, such as the sunspots that form on our sun.

We found in the previous problem that the Rayleigh resolution limit for Hubble was $2.8 \times 10^{-7}\ rad$. Everything we see is blurred by that amount, so anything we want to see needs to subtend a larger angle.

What angle would the entire star take up when we view it? Alpha Centauri is sun-like, so let's assume it has the same diameter of the Sun or about $14 \times 10^8\ m$. That means it subtends an angle of $\theta = s/r = (14 \times 10^8\ m)/(38 \times 10^{15}\ m) = 0.37 \times 10^{-7}\ rad$.

Compare that to the Rayleigh resolution limit of Hubble, which was $2.8 \times 10^{-7}\ rad$. We see that there's no way we can make out any details this small. The entire star is just a blob, so there's no hope of making out any smaller details like sun spots and any features on the hypothetical planet would be orders of magnitude smaller, so no chance of seeing any details there either.

Example 44: Hubble spy telescope

If we turn the Hubble towards Earth, what's the smallest feature we can resolve?

The Rayleigh resolution limit for Hubble was found to be $\theta = 2.8 \times 10^{-7}\ rad$. How close would two objects (or two points on an object) be to create such an angle?

$\theta = s/r$ where s is the size of the object, or the distance between the two points of interest. r is the distance from the telescope to the object. Hubble orbits at about 560 km above the Earth, so $s = r\theta = (560,000\ m)(2.8 \times 10^{-7}) = 0.157\ m$ or about 16 cm.

Any two points that close or closer can't be resolved. This effect appears as objects being 'blurred' by a filter of this size. As an example, below is a picture of a car and the same picture after being blurred this way.

Example 46: Spy Telescope

Suppose we want a spy satellite to be able to see things down to a 1 cm resolution, which would be about enough to read license plate numbers or possibly identify faces. How big does the aperture on this satellite have to be to allow for this? We're not considering how large (huge) the magnification would need to be, here we're focusing on how blurry the image will be due to diffraction.

The lowest 'safe' orbit before atmospheric drag gets too high is around 200 km above the surface, so a 1 cm separation this far away represents an angle of:

$\theta = s/r = (0.01\ m)/(200,000\ m) = 5 \times 10^{-8}\ rad$. We can write that as $0.5 \times 10^{-7}\ rad$ to compare with the Hubble's diffraction limit of 2.8×10^{-7} and clearly we'll need a telescope with a much larger aperture than Hubble has.

The diffraction-created angular resolution limit is $\theta = 1.22\lambda/D$ so $D = 1.22\lambda/\theta$ would give us the necessary diameter to be able to resolve something of a given size. We'll pick $\lambda = 550\ nm$ again (middle of the visible spectrum) to make an estimate. Doing so:

$D = (1.22)(550 \times 10^{-9}\ m)/(5 \times 10^{-8}\ rad) = 13.4\ m$ or about 45 feet across.

Without some trickery, I don't see how something that large could be put in orbit (it certainly won't fit in the space shuttle cargo bay), so it seems unlikely that a spy satellite would be able to read a license plate, let alone be able to identify a particular individual from space...

Example 48: TV screen pixel resolution

How many pixels on a screen are 'enough'? The higher the pixel density, the closer each pixel is to it's neighboring pixel, so the smaller the angle between them (from the vantage point of the person looking at the screen). Light from the screen has to get through the iris of our eyes, which is essentially a small circular hole with a diameter of about 3 mm. Eventually, neighboring pixels will get so close together that the diffraction blurring caused by the size of this aperture will no longer allow us to perceive them as separate dots - they'll blur together and we'll just see a continuous picture.

Rayleigh limit for the eye

$\sin\theta = 1.22\lambda/D$ where the diameter of the 'hole' (the iris) is about 3 mm and we'll pick a wavelength in the middle of the visible spectrum of $\lambda = 550\ nm$. $\lambda << D$ here, so we'll use the small angle approximation: $\theta = 1.22\lambda/D = (1.22)(550\times10^{-9}\ m)/(3\times10^{-3}\ m) = 2.2\times10^{-4}\ rad$.

That's the smallest separation angle the eye can resolve.

Pixel separation

Suppose we have a fairly large HDTV (1920x1080) resolution screen that's 1.2 m across. That means pixels are separated by a distance of $\Delta x = (1.2\ m)/(1920) = 6.25\times10^{-4}\ m$ or about 0.6 mm.

If we look at this TV from 3 meters away, this separation distance from one pixel to the next represents an angle of: $\theta = s/r = (6.25\times10^{-4}\ m)/(3\ m) = 2.1\times10^{-4}\ rad$ which is right at the eye's limit. We might just barely be able to detect there are pixels present, instead of a continuous picture.

If we look closely enough at each pixel, we see that it's not a single point though. Instead, it's formed from 3 or 4 smaller elements, each one responsible for a single color (red, green, blue, for example), so the separation between elements is slightly smaller than what we computed above, making them even harder to resolve.

Example 50: Tablet/phone resolution

How many pixels on a phone or tablet screen are enough that we can't see the individual pixels making up the image?

The closer you bring the screen to your eye, the larger the angle between adjacent pixels so the easier it should be to detect their presence, but recall that the 'near-point' is the closest you can focus on something. Any closer and the image forms off the retina and is out of focus, creating it's own form of blurring.

So suppose we hold the screen 25 cm from our eye. What pixel density do we need to not be able to see the individual pixels?

The Rayleigh limit for the eye we found earlier to be 2.2×10^{-4} rad. At a distance of 25 cm that represents a physical separation between pixels of $s = r\theta = (0.25\ m)(2.2 \times 10^{-4}\ rad) = 5.5 \times 10^{-5}\ m$.

How many pixels/meter does this represent? We just found the distance between each pixel, so the pixel density would be the inverse of that or $18,200$ $pixels/meter$ or about 182 $pixels/cm$ or 460 $pixels/inch$.

A typical smartphone screen these days is around 4.5 inches by 2.5 inches which would mean about 2070 pixels by 1150 pixels - pretty close to the 1080p HDTV standard of 1920x1080 pixels.

Typical tablets are larger, but are also held farther away from your eye, so the angle between pixels remains about the same but if you bring the tablet closer to your eye, the pixels can often be clearly seen.

Example 52: Digital camera resolution

Let's look at the 'mega-pixels' usually quoted in the context of digital cameras.

Light from outside has to pass through the lens on its way to the sensor, so we have diffraction limiting going on again. If the aperture of a professional DSLR camera has a diameter of 4 cm, and the focal length of the lens is 50 cm, how close do the pixels in the sensor need to be to each other? If a typical sensor is a about 3 centimeters by 2 cm in size, how many pixels does this represent?

The Rayleigh limit for an aperture is $\sin\theta = 1.22\lambda/D$ so here we have $D = 4\ cm$ and we'll do the usual thing of just picking a wavelength in the middle of the visual spectrum: $\lambda = 550\ nm$. The RHS of the equation will be tiny, so we can use the approximation that $\sin\theta \approx \theta$ and: $\theta = 1.22\lambda/D = (1.22)(550 \times 10^{-9}\ m)/(0.04\ m) = 1.7 \times 10^{-5}\ rad$

Unless the object is pretty close, the distance from the lens to the sensor will be about the focal length of the lens, or 50 cm so the angular separation we just computed corresponds to a physical separation from $s = r\theta$ of $s = (0.5\ m)(1.7 \times 10^{-5}) \approx 8 \times 10^{-6}\ m$.

This is about $120,000\ pixels/meter$ or $1200\ pixels/cm$.

Our sensor is 3x2 centimeters in size, so multiplying by the pixel density we just found, apparently we need a 3600x2400 array of pixels in the sensor: about 8 to 9 megapixels. At this resolution, we are at the limit of still being (just barely) able to detect the pixel-ness in the image, so high-end digital cameras usually squeeze the pixels a bit closer together yielding 14 to 16 megapixels (or higher) in the sensor.

Relation to phone cameras

The cameras in phones have *really* small apertures (a few millimeters) so the Rayleigh limit is reached much more quickly, but they also have *very* short focal lengths (less than a centimeter). Let's redo the analysis above for this type of camera.

Suppose we have a little phone camera with $D = 1\ mm$ and $f = 5\ mm$. What pixel density do we need now?

Rayleigh limit: $\theta = 1.22\lambda/D = (1.22)(550 \times 10^{-9}\ m)/(1 \times 10^{-3}\ m) = 6.8 \times 10^{-4}\ rad$.

If the sensor is 5 mm away from the lens, that means the pixel spacing at this Rayleigh limit would be: $s = r\theta = (5 \times 10^{-3}\ m)(6.8 \times 10^{-4}) = 3.5 \times 10^{-6}\ m$.

That's how close each pixel is to the next; the pixel density (pixels/meter) would be the inverse of that or roughly around $300,000\ pixels/meter$ or $3000\ pixels/cm$.

Physically, the image sensors in phones are about 10x smaller in each dimension than those in high-end DSLR cameras, so the sensor in the phone is probably around 0.3 cm by 0.2 cm, which turns into 900 by 600 pixels or only around a half a megapixel.

As with the DLSR example, this would be the point where we could still just make out the pixel-ness in the image, so we'd want to do better. If we doubled the pixels in each direction, we'd definitely be under the Rayleigh resolution limit, and we might end up with a 2 megapixel camera. Anything beyond that (in a phone anyway) seems like marketing overkill.