One end of a nylon rope is tied to a stationary support at the top of a vertical mine shaft of depth 80.0 m. The rope is made taut by a box of mineral samples with mass 20.0 kg attached at the lower end. The mass of the rope is 2.00 kg. The geologist at the bottom of the mine signals to his colleague at the top by jerking the rope sideways. (Do not neglect the weight of the rope.).

(a) What is the wave speed at the bottom of the rope? At the middle of the rope? At the top of the rope?

Instead of a single pulse, suppose the geologist wiggles the rope at a frequency of 2 Hz with a (transverse) displacement of 4 cm. (Note: for the remaining parts, ignore the change in tension due to the mass of the rope and just use the tension that you calculated above at the bottom of the rope.)

(b) What will be the wavelength of the waves travelling up the rope?

(c) What will the maximum transverse acceleration be at any point on the rope?

(d) How much power does the geologist need to be exerting?

Note: that's material from chapter 15. Here are some chapter 14 type questions that could be asked about the same scenario:

Suppose the geologist pulls the crate off to the side by 20 cm and lets the crate swing back and forth on the rope.

- What would the period of this motion be?
- How fast is the crate moving as it swings past the lowest point on it's motion? (Use the methods of this chapter; not the PH2213 way of doing this, although you can use that to check this part via CoE.)

Hints:

- We don't know the size or geometry of the crate, so are forced to use the point-mass approximation.
- The offset is very small compared to the length of the rope, so we're clearly in the 'small angle' regime, meaning that the 'equation of motion' giving the angle as a function of time would be $\theta(t) = \theta_o \cos(\omega t)$. What does that tell you about the maximum angular speed, which the crate will reach as it passes through the lowest point, and how would you relate that to the linear speed at that point?

(a) What is the wave speed at the bottom of the rope? At the middle of the rope? At the top of the rope?

The wave speed on the rope is given by $v = \sqrt{F/\mu}$, so depends on the tension in the rope at various points. Since we want to include the mass of the rope in our calculations, the tension will be changing. The mass per unit length μ is fixed though and is $\mu = M/L = (2.00\ kg)/(80\ m) = 0.0250\ kg/m$.

At the very bottom of the rope, looking at $\Sigma F = 0$ we have the weight of the box of samples pulling down and the tension in the rope at that point pulling up, so the tension must have a value of $(20\ kg)(9.8\ m/s^2) = 196\ N$. The velocity of a transverse signal down at the bottom of the rope, then, is $v_{bottom} = \sqrt{F/\mu} = \sqrt{(196)/(0.025)} = \mathbf{88.54}\ m/s$.

At the middle of the rope, we have the mass of the box ($20\ kg$) plus half the mass of the rope (or an additional $1\ kg$) pulling down, and the tension in the rope at that point pulling up. So the force down at this point (and therefore the tension in the rope at that point) is $(21\ kg)(9.8\ m/s^2) = 205.8\ N$. The velocity of a transverse signal at the middle of the rope, then, is $v_{middle} = \sqrt{F/\mu} = \sqrt{(205.8)/(0.025)} = \mathbf{90.73}\ m/s$.

At the top of the rope, we have the mass of the box ($20\ kg$) plus the full mass of the rope (or an additional $2\ kg$) pulling down, and the tension in the rope at that point pulling up. So the force down at this point (and therefore the tension in the rope at that point) is $(22\ kg)(9.8\ m/s^2) = 215.6\ N$. The velocity of a transverse signal up at the top of the rope, then, is $v_{top} = \sqrt{F/\mu} = \sqrt{(215.6)/(0.025)} = \mathbf{92.87}\ m/s$.

(Note that the wave travels faster and faster as it moves up the rope, being about 5 percent faster at the top, compared to the bottom.)

(b) What will be the wavelength of the waves travelling up the rope?

We know the wave speed and frequency, so this is straightforward. $v = \lambda/T = \lambda f$ so (using the wave speed at the bottom of the rope), $88.54\ m/s = (\lambda)(2\ s^{-1})$ or $\lambda = 88.54/2 = 44.27\ m$ (which is about half the length of the rope, so if we took a snapshot of the rope's transverse motion, we'd see about 2 full sine wave shapes on it).

(And note that as the wave travels up the rope, since the wave speed v gets slightly faster and $\lambda = v/T$ then the wavelength also gets slightly longer.)

(c) What will the maximum transverse acceleration be at any point on the rope?

The wave travelling up the rope can be written as $D(x,t) = A\sin(kx - \omega t)$ where D is giving the transverse displacement (how far a particular point on the rope is moving back and forth as the wave passes through that point). The transverse velocity will be the **time** derivative of this 'position' (partial derivative here), so $v_y = -A\omega\cos(kx - \omega t)$ and another (time) partial derivative will give us the transverse acceleration: $a_y = -A\omega^2\sin(kx - \omega t)$.

The **maximum** acceleration then will be the amplitude of that function or $a_{max} = A\omega^2$.

For this particular wave, $A = 4\ cm = 0.04\ m$ and $\omega = 2\pi f = (2)(\pi)(2\ s^{-1}) = 12.566\ s^{-1}$ so $a_{max} = A\omega^2 = (0.04\ m)(12.566\ s^{-1})^2 = 6.32\ m/s^2$ (so slightly less than g).

(Continued...)

(d) How much power does the geologist need to be exerting?

A sinusoidal wave travelling along a wire is carrying power of $P = 2\pi^2 \mu v f^2 A^2$ where $v = \sqrt{F_T/\mu}$.

We already have all those values: $\mu = (2\ kg)/(80\ m) = 0.025\ kg/m$, $f = 2\ Hz = 2\ s^{-1}$, $A = 4\ cm = 0.04\ m$ and we found in the first part that $v = 88.54\ m/s$ (at the bottom of the rope: we'll come back to that), so:

$P = 2\pi^2 \mu v f^2 A^2 = (2)(\pi)^2(0.025)(88.54)(2)^2(0.04)^2 = 0.28\ Watts.$

The geologist needs to be expending (at least) that same amount of power to generate these waves.

Note: in the first part, we found that v increases as we move up the rope (due to the tension changing). Assuming we aren't losing power due to internal friction in the rope, if v is increasing then something else has to change to maintain the same rate of power. The frequency will remain the same: at any point on the rope, waves aren't piling up: as many waves are leaving that point as arriving, so the factor that will change is the amplitude of the wave itself. Since everything else in the equation is constant, the factor $\boxed{vA^2}$ must be constant.

Comparing the top and the bottom: $(vA^2)_{top} = (vA^2)_{bottom}$ or rearranging: $A_{top} = A_{bottom} \times \sqrt{v_{bot}/v_{top}}$.

Using the values we found above, the 4 cm amplitude of the wiggles at the bottom of the rope end up as $A_{top} = (4\ cm) \times \sqrt{88.54/92.87} = 3.905\ cm$, so the wave amplitude is slightly decreasing as the wave moves up the rope.

See next page for the 'pendulum' part of the problem.

Pendulum part of the problem :

Version 1 : The original displacement represents an angle of $\theta = l/r = (0.20\ m)/(80\ m) = 0.0025\ rad$ or about 0.143^o, so this is **very** small angle oscillation. We can approximate the motion as $\theta(t) = \theta_o \cos(\omega t)$ with $\theta_o = 0.0025\ rad$. Careful here though: the ω in that equation represents the **angular frequency** of the oscillations of this pendulum (and **not** the angular speed, which uses the same symbol unfortunately).

The angular frequency of this pendulum will be $\omega = \sqrt{g/L} = \sqrt{9.8/80} = 0.35\ s^{-1}$, representing a period of $T = 2\pi/\omega = 17.95\ sec.$

The angular speed of the pendulum would be $d\theta/dt$. Note I avoided writing that as $\omega = d\theta/dt$ because **this** ω is angular speed, but the ω we just found above is the angular frequency and they're not the same thing. Ugh.

In any event: $(angular\ speed) = d\theta/dt = -\theta_o \omega sin(\omega t)$ so the angular speed is varying between $\pm\theta_o\omega = (0.0025)(0.35) = \pm 0.000875\ rad/s$. Multiplying this by the length of the pendulum gives us the linear speed: $v = R\omega = (80\ m)(0.000875\ rad/s) = 0.07\ m/s$.

Version 2 : As the crate moves back and forth, it's barely moving vertically at all, so let's write this directly in terms of the horizontal displacement, which has an amplitude of 20 cm. Then $x(t) = (0.2\ m)\cos(\omega t)$ and the horizontal velocity would be $v = dx/dt = -(0.2\ m)(0.35\ s^{-1})\sin(\omega t)$ so the velocity varies between $\pm 0.07\ m/s$. (Same result.)

Version 3 : A perhaps better approach is to use conservation of energy. If you go back to PH2213, if the 'point mass' on the end of the string of length L is displaced by an initial angle of θ, that mass has moved vertically upward from its lowest point by $h = L(1-\cos\theta) = (80\ m)(1-\cos(0.0025\ rad)) = 0.00025\ m$, and that tiny amount of $U_g = mgh$ turns into $K = \frac{1}{2}mv^2$ as the mass passes through the lowest point on its trajectory, leading to $v = \sqrt{2gh} = 0.0699999..\ m/s$.

The starting angle is so small here that all these basically lead to the same 7 cm/s speed as the crate passes through the lowest point.